Calculus III Notes

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Contents

1 Three-Dimensional Coordinate Systems

1.1 Vectors in the Plane

The first goal for calculus 3 is to define the rectangular coordinate plane in 3-space.

In 2-space, this is the normal coordinate axis with coordinates (x, y).

In 3-space, you now have (x, y, z). We have an xy-plane, a yz-plane, and an xz-plane. Also we have octants in 3-space similar to quadrants in 2-space.

The z-axis is determined by the right-hand rule: if you curl the fingers on your right hand from the positive x-axis to the y-axis, your thumb points in the direction of the positive z-axis.

Exercise Graph $(2, 3, 4)$.

When graphing points in 3-space, it might be easier to draw a prism to see the three-dimensional components easier.

Exercise Graph $(-3, 2, -6)$.

In 3-space the distance formula is simliar to 2-space. It is the following:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Example

Find the distance between $(2, 3, 4)$ and $(-3, 2, -6)$.

Plug in the numbers into the formula and the answer gives $d = \sqrt{126}$

Similar to circles in 2-space, there are spheres in 3-space. The equation of a sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Example

Write the equation of a sphere with a diameter having endpoints $(-1, 2, 1)$ and $(0, 2, 3)$.

We can find the center with the midpoint formula, and the center is $(-\frac{1}{2}, 2, 2)$.

Using the $(-1, 2, 1)$ endpoint and the center calculated above, we can determine the radius to be $\sqrt{5/4}$.

Therefore the equation of the sphere is $(x + \frac{1}{2})^2 + (y - 2)^2 + (z - 2)^2 = \frac{5}{4}$.

Example

Find the center and radius of the sphere $x^2 + y^2 + z^2 - 2x - 4y + 8z + 17 = 0$.

When we complete the square for this, we end up getting $(x - 1)^2 + (y - 2)^2 + (z + 4)^2 = 4$.

Therefore, the center is $(1, 2, -4)$ and the radius is 2.

Theorem 1.1

An equation of the form $x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$ represents a sphere, a point, or has no graph.

We would get a point if the radius gives you 0, and if the radius is negative, then there is no graph.

Exercise Graph $y = 3$ in $\mathbb{R}^3$.

Exercise Describe and sketch the surface in $\mathbb{R}^3$ represented by $y = x$.

Exercise Graph $x^2 + z^2 = 1$.

Example

Describe the graph of $1 \leq x^2 + y^2 + z^2 \leq 4$. What if $z \leq 0$?

This will represent the region between the spheres and centered at the origin with radii of 1 and 2. (This is a sphere with center cut out).

The condition $z \leq 0$ will give us a hemisphere, it will only give the lower half of the figure.

1.2 Vectors

Definition: Vector

A vector indicates a quantity that has both magnitude and direction.

Examples include displacement, velocity, or force.

Some ways to notate vectors are **u**, $\overrightarrow{u}$, $\vec{u}$, $\overrightarrow{AB}$, $\vec{AB}$. For the last two of these, these are read as a vector starting at A, heading towards B.

We say that 2 vectors are equivalent (or equal) if they have the same length and direction.

0 vector ($\mathbf{0}$) has a length of 0 and no direction. (Note that **0** is a vector because it is bolded.)

Definition: Vector Addition

If **u** and **v** are positioned so that the initial point of **v** is at the terminal point of **u**, then $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of **u** to the terminal point of **v**.

Note that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

Definition: Scalar Multiplication

If c is a scalar and **v** is a vector, then $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of **v** and whose direction is the same as **v** if $c > 0$ and is opposite if $c < 0$.

Using the two definitions above, we can find the difference $\mathbf{u} - \mathbf{v}$. From the above definitions, we can rewrite this as $\vec{u} + (-\vec{v})$, which is equivalent to $\vec{u} - \vec{v}$.

Exercise Sketch $\mathbf{a} - 2\mathbf{b}$ and define **a** and **b** as you wish.

Coordinate Systems: Generally we place the initial point at the origin and then express a vector as $\mathbf{a} = \langle a_1, a_2 \rangle$ or $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ where (a_1, a_2) and (a_1, a_2, a_3) are terminal points.

Given points $A(x_1, y_1)$ and $B(x_2, y_2)$, vector $\mathbf{a} = \mathbf{AB}$ is $\mathbf{a} = \langle x_2 - x_1, y_2 - y_1 \rangle$. (In 3-space the idea is similar.)

Example

Express vector $\overrightarrow{P_1P_2}$ in bracket notation if $P_1(1,3)$ and $P_2(4,-2)$.

Note that we start at the terminal point. Therefore we have $\overrightarrow{P_1P_2} = \langle 4-1, -2-3\rangle = \langle 3,-5\rangle$.

Note that the vector between the two points and the vector found have the same direction and same length.

Arithmetic Operations: If $\vec{v} = \langle v_1, v_2\rangle$ and $\vec{w} = \langle w_1, w_2\rangle$, then

- $\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2\rangle$
- $\vec{v} - \vec{w} = \langle v_1 - w_1, v_2 - w_2\rangle$
- $k\vec{v} = \langle kv_1, kv_2\rangle$

Example

If $\vec{a} = \langle -2, 0, 1\rangle$ and $\vec{b} = \langle 3, 5, -4\rangle$, find $\vec{a} + \vec{b}$ and $\vec{b} - 2\vec{a}$.

Finding $\vec{a} + \vec{b}$ is simple just add them together to get $\langle 1, 5, -3\rangle$.

To find $\vec{b} - 2\vec{a}$, multiply $\vec{a}$ by 2 to get $\langle -4, 0, 2\rangle$. Then subtracting gives you $\langle 7, 5, -6\rangle$.

Properties of Vectors:

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3. $\vec{a} + \mathbf{0} = \vec{a}$
4. $\vec{a} + (-\vec{a}) = \mathbf{0}$ (Additive Inverse)
5. $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
6. $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
7. $(cd)\vec{a} = c(d\vec{a})$
8. $1\vec{a} = \vec{a}$

Example

Prove property #2 from above.

Let $\vec{a} = \langle a_1, a_2\rangle$, $\vec{b} = \langle b_1, b_2\rangle$, and $\vec{c} = \langle c_1, c_2\rangle$.

From the left side of property 2, we have $\langle a_1, a_2\rangle + (\langle b_1, b_2\rangle + \langle c_1, c_2\rangle)$.

From this, we can simplify to get $\langle a_1 + a_2\rangle + \langle b_1 + c_1, b_2 + c_2\rangle$.

This gives us $\langle a_1 + (b_1 + c_1), a_2 + (b_2 + c_2)\rangle$.

From the associative law we can rewrite this as $\langle (a_1 + b_1) + c_1, (a_2 + b_2) + c_2\rangle$.

This is $\langle a_1 + b_1, a_2 + b_2\rangle + \langle c_1 + c_2\rangle$, which is equivalent to $(\vec{a} + \vec{b}) + \vec{c}$. $\square$

Unit Vectors: A unit vector is a vector with a length of 1.

In 2-space, define $\vec{i} = \mathbf{i} = \langle 1, 0\rangle$ and $\vec{j} = \mathbf{j} = \langle 0, 1\rangle$ and 3-space, $\vec{i} = \mathbf{i} = \langle 1, 0, 0\rangle$, $\vec{j} = \mathbf{j} = \langle 0, 1, 0\rangle$, and $\vec{k} = \mathbf{k} = \langle 0, 0, 1\rangle$.

$\vec{i}, \vec{j}$, and $\vec{k}$ are called unit or standard basis vectors. All have length 1 and point in the positive direction on the $x-$,$y-$, and $z-$ axes.

For example, $\vec{a} = \langle 1, 3, -4\rangle$ can be expressed as $\vec{a} = \vec{i} + 3\vec{j} - 4\vec{k}$.

Example

If $\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}$, and $\vec{b} = 4\vec{i} + 7\vec{k}$, find $2\vec{a} + 3\vec{b}$.

Adding them together gives $14\vec{i} + 4\vec{j} + 15\vec{k}$.

