1 Force and Translational Dynamics

1.1 Systems and Center of Mass

- System properties are determined by the interactions between objects within the system.

- If the properties or interactions of the constituent objects within a system are not important in modeling the behavior of a macroscopic system, the system can itself be treated as a single object.

- Systems may allow interactions between constituent parts of the system and the environment, which may result in the transfer of energy or mass.

- For objects with symmetrical mass distributions, the center of mass is located on lines of symmetry.

- The location of a system's center of mass along a given axis can be calculated using the equation:

$$x_{cm} = \frac{\sum m_i x_i}{M}$$

- For a nonuniform solid that can be considered as a collection of differential masses, $\mathrm{d}m$, the solid's center of mass can be calculated using

$$x_{cm} = \int x \mathrm{d}m / M$$

- A system can be modeled as a singular object that is located at the system's center of mass.

> **Example**
>
> A triangular rod of length L and mass M has a nonuniform linear mass density given by the equation $\lambda = \gamma x^2$ where $\gamma = \frac{3M}{L^2}$ and x is the distance from the left end of the rod. Determine the horizontal location of the center of the mass relative to point P. Express your answer in terms of L.
>
> From $\lambda = \frac{\mathrm{d}m}{\mathrm{d}l}$ we know that $\mathrm{d}m = \lambda \mathrm{d}l$. Plugging this into the center of mass formula $x_{cm} = \frac{\int \mathrm{d}m}{M}$ gives us $x_{cm} = \frac{\gamma \int_0^L x^3 \mathrm{d}x}{M} = \frac{3}{4}L$.

Exercise
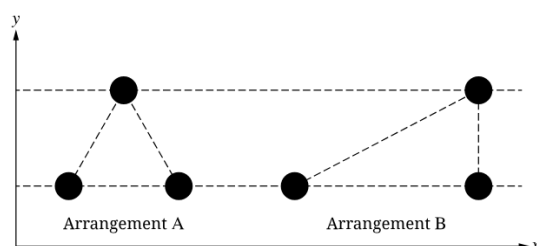
Six identical uniform spheres are arranged on a set of coordinate axes in two different triangular arrangements, A and B, as shown. How does the y-coordinate of the center of mass of the three spheres in arrangement A, $y_{cm,A}$ compare to the y-coordinate of the center of mass of the three spheres in arrangement B, $y_{cm,B}$?

Exercise
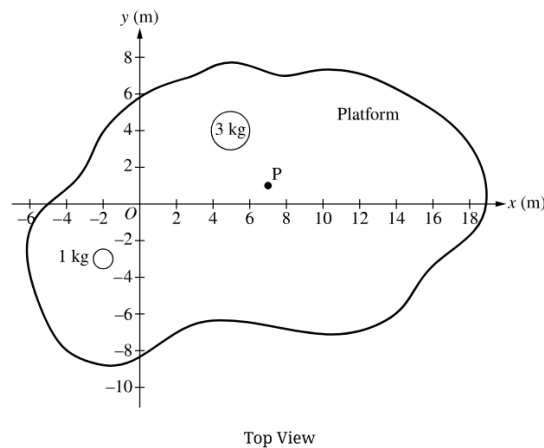
Top View

The center of mass of an irregularly shaped platform is balanced on a pivot Point P with coordinates (7.0 m, 1.0 m). Two rocks are then placed on top of the platform, as shown in the top view. One rock has a mass of 1.0 kg and is located at (-2.0 m, -3.0 m), and the second rock has a mass of 3.0 kg and is located at (5.0 m, 4.0 m). At what coordinates should a third rock of mass 4.0 kg be placed such that the three rock-platform system is balanced.

1.2 Forces and Free-Body Diagrams

- Forces are vector quantitites that describe interactions between objects or systems.

- Contact forces describe the interaction of an object or system touching another object or system.

- Free-body diagrams (FBDs) are useful tools for visualizing forces exerted on a single object or system and for determining the equations that represent a physical situation.

- The FBD of an object or system shows each of the forces exerted on the object or system by the environment.

- Forces exerted on an object or system are represented as vector originating from the center of mass, such as a dot.

- Choose a coordinate system such that one axis is parallel to the acceleration of the object or system.

Example

A skier of mass M is skiing down a frictionless hill that makes an angle θ with the horizontal, as shown in the diagram. The skier starts from rest at time $t = 0$ and is subject to a velocity-dependent drag force due to air resistance of the form $F = -bv$, where v is the velocity of the skier and b is a positive constant. Express all algebraic answer in terms of M, b, θ, and fundamental constants. Draw a dot that represents the skier, and draw a free-body diagram indicating and labeling all of the forces that act on the skier while the skier descends the hill.