The norm (or magnitude or length) of a vector is defined as

$$|\vec{v}| = \|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$$

Example

Let $\vec{u} = \vec{i} - 3\vec{j} + 2\vec{k}$ and $\vec{v} = \vec{i} + \vec{j}$. Find:

- $\|\vec{u}\| + \|\vec{v}\|$ Use the above formula to get $\sqrt{14} + \sqrt{2}$.
- $\|\vec{u} + \vec{v}\|$. First add the two vectors to get $\langle 2, -2, 2\rangle$. The norm of this is $2\sqrt{3}$.
- $\frac{1}{\|\vec{v}\|}\vec{v}$ We previously found the magnitude of $\vec{v}$ to be $\sqrt{2}$. So we simply have $\frac{1}{\sqrt{2}}\langle 1, 1, 0\rangle$. This is $\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$.
- $\|\frac{1}{\|\vec{v}\|}\vec{v}\|$. The norm of $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\rangle$ is 1.

The vector found in the third part of the previous example is known as a unit vector because it has a norm of 1.

The process of obtaining a unit vector with the same direction is called normalizing $\vec{v}$.

Example

Find the unit vector in the direction of $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$.

The norm of a is $\|\vec{a}\| = \sqrt{4 + 1 + 4} = 3$. Then normalizing $\vec{a}$ gives $\langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\rangle$.

Vectors in Polar Form: Any vector can be written in the following form

$$\vec{v} = \|\vec{v}\|\langle \cos\theta, \sin\theta\rangle$$

We know that $\|\vec{v}\|$ is the magnitude of the vector and $\langle cos\theta, \sin\theta\rangle$ is the direction.

Example

Find the angle that $\vec{v} = \langle -\sqrt{3}, 1\rangle$ makes with the positive x-axis.

We know we can write this as $\langle -\sqrt{3}, 1\rangle = \|\vec{v}\|\langle \cos\theta, \sin\theta\rangle$.

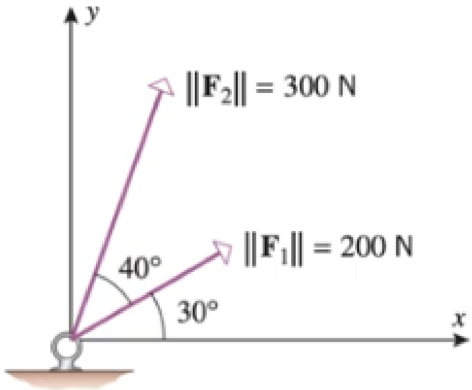
The magnitude of $\vec{v}$ is 2, so simplifying a little gives us $\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\rangle = \langle \cos\theta, \sin\theta\rangle$.

From this, we can see that θ is the θ when $\cos\theta = -\frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$. So, $\theta = \frac{5\pi}{6}$.

Forces are often represented by vectors because they have a length and direction. If two forces are applied to the same point, they are concurrent. The two forces together form the resultant force, $\vec{F_1} + \vec{F_2}$.

Example

Suppose two forces are applied to an eye bracket. Find the magnitude of the resultant and the angle that it makes with the positive x-axis.



From the diagram, we can see that $\vec{F_1} = 200\langle \cos 30°, \sin 30° \rangle = \langle 100\sqrt{3}, 100 \rangle$.

For $\vec{F_2}$, we get $\vec{F_2} = \langle 300\cos 70°, 300\sin 70° \rangle$.

Adding them gives $\vec{F} = \langle 100\sqrt{3} + 300\cos 70°, 100 + 300\sin 70° \rangle$.

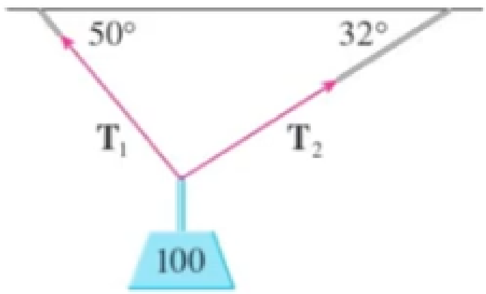
This approximates to $\langle 275.8, 381.9 \rangle$. The magnitude of this approximates to $\|\vec{F}\| \approx 471$ N.

If we let $100\sqrt{3} + 300\cos 70°$ be $\|\vec{F}\|\cos\theta$, then we can determine the angle.

When we find θ in $\cos\theta = \frac{100\sqrt{3}+300\cos 70°}{471}$, we get $\theta \approx 54.2°$.

Example

A 100-lb weight hangs from two wires. Find the forces (tensions) $\vec{T_1}$ and $\vec{T_2}$ in both wires and the magnitude of those tensions.



Note that $\vec{T_1}$ does not have an angle of $50°$, rather it has a degree of $130°$ from the point.

Therefore, we have $\vec{T_1} = |\vec{T_1}|\langle \cos 130°, \sin 130° \rangle$.

Note we can rewrite this with to be in the first quadrant as $|\vec{T_1}|\langle -cos 50°, \sin 50° \rangle$.

We also have $\vec{T_2} = |\vec{T_2}|\langle \cos 32°, \sin 32° \rangle$.

We also see that the weight of the block is $\vec{W} = 100\langle 0, -1 \rangle = \langle 0, -100 \rangle$, so we see to balance this out, both tension forces need to equal $\langle 0, 100 \rangle$.

Therefore, we see that $-|\vec{T_1}| \cos 50° + |\vec{T_2}| \cos 32° = 0$ (addition of the $\vec{i}$ components).

We also have $|\vec{T_1} \sin 50° + |\vec{T_2}| \sin 32° = 100$ ($\vec{j}$ components).

Solving for $|\vec{T_1}|$ and $|\vec{T_2}|$ from this gives us 85.64 lbs and 64.91 lbs respectively.

Plugging these values back in gives us the tension vectors.

$\vec{T_1} \approx \langle -55.05, 65.60 \rangle$

$\vec{T_2} \approx \langle 55.05, 34.40 \rangle$

1.3 The Dot Product

Definition: The Dot Product

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the dot product $\vec{a} \cdot \vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b + 1 + a_2 b_2 + a_3 b_3$$

This is called the scalar or inner product. (Note: 2-space is a similar idea)

Example

$(\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (2\vec{j} - \vec{k})$

Using the dot product formula gives you 7.

Properties of dot products:

1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$

Example

Prove the first property from above.

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$.

The dot product of $\vec{a}$ and $\vec{a}$ gives us $a_1^2 + a_2^2 + a_3^2$.

The magnitude of this is $\sqrt{a_1^2 + a_2^2 + a_3^2}^2$ which is equal to $|\vec{a}|^2$ □

Example

Prove the third property from above.

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, likewise for $\vec{b}$ and $\vec{c}$.

Then when we do $\vec{a} \cdot (\vec{b} + \vec{c})$ we get

$$\begin{aligned} &\langle a_1, a_2, a_3 \rangle \cdot (\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle) \\ &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\ &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\ &= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \end{aligned}$$

□

Theorem 1.2

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

Corollary 1.3

$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

Example

Find the angle between $\vec{u} = \vec{i} - 2\vec{j} + 2\vec{k}$ and $\vec{v} = -3\vec{i} + 6\vec{j} + 2\vec{k}$.

The dot product of the two gives -11.

The magnitude of $\vec{u}$ is 3 and the magnitude of $\vec{v}$ is 7.

We see that $\cos\theta = -\frac{11}{3 \cdot 7}$, so $\theta = 2.12$ radians or $121.6°$.

Recall, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$. Since $|\vec{a}||\vec{b}|$ is always positive, the sign of the dot product is determined by $\cos\theta$.

- If $\vec{a} \cdot \vec{b} > 0$, then the angle is acute.
- If $\vec{a} \cdot \vec{b} < 0$, then the angle is obtuse.
- If $\vec{a} \cdot \vec{b} = 0$, then the vectors are orthogonal.

To determine if two vectors are parallel, the vectors have to be scalar multiples of each other.

Definition: Direction Angles

The direction angles α, β, and $\gamma([0,\pi])$ are the angles that $\vec{a}$ makes with the positive $x-$, $y-$, and $z-$ axes. Their cosines are called direction cosines.

$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{\Longrightarrow} \cos\alpha = \frac{a_1}{|\vec{a}|}$. Likewise, $\cos\beta = \frac{a_2}{|\vec{a}|}$ and $\cos\gamma = \frac{a_3}{|\vec{a}|}$.

Notice that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Also $\vec{a} = |\vec{a}|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$, which can be expressed as $\frac{\vec{a}}{|\vec{a}|} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle$.

So, the direction cosines form the unit vector in the direction of $\vec{a}$.

Example

Find the direction cosines of $\vec{a} = \langle 2, -4, 4\rangle$ and approximate the direction angles to the nearest degree.

The magnitude of $\vec{a}$ is 6.

From the formulas, we can find that $\cos\alpha = \frac{1}{3}$, $\cos\beta = -\frac{2}{3}$, and $\cos\gamma = \frac{2}{3}$.

Finding the angles gives us $\alpha \approx 71°$, $\beta \approx 132°$, and $\gamma \approx 48°$.

Example

Find the angle between a diagonal of a cube and one of its edges.

Let's call the vector from the diagonal to the edge as $\vec{d}$ and the length of the edge be a.

We get $\vec{d} = \langle a, a, a\rangle$ as a result. The magnitude of $\vec{a}$ ends up being $\sqrt{3a^2}$.

We get that $\cos\alpha = \frac{a}{\sqrt{3a^2}} = \frac{1}{\sqrt{3}}$.

This approximates $\alpha \approx 0.955$ radians or $54.7°$.

$\text{proj}_{\vec{a}}\vec{b}$ is the vector projection of $\vec{b}$ onto $\vec{a}$

$\text{comp}_{\vec{a}}\vec{b}$ is the scalar projection of $\vec{b}$ onto $\vec{a}$ (a signed magnitude of the vector projection).

If we look at the scalar projection, $\text{comp}_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta$ and remember that $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$. Therefore $\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

The projection will just be the magnitude (which is the scalar projection) multiplied by the unit vector: $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\left(\frac{\vec{a}}{|\vec{a}|}\right)$.

This is equal to $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$.

Example

Find the scalar and vector projections of $\vec{b} = \langle 1, 1, 2\rangle$ onto $\vec{a} = \langle -2, 3, 1\rangle$.

The dot product of the vectors is 3, the magnitude of $\vec{a} = \sqrt{14}$.

From the formula above, the scalar projection is $\frac{3}{\sqrt{14}}$.

The vector projection gives $\frac{3}{(\sqrt{14})^2}\langle -2, 3, 1\rangle$, simplifying to $\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\rangle$.

Previously, you learnt work is defined as $W = F \cdot d$ (this only applies when the force is directed along the line of motion).

Now we can define work as $W = (|\vec{F}|\cos\theta)|\vec{d}|$. Moving stuff around gives $W = |\vec{F}||\vec{D}|\cos\theta$.

Now we see that $W = \vec{F}\cdot\vec{d}$. (Work is constant which means it should result in a scalar.)

Example

A wagon is pulled horizontally by exerting a constant force of 10 lb on the handle at an angle of $60°$ with the horizontal. How much work is done in moving the wagon 50 feet?

We can easily see that the displacement vector is $\vec{d} = \langle 50, 0 \rangle$.

The force vector we must divide into components, so we get $\vec{F} = 10\langle \cos 60°, \sin 60° \rangle = \langle 5, 5\sqrt{3} \rangle$.

So the dot product of both vectors gives us 250 ft-lb.

1.4 The Cross Product

To the interested reader that has gotten this far, review how to find the determinant of a matrix. *Exercise* $\begin{vmatrix} 4 & -2 \\ -5 & -1 \end{vmatrix}$

Exercise $\begin{vmatrix} 3 & -1 & 4 \\ 2 & -2 & 5 \\ 4 & -1 & 0 \end{vmatrix}$

Remember: when you see the straight vertical bars, you are trying to find the determinant.

Also remember

- If any two rows are the same, the determinant is zero.
- Interchanging two rows in a determinant multiplies the value by -1.

Definition: Cross Product

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of $\vec{a}$ and $\vec{b}$ is:

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

OR

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Notice: This is a vector.

Example

$\vec{u} = \langle 1, 2, -2 \rangle$ and $\vec{v} = \langle 3, 0, 1 \rangle$. Find $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$.

First set up $\vec{u} \times \vec{v}$.

This is $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix}$.

This is $\vec{i}\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}$.

This gives you a vector $\vec{u} \times \vec{v} = 2\vec{i} - 7\vec{j} - 6\vec{k}$.

Now when we set up $\vec{v} \times \vec{u}$, notice that the two rows will be interchanged. Therefore $\vec{v} \times \vec{u} = -2\vec{i} + 7\vec{j} + 6\vec{k}$.

Important: $\vec{a} \times \vec{b}$ is orthogonal to BOTH $\vec{a}$ and $\vec{b}$. (If you are asked to find a vector orthogonal to both $\vec{a}$ and $\vec{b}$, this is useful.)

Use the Right-Hand Rule to find direction. If the fingers of your right hand curl in the direction of rotation (less than $180°$) from $\vec{a}$ to $\vec{b}$, then your thumb points in the direction of $\vec{a} \times \vec{b}$.

Properties of the Cross Product:

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

A few more important things:

- $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta$ (Similar to the dot product)
- $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$. This is because $\sin\pi = 0$.
- The length of $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

Example

Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

We need to find $\overrightarrow{PQ} \times \overrightarrow{PR}$ to find the vector perpendicular to the plane.

Finding $\overrightarrow{PQ}$ gives $\langle -3, 1, -7\rangle$ and finding $\overrightarrow{PR}$ gives $\langle 0, -5, -5\rangle$.

The cross product of this is $-40\vec{\imath} - 15\vec{\jmath} + 15\vec{k}$. (All scalar multiples of this are also perpendicular.)

Example

Find the area of a triangle determined by $P_1(2, 2, 0), P_2(-1, 0, 2)$, and $P_3(0, 4, 3)$.

First you want to find the area of the parallelogram, then divide it by two.

Previously, the area of the parallelogram was determined by $\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\|$.

Do a similar process as the last example, the cross product is $\langle -10, 5, -10\rangle$.

The magnitude of this vector is 15, so the area of the triangle is $15/2$.

Now the scalar triple product: we have $\vec{a} \cdot (\vec{b} \times \vec{c})$. This gives a scalar.

Note this is $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

The absolute value of this scalar triple product gives the volume of a parallelpiped determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$. (Note: absolute value, not magnitude.)

Example

Use the scalar triple product to show that $\vec{a} = \langle 1, 4, -7\rangle$, $\vec{b} = \langle 2, -1, 4\rangle$, and $\vec{c} = \langle 0, -9, 18\rangle$ are coplanar.

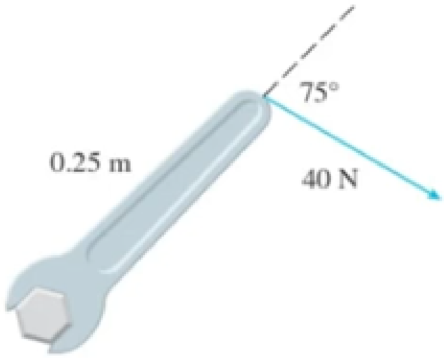
The scalar triple product of this will give a determinant of 0. The volume of the parallelpiped therefore is 0, which means that all three vectors are coplanar (meaning on the same plane).

Torque measures the tendency of a body to rotate about the origin. The direction of the torque vector indicates the axis of rotation.

$\vec{\tau} = \vec{r} \times \vec{F}$ where $\vec{r}$ represents position and $\vec{F}$ represents force.

Example

A bolt is tightened by applying a 40-N force to a 0.25 m as shown. Find the magnitude of the torque about the center of the bolt.



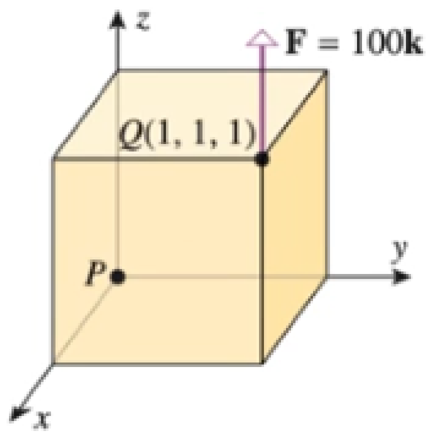
Remember: the $\vec{r}$ value must be in meters.

We know that $|\vec{tau}| = |\vec{r}||\vec{F}|\sin\theta$ (Cross Product)

So plug in numbers to get $|\vec{\tau}| \approx 9.66$ N-m.

Example

The figure shows a force of 100-N applied in the positive z-direction at the point $Q(1, 1, 1)$. Assuming the cube is free to rotate about $P(0, 0, 0)$, find the scalar moment (aka torque) of the force about P.



We know that $\vec{F} = \langle 0, 0, 100\rangle$ from the diagram.