The correct answer will be $F_g = mg$ pointing downwards, a normal force an angle and the force $-bv$ perpendicular to this force.

Exercise
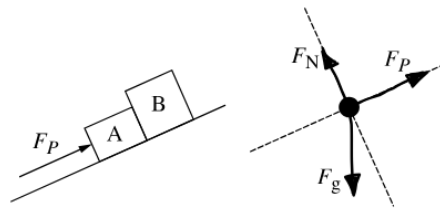
Two different blocks, A and B, are next to each other on an inclined smooth surface which has negligible friction. An applies force, F_P, pushes Block A as shown and the blocks move up the ramp. A student sketch of the free-body diagram representing the forces is given. What changes should be made to this free-body diagram?

Exercise A block is at rest on a desk's horizontal surface. A student correctly identifies the force exerted on the block as the force of Earth on the block and the force of the desk on the block. A book then is placed between the block and the desk. Which objects exert forces of equal magnitude on the block after the book has been introduced?

1.3   Newton's Third Law

Newton's third law describes the interaction of two objects or systems in terms of the paired forces that exerts on the other.

$$\vec{F}_{\text{A on B}} = -\vec{F}_{\text{B on A}}$$

Interactions between objects within a system do not influence the motion of a system's center of mass.

Tension is the macroscopic net results of forces that infinitesimal segments of a string, cable, chain or similar systme exert on each other in response to an external force.

- An ideal string has negligible mass and does not stretch when under tension.

- The tension in an ideal string is the same at all points within the string.

- In a string with nonneglibible mass, tension may not be the same at all points within the string.

- An ideal pulley that has negligible mass and rotates about an axle through its center of mass with negligible friction.


Example
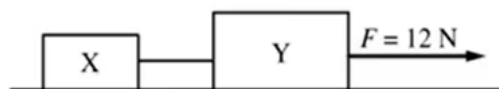
Blocks X and Y of masses 3.0 kg and 5.0 kg, respectively, are connected by a light string and are both on a level horizontal surface of negligible friction. A force $F = 12$ N is exerted on Block Y, as shown in the figure above. What is the tension in the string connecting the two blocks?

After drawing a free body diagram, we see that the $\sum F_x = ma_x$ and we can find that $a_x = 1.5 \text{ m/s}^2$.

We alsk now that $F_T = ma_x$, so $F_T = 4.5$ N.

Exercise A cart moving to the right collides with a stationary block, resulting in the two objects sliding together along the horizontal surface until coming to a stop. During the collision, the cart exerts a force F_1 on the block, the surface exerts a force of friction F_2 on the block, and the block exerts a force F_3 on the cart. Which two forces are equal during the collision?

Exercise
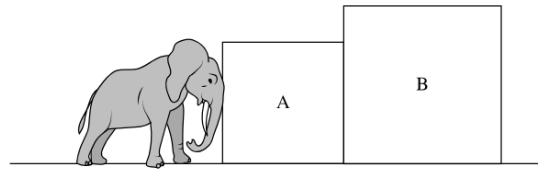
An elephant pushes two heavy boxes across a rough surface. The force that Box A exerts on Box B is F_{AB} and the force that Box B exerts on Box A is F_{BA}. What must be true of the two boxes to support that $|F_{AB}| = |F_{BA}|$?

1.4 Newton's First Law

The net force on a system is the vector sum of all forces exerted on the system.

Translational equilibrium is the configuration of forces that the net force exerted on a system is zero.

$$\sum F = 0$$

Newton's first law states that if the net force exerted on a system is zero, the velocity of that system will remain constant.

Forces may be balanced in one dimension but unbalanced in another.

Example
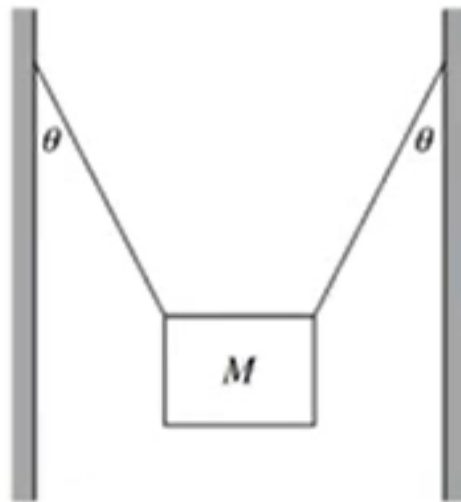
A heavy sign of mass M is held at rest by two supporting wires between two buildings, with each wire making an angle θ with the vertical, as shown in the figure. What is the tension in each wire?