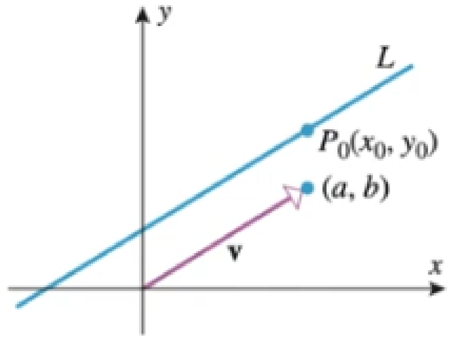
The vector $\vec{r}$ will be $\overrightarrow{PQ} = \langle 1, 1, 1\rangle$.

So the cross product of the two vectors gives the torque vector.

This vector is $\langle 100, -100, 0\rangle$.

1.5 Equations of Lines and Planes

To write the equation of a line, you need a point and the position.



In this figure, we know P_0 is a point on the line. The direction of the line is determined by a parallel vector $\vec{v}$. If $\vec{a} = \vec{P_0}P$, then $\vec{a} = t\vec{v}$.

Therefore, the vector equation of a line is $\vec{r} = \vec{r}_0 + t\vec{v}$. Each value of t gives a different point on the line. For $t > 0$, points are to the right.

Another expression is if we let $\vec{r} = \langle x, y, z\rangle$, $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$, and $\vec{v} = \langle a, b, c\rangle$, then we can write this as

$$\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$$

This leads us to the parametric equation of a line. The parametric equations of a line through (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c\rangle$ are:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

Example

Find a vector and parametric equations for the line that passes through $(4, 2)$ and is parallel to $\vec{v} = \langle -1, 5\rangle$. Then find 2 other points on that line.

We have the vector $\vec{r} = \langle 4, 2\rangle + t\langle -1, 5\rangle$. This is the vector equation. This can also be written as $\vec{r} = (4\vec{i} + 2\vec{j}) + t(-\vec{i} + 5\vec{j})$. Also this can be written as $\vec{r} = (4 - t)\vec{i} + (2 + 5t)\vec{j}$.

For the parametric, we have $x = 4 - t$ and $y = 2 + 5t$.

Note: These equations are not unique. You can choose a different point or parallel vector.

To find 2 other points, just plug in t values.

For $t = 1$, we can get $x = 3$, and $y = 7$, so the point is $(3, 7)$.

Example

Find parametric equations of the line passing through $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$. Where does this line intersect the xy-plane?

We have $\overrightarrow{P_1P_2} = \langle 3, -4, 8\rangle$.

We can find the parametric equations as $x = 2 + 3t, y = 4 - 4t$, and $z = -1 + 8t$ using P_1. (Note any scalar multiple of $\overrightarrow{P_1P_2}$ will be parallel to the line).

This will intersect the xy-plane when $z = 0$. So if we let $-1 + 8t = 0$, $t = \frac{1}{8}$.

Plugging this in x and y gives the point $\left(\frac{19}{8}, \frac{7}{2}, 0\right)$ as the intersection point.

A third representation is symmetric equations. Setting the parametric equations and solving for t gives the following:

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example

For the last example, write the symmetric equations for the line.

This is just $\frac{x-2}{3} = \frac{y-4}{-4} = \frac{z+1}{8}$.

What happens if we limit t? You get a line segment.

Example

Find parametric equations for the line segment joining $P(2, 4, -1)$ and $Q(5, 0, 7)$.

First, $\overrightarrow{PQ} = \langle 3, -4, 8 \rangle$.

The parametric equations are $x = 2 + 3t, y = -4 + 4t$, and $z = -1 + 8t$. We have to limit t though as $0 \leq t \leq 1$ for this to work.

A different representation for $\vec{r}(t) = (1 - t)\vec{r}_0 + t\vec{r}_1 = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$.

So for this example, this would be $(1 - t)\langle 2, 4, -1 \rangle + t\langle 5, 0, 7 \rangle$.

Example

Consider the following lines. Are they parallel? Do they intersect?

$$L_1 \quad x = 1 + 4t \quad y = 5 - 4t \quad z = -1 + 5t$$

$$L_2 \quad x = 2 + 8t \quad y = 4 - 3t \quad z = 5 + t$$

Writing these in vector form gives L_1 as $\langle x, y, z \rangle = \langle 1, 5, -1 \rangle + t\langle 4, -4, 5 \rangle$, and L_2 as $\langle x, y, z \rangle = \langle 2, 4, 5 \rangle + t\langle 8, -3, 1 \rangle$.

We can see that the t terms are not scalar multiples of each other, so the lines are not parallel.

To find intersection points, we need to see if the points of the two lines can ever equal each other.

This means we have the equations $1 + 4t_1 = 2 + 8t_2$, $5 - 4t_1 = 4 - 3t_2$, and $-1 + 5t_1 = 5 + t_2$.

Solving for t_1 in this system gives us $t_1 = \frac{1}{4}$ and $t_2 = 0$.

This means when $t_1 = \frac{1}{4}$ and $t_2 = 0$, then L_1 and L_2 have the same x, y.

What we can see from this, is that for the last equation for z, they do not equal each other. That means both lines have a different z value. That means the line skews because they are not parallel or intersect.

Unlike a line, a vector parallel to a plane is not enough information to determine a plane. Instead we need a vector perpendicular to the plane.

The equation of a plane is $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$. (This is the vector equation of a plane.)

An alternate form is $\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$. This is also $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. (This is the scalar equation of a plane).

Note that "an equation" means that the equation is not unique.

Example

Find an equation of the plane passing through $(3, -1, 7)$ and perpendicular to the vector $\vec{n} = \langle 4, 2, -5 \rangle$.

The equation is $\langle 4, 2, -5 \rangle \cdot \langle x - 3, y + 1, z - 7 \rangle = 0$.

This can be written as $4(x-3) + 2(y+1) - 5(z-7) = 0$ or $4x + 2y - 5z + 25 = 0$. The second equation of this is known as the linear equation.

To graph this, we need to find 3 points (that are not co-linear). To find 3 points, you would need to come up with the intercepts, and then graph those and that gives you the plane.

Example

Find an equation of the plane that passes through $P(1, 3, 2)$, $Q(3, -1, 6)$ and $R(5, 2, 0)$.

If we find the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, this will find the vector perpendicular to the plane.

We can find that $\overrightarrow{PQ} = \langle 2, -4, 4 \rangle$ and $\overrightarrow{PR} = \langle 4, -1, -2 \rangle$ and that $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$.

The cross product is $\langle 12, 20, 14 \rangle$. Dividing by 2 gives us $\langle 6, 10, 7 \rangle$, which is also perpendicular to the plane.

So an equation of the plane is $6(x-1) + 10(y-3) + 7(z-2) = 0$ or $6x + 10y + 7z = 50$. (There are many ways to represent this.)

Example

Consider the planes $x + y + z = 1$ and $x - 2y + 3z = 1$. Find the angle between the two planes and find symmetric equations for the line of intersection of the two planes.

We have $\vec{n}_1 = \langle 1, 1, 1 \rangle$ and $\vec{n}_2 = \langle 1, -2, 3 \rangle$.

We know that $\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|}$.

So plugging in numbers gives $\theta \approx 72°$.

The two planes of the two normal vectors will form a parallelogram. Remember $\vec{n}_1$ and $\vec{n}_2$ are normal vectors to the plane.

To find symmetric equations we need a point on the line and a parallel vector.

We will choose the point where the lines intersects the xy-plane ($z = 0$), so $x + y = 1$ and $x - 2y = 1$. Solving this system gives $x = 1$ and $y = 0$.

Both planes have the point $(1, 0, 0)$. If line L lies on both planes, then the line must be perpendicular to both $\vec{n}_1$ and $\vec{n}_2$.

The cross product of the two vectors is $\langle 5, -2, -3 \rangle$.

So the symmetric equation is $\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$. (There are other representations.)

Theorem 1.4

The distance between a point $P(x_1, y_1, z_1)$ and plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example

Find the distance between parallel planes $x + 2y - 2z = 3$ and $2x + 4y - 4z = 7$.

We have $\vec{n}_1 = \langle 1, 2, -2 \rangle$ and $\vec{n}_2 = \langle 2, 4, -4 \rangle$.

We can see that the point $(3, 0, 0)$ is on the first plane (letting $y = z = 0$).

For the second equation we have $2x + 4y - 4z - 7 = 0$.

From this, we have all the values needed now.

Plugging into the distance formula gives $\frac{1}{6}$.

Example

From a previous example, you found the lines are skew. Find the distance between them.

$$L_1 \qquad x = 1 + 4t \quad y = 5 - 4t \quad z = -1 + 5t$$

$$L_2 \qquad x = 2 + 8t \quad y = 4 - 3t \quad z = 5 + t$$

First find a point on L_1. We get a point $(1, 5, -1)$ (when $t = 0$).

We need to find a plane through L_2 and need a point and a perpendicular vector.

We know a point would be $(2, 4, 5)$. To find the perpendicular vector, we need to find a cross product of $\vec{L}_1 \times \vec{L}_2$.

The cross product is $\langle 11, -36, 20 \rangle$. So the equation of the plane is $11(x-2)-36(y-4)+20(z-5) = 0$.

This can be written as $x - 36y + 20z + 22 = 0$.

Now we can find the distance since we have the plane and the point.

The distance is roughly $D \approx 3.918$ units.

1.6 Cylinders and Quadric Surfaces

Definition

A cylinder is a surface that consists of all lines parallel to a given line and passing through a given plane curve (i.e - in 3 space, the equation only has 2 variables).

For example, sketching $z = x^2$ in 3-space. This is essentialy a parabola on the xz-plane along a parallel line, and it will look like a piece of paper being in the process of being folded.

Exercise Sketch $y^2 + z^2 = 1$ in 3-space.

Definition

A quadric surface is the graph of a second-degree equation in 3 variables

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

Exercise Graph $z = x^2 + y^2$. (Hint $z \geq 0$.)

There are 6 different quadric surfaces.

First, the ellipsoid has the general equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The intercepts of this are $(0, 0 \pm c)$, $(0 \pm b, 0)$. and $(\pm a, 0, 0)$. For the traces, we can see that if we let x, y, or z be a number, they will give ellipses. The shape of this looks like a football (American).

Next, the cone has the general equation $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. The axis is that $\frac{z^2}{c^2}$ term, but it can be y^2 or z^2. This has one intercept, $(0, 0, 0)$. The traces are z is some number, it gives an ellipse, if x or y is some number, you get a hyperbola.

Next, the elliptic paraboloid has the equation $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. Likewise with the cone, the $\frac{z}{c}$ term just tells you what axis (it can be x or y.) The intercept is the same as the cone, $(0, 0, 0)$. For the traces, z gives an ellipse, and x and y gives parabolas.

Next the hyperboloid of one sheet. The equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. (The term with a minus is the axis.) You have to find the intercept for this one. z gives an ellipse, and x and y gives hyperbolas.

Next, hyperboliods of 2 sheets. The equation is $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. (The term with the plus is the axis.) You have to find the intercept, and z gives an ellipse ($z > c$), and x and y gives ellipses.

Lastly, the hyperbolic paraboloid. The formula is $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$. You have to find the intercept. x and y gives parabolas and z gives a hyperbola. (This should kind of look like a saddle (on a horse).)

Example

Let's look at $y^2 = x^2 + \frac{z^2}{4}$.

The intercept is $(0, 0, 0)$.

We can assume that y is the axis we are drawing on.

When we look at the traces, they all give ellipses. For example, $y = 1$ gives $1 = x^2 + \frac{z^2}{4}$, $y = 2$ gives $4 = x^2 + \frac{z^2}{4}$ or $1 = \frac{x^2}{4} + \frac{z^2}{16}$.

When we let $x = 0$, then we can get $y^2 = \frac{z^2}{4}$, which gives $\pm y = \pm\frac{z}{2}$, which is two perpendicular lines essentially.

This is an elliptic cone.

Example

Let's look at $4x^2 + 4y^2 + z^2 + 8y - 4z = -4$.

Completing the square gives $x^2 + (y + 1)^2 + \frac{(z-2)^2}{4} = 1$.

The intercepts are $(0, 0, 2)$, $(0, -1, 0)$ as the two intercepts, (there are no x-intercepts.) From what we also see, we can see the center of the graph should be $(0, -1, 2)$.

Hopefully, you can see that all the traces are ellipses.

Cool! We get an ellipsoid.

Example

Lastly, graph $z = \frac{y^2}{4} - \frac{x^2}{9}$.

Starting with intercepts, we get $(0, 0, 0)$ as the only intercept.

For our traces, x and y give parabolas, and z gives a hyperbola.

This gives a hyperbolic paraboloid.

For a brief review on conic sections: the following.

Circles: A circle is the set of all points in the plane equidistant from a fixed point. The standard equation is $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center, and r is the radius.

Ellipses: An ellipse is the set of all points in the plane the sum of whose distances from two fixed points (the foci) is constant. The standard equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$. If $a>b$, we get the foci as $(h\pm c,k)$, and if $b>a$, we get the foci as $(h,k\pm c)$. For both of these, the center will be (h,k), and the vertices are the endpoints of the major axis. Use c to find the coordinates of each focus. The foci are located on the major axis and are each c units away from the center. If $a=b$, then the ellipse is just a circle, and a and b will equal r. The foci of a circle are located at the same point - the center. Eccentricity of an ellipse is $\frac{c}{a}$. (For a circle $e=0$.)

Hyperbolas: A hyperbola is the set of all points in the plane the difference of whose distances from two fixed points (the foci) is constant. The standard equation are the following:

- For hyperbolas that open left and right: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, the foci will be $(h\pm c,k)$
- For hypberolas that open up and down: $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$, the foci will be $(h,k\pm c)$

The center is (h,k) for both and the vertices are the turning points of the brances of the hyperbola. Use a and b to create the central rectangle around the center of the hyperbola. The diagonals of this rectangle form the asymptotes. The equation for the asymptotes are $y-k=\pm\frac{b}{a}(x-h)$. Use the value $c=\sqrt{a^2+b^2}$, to find the coordinates of each focus. The branches of a hyperbola will always bend towards the foci and away from the center.

Parabolas: A parabola is the set of all points in the plane equidistant from a fixed line (the directrix) and a fixed point (the focus). The graph of a parabola will always bend towards its focus and away from its directrix. The coordinates of the vertex are (h,k), the distance from the vertex to both the focus and directrix is given by $|p|$. For equation for a parabola that opens up and down is $(x-h)^2=4p(y-k)$, and for one that opens left or right it is $(y-k)^2=4p(x-h)$. If $p>0$, the parabola either opens up or right, and for $p<0$, the opposite happens.

2 Vector-Valued Functions

2.1 Vector Functions and Space Curves

Review: Parametric Curves

- $x = f(t)$
- $y = g(t)$
- $z = h(t)$

These represent a curve in 3-space (for 2-space, it is just x and y.)

The above represents a path in space that is traced in a specific direction as t increases (orientation). The domain is $(-\infty, \infty)$, unless specified otherwise.

Definition

$$\vec{r} = \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

At any given t value, $\vec{r}$ represents a vector whose initial point is at the origin and terminal point is $(f(t), g(t), h(t))$.

The domain is $(-\infty, \infty)$ and the range is the set of vectors.

Graphs of vector-valued functions: curve that is traced by connecting tips of "radius vectors".

Example

Graph $\vec{r}(t) = 2\cos t\vec{i} - 3\sin t\vec{j}$ for $0 \leq t \leq 2\pi$.

We could write this as $x = 2\cos t$ and $y = -3\sin t$ (parametric).

We could instead write a table.

t	x	y
0	2	0
$\pi/2$	0	-3
π	-2	0
$3\pi/2$	0	3
2π	2	0

As you draw this, you can see that this will be an ellipse.

Example

$$\vec{r}(t) = \langle 4\cos t, 4\sin t, t \rangle$$

We should know that since there are trig things in here, that we go from 0 to 2π, and if we put this on a table, we can see that x and y will give you a circle from the table. The z is moving up though, so basically the function will just be circling around a cylinder of radius 2.

Example

Find a vector and parametric equations for the line segment that joins $A(1,-3,4)$ to $B(-5,1,7)$.

We have $\vec{r} = \vec{AB} = \langle -6,4,3\rangle$. So $\vec{r}(t) = \langle 1-6t, -3+4t, 4+3t\rangle$, and we want to put the bound $0 \leq t \leq 1$

The parametrics are $x(t) = 1-6t, y(t) = -3+4t$, and $z = 4+3t$, with $0 \leq t \leq 1$.

Example

Find a vector function that represents the curve of intersection of $x^2+y^2=1$ and $y+z=2$.

$x^2+y^2=1$ is a cylinder and $y+z=2$ is a plane.

We can represent $x^2+y^2=1$ as $x=\cos t$ and $y=\sin t$, with bounds $0 \leq t \leq 2\pi$.