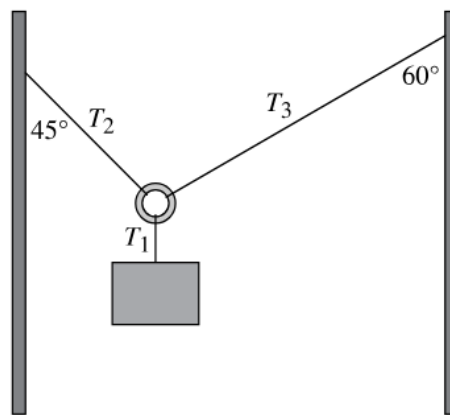Drawing the free body diagram of the system results in the following:

In the x-direction, we get $T = T$.

In the y-direction we get $2T \cos \theta = Mg$.

Solving this for T gives $T = \frac{Mg}{2 \cos \theta}$

Exercise An object is moving while a constant force is exerted on it. Could the addition of a force of the same magnitude cause the object to move with a constant velocity? Why or why not?

Exercise

A heavy block is suspended by a string which is attached to a plastic ring. The ring is attached to two other strings which are tied to vertical supports at the angles shown. The masses of the ring and strings are negligible. Compare the magnitudes of the tensions in the strings T_1, T_2, and T_3.

1.5 Newton's Second Law

Unbalanced forces are a configuration of forces such that the net force exerted on a system is not equal to zero.

Newton's second law of motion states that the acceleration of a system's center of mass has a magnitude proportional to the magnitude of the net force exerted on the system and is in the same direction of the force.

$$\sum F = ma = 0$$

The velocity of a system's center of mass will only change if a nonzero net external force is exerted on that system/
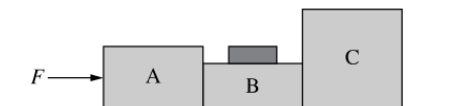
> **Example**
>
> An object of mass 10 kg starts from rest at time $t = 0$ and moves in a straight line. For time $t > 0$, the object's velocity as a function of time t is given by $v = 2t + 3t^2$, where v is in m/s and t is in seconds. What is the instantaneous net force that acts on the object at $t = 2$ s?
>
> The acceleration function is given by $2 + 6t$, so $a(2) = 14$ m/s^2.
>
> $F = ma$, so plugging in numbers gives 140 N.

Exercise

Three large blocks, A, B, and C, and a small block attached to Block B slide across a horizontal surface as a constant force F is exerted on Block A, as shown in the figure. There is negligible friction between the blocks and the horizontal surface. Block A pushes Block B with a force F_{AB}. The small block is then removed from Block B and attached to Block C and the same force F is exerted on Block A. How does F_{AB} compare in the second situation to the first situation and why?

Exercise
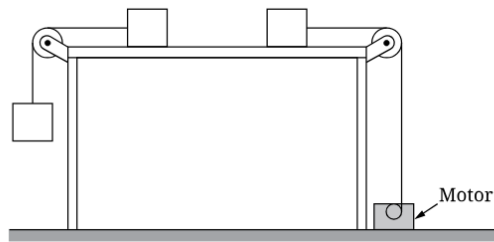
Two identical blocks are placed on a table as shown in the figure. The block on the left is attached to another identical block hanging over the edge of the table. The block on the right is attached to a motor pulling downward with a constant tension equal to the weight of one block. The mass of the strings and friction between the blocks and table are negligible and the pulleys are ideal. How do the magnitudes of the acceleration of the blocks compare and why?

1.6 Gravitational Force

Newton's law of universal gravitation describes the gravitational force between two objects as directly proportional to each of their masses and inversely proportional to the square of the distance between their centers.

$$F_G = \frac{Gm_1m_2}{d^2}$$

A field models the effects of a noncontact force exerted on an object at various positions in space.
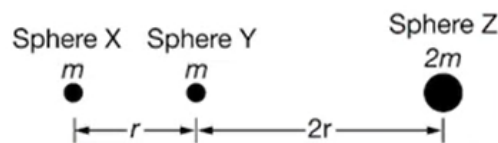
The magnitude of the gravitational field created by a system of mass M at a point in space is equal to the ratio of the gravitational force exerted by the system on a test object of mass m to the mass of the test object.

$$\vec{g} = \frac{\vec{F}_g}{m}$$

If a system is accelerating, the apparent weight of the system is not equal to the magnitude of the gravitational force exerted on the system.