$y+z=2$ can be represented as $z=2-y$ or $z=2-\sin t$ with $0 \leq t \leq 2\pi$.

So $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + (2-\sin t)\vec{k} = \langle \cos t, \sin t, 2-\sin t\rangle$ with $0 \leq t \leq 2\pi$.

Example

Find the domain of $\vec{r}(t) = \langle \ln|t-1|, e^t, \sqrt{t}\rangle$.

The domain is all values of t for which $\vec{r}(t)$ is defined.

So we have $x = \ln|t-1|$, $y = e^t$ and $z = \sqrt{t}$.

For x, we have the domain as $(-\infty,1)\cup(1,\infty)$, for y we have the domain as $t \in \mathbb{R}$, and for z, we have $t \geq 0$, so combining them gives domain $[0,1)\cup(1,\infty)$.

Definition

If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, then $\lim_{t\to a}\vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle$ (as long as all 3 limits exist).

Example

Let $\vec{r}(t) = t^2\vec{i} + e^t\vec{j} - (2\cos\pi t)\vec{k}$. Find $\lim_{t\to 0}\vec{r}(t)$.

The limit of the $\vec{i}$ term is 0 as it goes to 0.

The limit of the $\vec{j}$ term is 1 as it approaches 0.

The limit of the $\vec{k}$ term is -2 as it approaches 0.

So the limit is $\lim_{t\to 0}\vec{r}(t) = \vec{j} - 2\vec{k}$

Example

Let $\vec{r}(t) = \left(\frac{4t^3+5}{3t^3+1}\right)\vec{i} + \left(\frac{1-\cos t}{t}\right)\vec{j} + \left(\frac{\ln(t+1)}{t}\right)\vec{k}$. Find $\lim_{t\to 0}\vec{r}(t)$.

For the first term, we get 5 as the limit.

For the other two, we will use L'Hopital's Rule.

Doing this and finding the limits should give that $\lim_{t\to 0}\vec{r}(t) = \langle 5,0,1\rangle$.

Continuity: A vector function $\vec{r}(t)$ is continuous at a if: $\lim_{t\to a}\vec{r}(t) = \vec{r}(a)$. (This is just AP Calculus BC)

2.2 Derivatives and Integrals of Vector Functions

Definition

If $\vec{r}(t)$ is a vector function, the derivative of $\vec{r}(t)$ with respect to t is

$$\vec{r'} = \vec{r}(t)' = \frac{d\vec{r}}{dt} = \frac{d}{dt}(\vec{r}(t)) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Geometrically, this would have $\vec{r}(t)$ as a vector tangent to the curve at the tip of $\vec{r}(t)$. It points in the direction of increasing parameter.

Theorem 2.1

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f, g, and h are differentiable functions, then

$$\vec{r'}(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Proof. Let $\vec{r}(t) = \langle x(t), y(t) \rangle$

By definition, $\vec{r'}(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$.

This is equal to $\lim_{h \to 0} \frac{[x(t+h)\vec{i} + y(t+h)\vec{j}] - [x(t)\vec{i} + y(t)\vec{j}]}{h}$.

Which is equal to

$$\left(\lim_{h \to 0} \frac{x(t+h)\vec{i} - x(t)\vec{i}}{h} \right) + \left(\lim_{h \to 0} \frac{y(t+h)\vec{j} - y(t)}{h} \right)$$

Taking out the $\vec{i}$ and $\vec{j}$, allows us to see that this equals to $x'(t)\vec{i} + y'(t)\vec{j}$. □□

Example

$\vec{r}(t) = \frac{1}{t}\vec{i} + e^{2t}\vec{j} - 2\cos \pi t\vec{k}$. Find $\vec{r}(t)$.

The derivative of this is simply $\langle \frac{-1}{t^2}, 2e^{2t}, 2\pi \sin \pi t \rangle$.

$\vec{r'}(t)$ refers to the tangent vector. The tangent line is the line through P that is parallel to $\vec{r'}(t)$.

Unit Tangent Vector: $\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}$.

Example

From the previous example, find the unit tangent vector at $t = 1$.

We know that $\vec{r'}(t) = \langle \frac{-1}{t^2}, 2e^{2t}, 2\pi \sin \pi t \rangle$.

From this, $\vec{r'}(1) = \langle -1, 2e^2, 0 \rangle$, and the magnitude of this is $\sqrt{1 + 4e^4}$.

Therefore, $\vec{T}(1) = \langle \frac{-1}{\sqrt{1+4e^4}}, \frac{2e^2}{\sqrt{1+4e^4}}, 0 \rangle$.

Exercise For the curve $\vec{r}(t) = \sqrt{t}\vec{i} + (2-t)\vec{j}$, find $\vec{r'}(t)$. Sketch $\vec{r}(1)$ and $\vec{r'}(1)$.

Example

Find parametric equations for the tangent line to the helix with equations $x = 2\cos t$, $y = \sin t$, and $z = t$ at the point $(0, 1, \pi/2)$.

We have $\vec{r}(t) = \langle 2\cos t, \sin t, t\rangle$, so $\vec{r'}(t) = \langle -2\sin t, \cos t, 1\rangle$.

We get $0 = 2\cos t$, $1 = \sin t$, and $\frac{\pi}{2} = t$, so we know that t is.

Plugging this in gives $\vec{r'}\left(\frac{\pi}{2}\right) = \langle -2, 0, 1\rangle$. This is the tangent vector.

So $\vec{r}(t) = \langle 0, 1, \frac{\pi}{2}\rangle + t\langle -2, 0, 1\rangle$.

Parametrically: $x = -2t$, $y = 1$, $z = \frac{\pi}{2} + t$.

Differentiation Rules:

1. $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$
2. $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$
3. $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
4. $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
5. $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$ (Order matters here)
6. $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\vec{u}'(f(t))$

Theorem 2.2

If $\vec{r}(t)$ is differentiable and $\|\vec{r}(t)\|$ is constant for all t, then $\vec{r}(t) \cdot \vec{r'}(t) = 0$.

This means they are orthogonal for all t.

Example

The graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ intersect at the origin. Find the degree measure of the acute angle between the tangent lines to the graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ at the origin.

We have $\vec{r}_1(t) = \langle \tan^{-1} t, \sin t, t^2\rangle$ and $\vec{r}_2(t) = \langle t^2 - t, 2t - 2, \ln t\rangle$.

$\vec{r}_1(t) = \langle 0, 0, 0\rangle$ at $t = 0$.

$\vec{r}_2(t) = \langle 0, 0, 0\rangle$ at $t = 1$.

We need the derivatives of the functions.

$\vec{r}_1'(t) = \langle \frac{1}{1+t^2}, \cos t, 2t\rangle$

$\vec{r}_2'(t) = \langle 2t - 1, 2, \frac{1}{t}\rangle$

$\vec{r}_1'(0) = \langle 1, 1, 0\rangle$ and $\vec{r}_2'(1) = \langle 1, 2, 1\rangle$.

If we want to find the angles between then we have to use the dot product.

We get $\cos\theta = \frac{1+2+0}{\sqrt{2}\cdot\sqrt{6}} = \frac{\sqrt{3}}{2}$.

So $\theta = \frac{\pi}{6}$.

Example

Calculate $\frac{d}{dt}\left[\vec{r}_1(t)\cdot\vec{r}_2(t)\right]$ and $\frac{d}{dt}\left[\vec{r}_1(t)\times\vec{r}_2(t)\right]$ by differentiating the product directly and using the formulas.

$$\vec{r}_1(t) = 2t\vec{i} + 3t^2\vec{j} + t^3\vec{k}$$
$$\vec{r}_2(t) = t^4\vec{k}$$

Directly:

The dot product $\vec{r}_1 \cdot \vec{r}_2 = t^7$. The derivative of this is $7t^6$.

Formula: The formula is $\vec{r}_1' \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_2'$.

Using this formula gives you $3t^4t^6 = 7t^6$.

Now for the cross product.

Directly: The cross product gives $\langle 3t^6 - 0, -(2t^5 - 0), 0\rangle = \langle 3t^6, -2t^5, 0\rangle$.

The derivative of this is $\langle 18t^5, -10t^4, 0\rangle$.

Formula: The formula is $\vec{r}_1' \times \vec{r}_2 + \vec{r}_1' \times \vec{r}_2'$.