Newton's shell law theorem describes the net gravitational force exerted on an object by a uniform spherical shell of mass.
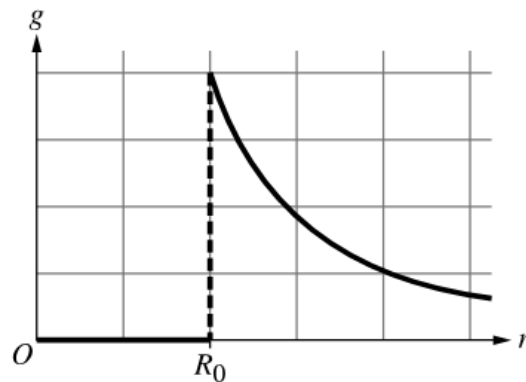
Example

Spheres X, Y, and Z have the masses and locations indicated in the figure above. What is the magnitude of the net gravitational force on sphere X due to the other two spheres?
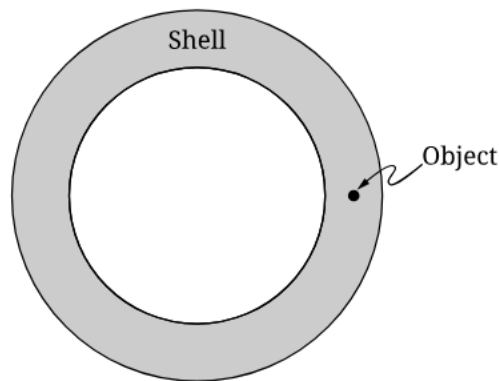
We have that $F_y = \frac{Gm^2}{r^2}$ and $F_z = \frac{1}{2}\frac{Gm^2}{r^2}$ so adding these two together gives $\frac{3}{2}\frac{Gm^2}{r^2}$.

Exercise

The gravitational field g of a spherically symmetric object of radius R_0 as a function of distance r from the object's center is shown in the graph. What best describes the object?

Exercise

A large spherical shell with a uniform mass distribution contains a small object within the thickness of the shell, as shown in the figure. At which locations could the object be moved to increase the magnitude of the gravitational force exerted on the object by the shell?

1.7   Kinetic and Static Friction

Kinetic friction occurs when two surfaces in contact move relative to each other.

- It opposes the direction of motion.

- The surface area of contact is not a factor.

The magnitude of the kinetic friction force exerted on an object is the product of the normal force the surface exerts on the object and the coefficient of kinetic friction.

$$f_k = \mu_k F_N$$

Static friction may occur between the contacting surfaces of two objects that are not moving relative to each other.

Static friction adopts the value and direction required to prevent an object from slipping or sliding on a surface.

$$f_s \leq \mu_s F_N$$

The coefficient of static friction is typically greater than the coefficient of kinetic friction for a given pair of surfaces.

Example
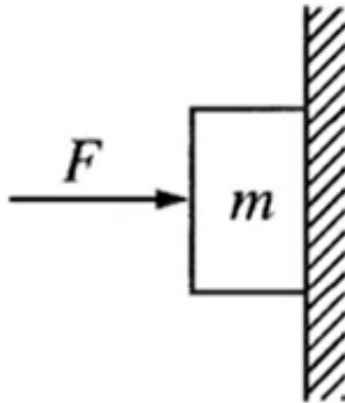
A horizontal force F pushes a block of mass m against a vertical wall. The coefficient of friction between the block and the wall is μ. What value of F is necessary to keep the block from slipping down the wall?

In the x direction the forces result in $F_N = F$.

In the y direction the forces end up with $f = F_g$ or $\mu F_N = mg = \mu F = mg$. The force is therefore $F = \frac{mg}{\mu}$.

Exercise
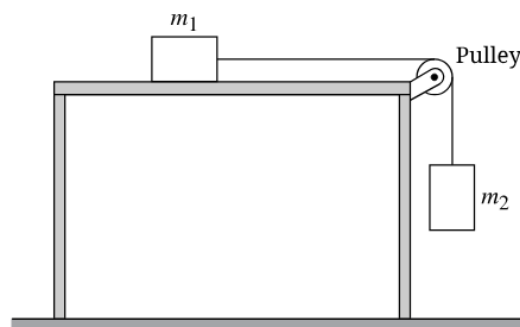
A block of mass m_1 rests on a rough horizontal tabletop, as shown in the figure. The block is connected by a string to a second block of mass m_2, which hangs below a pulley at the edge of the table. The coefficient of static friction between the tabletop and the first block is μ_s. The masses of the string and the pulley are negligible, and the pulley can rotate with negligible friction on its axle. What is the minimum mass m_2 that will cause the blocks to start moving?

Exercise A rectangular block is pushed by a constant force and accelerates along a rough horizontal surface. The block can be oriented to slide along any of three different sides, A, B, and C. Sides A, B, and C have surface areas S_A, S_B, and S_C, respectively where $S_A < S_B < S_C$. On which side should the block be placed to have the greatest magnitude of acceleration?

1.8   Spring Forces

An ideal spring has negligible mass and exerts a force that is proportional to the change in its length as measured from its relaxed length.

The magnitude of the force exerted by an ideal spring on an object is given by Hooke's Law:

$$F_{sp} = -k\Delta x$$

The force exerted on an object by a spring is always directed toward the equilibrium position of the object-spring system.

A collection of springs that exert forces on an object may behave as though they were a single spring with an equivalent spring constant.

- Springs in series: $\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots$

- Springs in parallel: $k_{eff} = k_1 + k_2 + \ldots$

Example

To illustrate a human soft tissue deformation, a science teacher uses two ideal springs and a small sphere. The sphere of mass m_s is attached to the free ends of the two springs. Then, the system is suspended vertically. The upper string has an equilibrium L_u and a spring constant k_u. The lower spring has an equilibrium length L_l and a spring constant k_l. The teacher fixes an additional small block of mass m_b to the free end of the lower spring. Find the expression of the system's total length.

The upper string is given as $F_{sp} = k_u \Delta x_u$. Plugging in total mass and gravity we get $(m_b + m_s)g = k_u \Delta x_u$. Solving for Δx_u gives $\Delta x_u = \frac{(m_b+m_s)g}{k_u}$

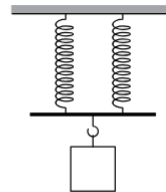The lower string is given by a similar approach and gives us $\Delta x_l = \frac{m_b g}{k_l}$.

The total length is therefore $L_T = L_l + L_u + \frac{(m_b+m_s)g}{k_u} + \frac{m_b g}{k_l}$.

Exercise When a block of mass M is hung vertically from a spring, the spring is stretched by a distance D compared to its unstretched length. If a second identical spring is connected in series with the first spring and a larger block of mass $2M$ is then hung vertically from the two-spring combination, by how much is the combination stretched compared to its unstretched length?

Exercise
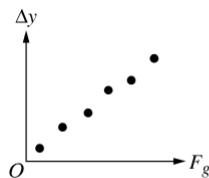
Data for Single Spring   Two-Spring Arrangement

Some students attach a single spring to a clamp and let the spring hang vertically. Objects of different mass are attached to the free end of the spring and allowed to hang at rest. The students measure the distance Δy the spring stretches from its equilibrium length for each object. The students produce the graph of Δy as a function of the weight F_g of the objects shown in the figure, and the slope of the best-fit line to the data is determined to be S_1. Next, the students take a second spring that is identical to the first and arrange the two springs as shown in the two-spring arrangement next to the graph. Once again, the objects of different mass are attached to the two-spring arrangement, Δy is measured, and the data is plotted on another graph showing Δy as a function of F_g. What best describes the slope of the best-fit line to the data collected for the two-spring arrangement?

1.9 Resistive Forces

A resistive force is defined as a velocity-dependent force in the opposite direction of an object's velocity.

$$F_R = -kv[F_R = -bv^2]$$

Applying Newton's second law to an object upon which a resistive force is exerted results in a differential equation for velocity.

- The differential portion of a=the equation comes from substituting in $a = \frac{dv}{dt}$

Terminal velocity is defined as the maximum speed achieved by an object moving under the influence of a constant force and a resistive force that are exerted on the object in opposite directions.

- For a falling object, this occurs when the air resistance equals the weight of the object.