You should get the same answer.

$$\int_a^b \vec{r}(t) = \lim_{n\to\infty}\sum_{i=1}^{n} \vec{r}(t_i^*)\Delta t$$

Or, more helpfully

$$\int_a^b \vec{r}(t)dt = \left(\int_a^b f(t)dt\right)\vec{i} + \left(\int_a^b g(t)dt\right)\vec{j} + \left(\int_a^b h(t)dt\right)\vec{k}(t)$$

Example

Let $\vec{r}(t) = t^2\vec{i} + e^t\vec{j} - 2\cos\pi t\vec{k}$. Find $\int_0^1 \vec{r}(t)dt$.

Integrating each component and plugging in the limits of integration results in $\int_0^1 \vec{r}(t)dt = \frac{1}{3}\vec{i} + (e^t)\vec{j}$.

Example

Find $\int(2t\vec{i} + 3t^2\vec{j})dt$.

Remember in an indefinite integral to add a constant at the end.

The result is $t^2\vec{i} + t^3\vec{j} + \vec{c}$.

Example

Find $\vec{r}(t)$ given that $\vec{r}'(t) = \langle 3, 2t\rangle$ and $\vec{r}(1) = \langle 2, 5\rangle$.

If we start by integrating, then $\vec{r}(t) = \langle 3t, t^2\rangle + \vec{c}$.

We have $\langle 2, 5\rangle = \langle 3, 1\rangle + \langle c_1, c_2\rangle$.

We get $\vec{c} = \langle -1, 4\rangle$ from this.

So $\vec{r}(t) = \langle 3t - 1, t^2 + 4\rangle$.

2.3 Arc Length and Curvature

Consider a curve given by parametric equations $x = x(t)$ and $y = y(t), a \leq t \leq b$.

Then arc length

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The Arc Length of a Vector Valued Function is the exact same idea

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_a^b \|\vec{r'}(t)\| dt$$

Example

Find the arc length of the portion of the curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$ from $(3, 0, 0)$ to $(-3, 0, 4\pi)$.

If we use $z = 4t$ we get $t = 0$ and $t = \pi$ from both points.

The integral is $L = \int_0^\pi \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} dt$.

This is equal to $\int_0^\pi \sqrt{25} dt = 5\pi$.

A curve can be represented by more than one function.

Example

Given $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle, 1 \leq t \leq 2$.

If we use $t = e^u$ then $\vec{r}_1(u) = \langle e^u, e^{2u}, e^{3u} \rangle, 0 \leq u \leq \ln 2$.

Both represent the same curve. These are called parametrizations of the curve. Both can be use dto find arc length (because arc length does not depend on the parameter).

Example

Find the length of the curve above using both parametrizations.

$\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$.

$\vec{r}_1'(t) = \langle 1, 2t, 3t^2 \rangle$.

Then we integrate $L = \int_1^2 \sqrt{1 + 4t^2 + 9t^4} dt \approx 7.075$.

For $\vec{r}_2(u) = \langle e^4, e^{2u}, e^{3u} \rangle$.

The derivative of this is $\vec{r}_2'(u) = \langle e^u, 2e^{2u}, 3e^{3u} \rangle$.

The integral is $\int_0^{\ln 2} \sqrt{e^{2u} 4e^{4u} + 9e^{6u}} du \approx 7.075$.

As you can see, they are the same.

We want to parametrize a curve in terms of arc length, s, rather than an arbitrary value in a particular coordinate system.

We first must recognize that $s(t) = \int_a^t |\vec{r'}(u)| du$.

This of course is equal to

$$\int_a^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du$$

We can also see that $\frac{ds}{dt} = |\vec{r'}(t)|$.

Example

Find the arc length parametrization of $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$ with reference point $(1, 0, 0)$ and the same orientation as the helix.

We know that $\frac{ds}{dt} = |\vec{r'}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$.

$s = s(t) = \int_0^t \sqrt{2}du = \sqrt{2}t$.

We get that $t = \frac{s}{\sqrt{2}}$ as a result.

Therefore $\vec{r}(s) = \cos\left(\frac{s}{\sqrt{2}}\right)\vec{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\vec{j} + \left(\frac{s}{\sqrt{2}}\right)\vec{k}$.

Arc length formula guarantees same orientation.

This is useful because let's say we need to move along the curve for a certain amount of units, well we can just plug in that value and find the point at which we are.

For example, $\vec{r}(5) \approx (-0.923, -0.384, 3.5636)$.

Example

Find the arc length parametrization of the curve below measured from $(0, 0)$ in the direction of increasing t.

$$\vec{r}(t) = \langle 1/3t^2, 1/2t^2 \rangle, t \geq 0$$

$\vec{r'}(t) = \langle t^2, t \rangle$ and the magnitude of this is $t\sqrt{t^2 + 1}$.

We are now integrating $s = \int_0^t u\sqrt{u^2 + 1}du$.

This gives you $\frac{1}{3}(u^2 + 1)^{3/2}$ from 0 to t.

Integrating this and solving for t gives you $t = \sqrt{(3s + 1)^{2/3} - 1}$.

Therefore the parametrization of this is $\vec{r}(s) = \langle \frac{1}{3}[(3s + 1)^{2/3} - 1]^{3/2}, \frac{1}{2}[(3s + 1)^{2/3} - 1] \rangle$.

Example

Let $\vec{r}(t) = \langle \ln t, 2t, t^2 \rangle$. Find

(a) $\|\vec{r'}(t)\|$

$\vec{r'}(t) = \langle \frac{1}{t}, 2, 2t \rangle$, so the magnitude of this is $\sqrt{\frac{1}{t^2} + 4 + 4t^2} = 2t + \frac{1}{t}$.

(b) $\frac{ds}{dt}$

This is the exact same thing as $\|\vec{r'}(t)\| = 2t + \frac{1}{t}$

(c) $\int_1^3 \|\vec{r'}(t)\|dt$

We are integrating $\int_1^3 (2t + \frac{1}{t})dt = 9 + \ln 3 - 1 - 0 = 8 + \ln 3$.

A parametrization is called smoth on I if $\vec{r'}(t)$ is continuous and $\vec{r'}(t) \neq 0$ on I (a smooth curve has smooth parametrization). Smooth means no sharp corners or cusps.

Example

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

Is $\vec{r}(t)$ smooth?

The derivative of the vector is $\langle -\sin t, \cos t, 1 \rangle$. This is continuous on $(-\infty, \infty)$ and this is not equal to $\vec{0}$, so $\vec{r}(t)$ is smooth.

Recall: $\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}$ (called unit tangent vector) indicated the direction of curve.

Curvature is as followed.

$$\kappa = \left| \frac{d\vec{T}}{ds} \right|$$

$\vec{T}$ has a constant length so κ is only affected by a change in direction.

Example

Show that the curvature of a circle with radius a is $1/a$.

$\vec{r}(t) = \langle a \cos t, a \sin t \rangle$.

The derivative $\vec{r'}(t) = \langle -a \sin t, a \cos t \rangle$.

$s(t) = \int_0^t \sqrt{a^2 \sin^2 u + a^2 \cos^2 u} du = \int_0^t adu$.

We get $s(t) = s = at$ so $t = \frac{a}{s}$.

The circle in terms of s is $\vec{r}(S) = \langle a \cos \frac{a}{s}, a \sin \frac{a}{s} \rangle$.

The derivative of this is $\langle -\sin \frac{a}{s}, \cos \frac{a}{s} \rangle$.

The magnitude of this is 1.

The unit tangent vector $\vec{T}(s) = \langle -\sin \frac{a}{s}, \cos \frac{a}{s} \rangle$.

The derivative of this vector is $\langle -\frac{1}{a} \cos \frac{a}{s}, -\frac{1}{a} \sin \frac{a}{s} \rangle$.

The magnitude of this vector is $\kappa = \frac{1}{a}$. A big radius means a small curvature.

The curvature of a straight line is $\kappa = 0$.

A circle has constant curvature.

Other formulas for κ are the following

$$\kappa = \left| \frac{d\vec{T}}{dS} \right| = \left| \frac{\frac{dT}{dt}}{\frac{dS}{dt}} \right|$$

$$\kappa = \frac{|\vec{T'}(t)|}{|\vec{r'}(t)|}$$

$$\kappa = \frac{|\vec{r'}(t) \times \vec{r''}(t)}{\vec{r'}(t)|^3}$$

$$\kappa(t) = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}}$$

Exercise Use another formula to calculate κ for $\vec{r}(t) = \langle a \cos t, a \sin t \rangle$.

Example

Find κ for $\vec{r'}(t) = \langle 2t, t^2, -\frac{1}{3}t^3 \rangle$.