Example
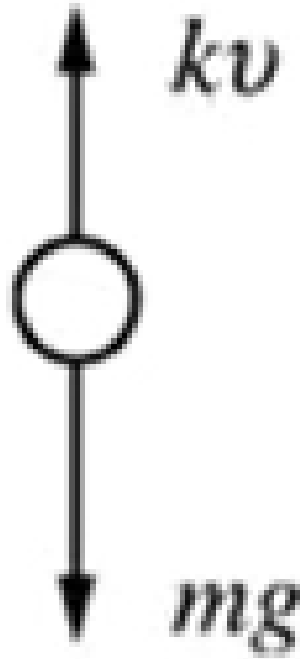
The object of mass m shown above is dropped from rest near Earth's surface and experiences a resistive force of magnitude kv, where v is the speed of the object and k is a constant. Derive an expression for the velocity of the object at any point in time. (Assume that the direction of the gravitational force is positive.)

We have that $\Delta F = ma$ so we have $mg - kv = ma$. We also have $mg - kv = m\frac{\mathrm{d}v}{\mathrm{d}t}$ as well as $mg - kv_T = 0$, so $v_T = \frac{mg}{k}$.

From $mg - kv = m\frac{\mathrm{d}v}{\mathrm{d}t}$ we can simplify this to $\int_0^t \mathrm{d}t = \int_0^{v(t)} \frac{\mathrm{d}v}{g - \frac{kv}{m}}$.

Solving this gives $t = -\frac{m}{k} \ln(1 - \frac{kv}{mg})$.

Simplifying for $v(t)$ gives $v(t) = \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right)$.

Exercise An object is released from rest and falls to the ground near Earth's surface. The resistive force exerted on the object is directly proportional to the speed of the object which results in a velocity function which includes the term $e^{-\frac{t}{\beta}}$, where β is a positive constant. What best describes the motion of the object if it falls for a time equal to β?

Exercise Two spheres, A and B, of identical size and surface material, but different masses, are dropped from rest near the surface of Earth. While falling, each sphere experiences a resistive force which is proportional to the sphere's velocity. What are the relationships of the magnitude of the initial acceleration a_0 of each sphere and of the terminal speed v_T of each sphere if $m_A < m_B$?

1.10 Circular Motion

Centripetal acceleration is the component of an object's acceleration directed toward the center of the object's circular path.

- The magnitude of the acceleration for an object moving in a circular path is the ratio of the object's tangential speed squared to the radius of the circular path.

$$a_c = v^2/r$$

Centripetal acceleration can result from a single force, more than one force, or components of forces that are exerted on an object in circular motion.

Tangential acceleration is the rate at which an object's speed changes and is directed tangent to the object's circular path.

$$a = \sqrt{a_c^2 + a_T^2}$$

The net acceleration of an object moving in a circle is the vector sum of the centripetal acceleration and tangential acceleration.

The revolution of an object traveling in a circular path at a constant speed (UCM) can be described using period and frequency.

$$v = \frac{2\pi r}{T} = 2\pi r f \qquad T = \frac{1}{f}$$

Example

A billiard ball (mass $m = 0.150$ kg) is attached to a light string that is 0.50 meters long and swung so that it travels in a horizontal, circular path of radius 0.40 m, as shown.
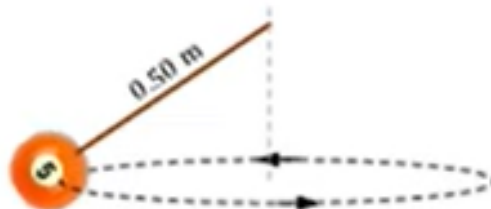
a. On the diagram, draw a free-body diagram of the forces acting on the billiard ball.

There will be a force T in the direction of the string, a_c pointing right from the billiard ball and F_g pointing downwards.

b. Calculate the force of tension in the string as the ball swings in a horizontal circle.

We know that $T \sin \theta = F_g$. From this we can determine that $T = 2.5$ N.

c. Determine the magnitude of the centripetal acceleration of the ball as it travels in the horizontal circle.

We know that $T \cos \theta = ma_c$, so solving for a_c gives us 13.3 m/s^2.

d. Calculate the period T (time for one revolution) of the ball's motion.

We know that $a_c = \frac{v^2}{r}$ so we can find that that $v = 2.30$ m/s. We also know that $v = \frac{2\pi r}{T}$, so solving for T gives 1.15.

Exercise An object of mass m is attached to the end of a spring. The string is spun around in a vertical circle of radius r. When the object is at the top of its path, the speed of the object is v and the string has a tension F_T. Write an expression for v at the top of the circular path.

Exercise Two small blocks, P and Q rotate without slipping on a horizontal disk with Block P being twice as far from the rotational axis of the disk as Block Q. The blocks are made of the same material and Block P is half the mass of Block Q. As the disk increases in speed, which block will be the first to begin to slide on the disk's surface?