The derivative $\vec{r'}(t) = \langle 2, 2t, -t^2 \rangle$.

$|\vec{r'}(t)| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2$

$\vec{T}(t) = \frac{\langle 2,2t,-t^2 \rangle}{t^2} + 2 = \langle \frac{2}{t^2+2}, \frac{2t}{t^2+2}, \frac{-t^2}{t^2+2} \rangle$.

$\vec{T'}(t) = \langle \frac{4t}{(t^2+2)^2}, \frac{-2t^2+4}{(t^2+2)^2}, \frac{-4t}{(t^2+2)^2} \rangle$.

$\|\vec{T'}(t)\| = \sqrt{\frac{16t^2+4t^4-16t^2+16+16t^2}{(t^2+2)^4}} = \frac{2}{t^2+2}$

$\kappa(t) = \frac{2/t^2+2}{t^2+2} = \frac{2}{(t^2+2)^2}$

We can also use the other formula using the cross product.

$\vec{r'}(t) = \langle 2, 2t, -t^2 \rangle$ and $\vec{r''}(t) = \langle 0, 2, -2t \rangle$.

The cross product of these two vectors will result in $\langle -4t^2 - -2t^2, -(-4t - 0), 4 - 0 \rangle = \langle -2t^2, 4t, 4 \rangle$.

The magnitude of this is $2(t^2 + 2)$, so $\kappa(t) = \frac{2(t^2+2)}{(t^2+2)^3} = \frac{2}{(t^2+2)^2}$.

Both ways give an equivalent answer.

There is one more curvature formula in terms of x rather than t.

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

Example

Find the curvature of the parabola $y = x^2$ at the points $(0, 0)$, $(1, 1)$, and $(2, 4)$.

So $f(x) = x^2$, $f'(x) = 2x$, and $f''(x) = 2$.

$\kappa(x) = \frac{|2|}{(1+(2x)^2)^{3/2}} = \frac{2}{(1+4x^2)^{7/2}}$

$\kappa(0) = 2$, $\kappa(1) \approx 0.18$, $\kappa(2) \approx 0.03$.

As $\kappa \to \infty$, $\kappa(x) \to 0$.

Radius of curvature: $\rho = \frac{1}{\kappa}$

We have also shown $\kappa = \frac{1}{a}$

Example

From the previous example, calculate the curvature at $(0, 0)$. Then draw a circle of curvature.

$\kappa(0) = 2$ and $\rho(0, 0) = \frac{1}{2}$.

At the point $(0, 0)$, κ is same as circle with radius $\frac{1}{2}$.

Recall the unit tangent vector, $\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}$ which points in the direction of increasing parameter.

The unit tangent vector is orthogonal to its derivative.

Unit normal vector $\vec{N}(t) = \frac{\vec{T'}(t)}{|\vec{T'}(t)|}$. This points inward towards the concave part of curve c.

Binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$.

$\|\vec{T} \times \vec{N}\| = \|\vec{T}\|\|\vec{N}\| \sin 90$. This is a also a unit vector.

Example

Find the unit tangent, unit normal, and binormal vectors for $\vec{r}(t) = \langle 3\sin t, 3\cos t, 4t\rangle$.

$\vec{r'}(t) = \langle 3\cos t, -3\sin t, 4\rangle$.

$\|\vec{r'}(t)\| = 5$

$\vec{T}(t) = \langle \frac{3}{5}\cos t, -\frac{3}{5}\sin t, \frac{4}{5}\rangle$.

$\vec{T'}(t) = \langle -\frac{3}{5}\sin t, -\frac{3}{5}\cos t, 0\rangle$

$\|\vec{T'}(t)\| = \frac{3}{5}$

$\vec{N}(t) = \langle -\sin t, -\cos t, 0\rangle$.

$\vec{B}(t) = \vec{T}\times\vec{N} = \langle \frac{4}{5}\cos t, -\frac{4}{5}\sin t, -\frac{3}{5}\rangle$

Another wayt to find $\vec{B}(t)$ is the following

$$\vec{B}(t) = \frac{\vec{r'}(t)\times\vec{r''}(t)}{\|\vec{r'}(t)\times\vec{r''}(t)\|}$$

Example

Consider $\vec{r}(t) = \langle t, \frac{\sqrt{2}}{2}t^2, \frac{1}{3}t^3\rangle$. Find $\vec{T}, \vec{N}$ at $t = 2$.

$\vec{r'}(t) = \langle 1, \sqrt{2}t, t^2\rangle$

$\|\vec{r'}(t)\| = \sqrt{1+2t^2+t^4} = t^2+1$

$\vec{T}(t) = \langle \frac{1}{1+t^2}, \frac{\sqrt{2}t}{1+t^2}, \frac{t^2}{1+t^2}\rangle$

$\vec{T}(2) = \langle \frac{1}{5}, \frac{2\sqrt{2}}{5}, \frac{4}{5}\rangle$

Now to find $\vec{N}(2)$.

$\vec{T'}(t) = \langle \frac{-2t}{(1+t^2)2}, \frac{(1+t^2)\sqrt{2}-2t(\sqrt{2}t)}{(1+t^2)^2}, \frac{2t(1+t^2)-t^2(2t)}{(1+t^2)^2}\rangle = \langle \frac{-2t}{(1+t^2)^2}, \frac{-2t^2+2}{(1+t^2)^2}, \frac{2t}{(1+t^2)^2}\rangle$

$\vec{N}(t) = \frac{\vec{T'}(t)}{|\vec{T'}(t)|}$

We should instead of finding the magnitude, find $\vec{T'}(2) = \langle \frac{-4}{25}, \frac{-8+\sqrt{2}}{25}, \frac{4}{25}\rangle$

The magnitude of this is $\|\vec{T'}(2)\| = \sqrt{\frac{16}{625}+\frac{64-16\sqrt{2}+2}{625}+\frac{16}{625}} = \frac{\sqrt{98-16\sqrt{2}}}{25}$

So $\vec{N}(2) = \frac{\langle \frac{-4}{25}, \frac{-8+\sqrt{2}}{25}, \frac{4}{25}\rangle}{\sqrt{98-16\sqrt{2}}225}$

This is equal to $\langle \frac{-4}{\sqrt{98-16\sqrt{2}}}, \frac{-8+\sqrt{2}}{\sqrt{98-16\sqrt{2}}}, \frac{4}{\sqrt{98-16\sqrt{2}}}\rangle$.

A normal plane contains $\vec{N}$ and $\vec{B}$. It contains all lines perpendicular to $\vec{T}$.

The osculating plane contains $\vec{T}$ and $\vec{N}$. It is related to the circle of curvature or osculating circle.

The rectifying plane contains $\vec{T}$ and $\vec{B}$.

To find the equation of a plane you need a point and a perpendicular vector.

Example

Find the equations of the normal and osculating plantes at $(3, 0, 2\pi)$ for the following:

$$\vec{T}(t) = \langle \frac{3}{5}\cos t, -\frac{3}{5}\sin t, \frac{4}{5} \rangle$$

$$\vec{N}(t) = \langle -\sin t, -\cos t, 0 \rangle$$

$$\vec{B}(t) = \langle \frac{4}{5}\cos t, -\frac{4}{5}\sin t, -\frac{3}{5} \rangle$$

The normal plane has point $(3, 0, 2\pi)$ and normalvector at $\frac{\pi}{2}$ is $\vec{T}\left(\frac{\pi}{2}\right) = \langle 0, -\frac{3}{5}, \frac{4}{5} \rangle$.

We have $0(x-3) + \frac{-3}{5}(y-0) + \frac{4}{5}(z-2\pi) = 0$ and this gives $\frac{-3}{5}y + \frac{4}{5}z = \frac{8}{5}\pi$.

The osculating plane we need the binormal vector. $\vec{B}\left(\frac{\pi}{2}\right) = \langle 0, -\frac{4}{5}, \frac{3}{5} \rangle$.

$0(x-3) + -\frac{4}{5}(y-0) + \frac{3}{5}(z-2\pi) = 0$ so we get $-\frac{4}{5}y - \frac{3}{5}z = -\frac{6}{5}\pi$

Example

Consider the ellipse given by

$$\vec{r}(t) = 2\cos t\vec{i} + 3\sin t\vec{j}, 0 \le t \le 2\pi$$

Note: $\kappa(t) = \frac{6}{[4\sin^2 t + 9\cos^2 t]^{3/2}}$

Find and draw the osculating circles at $(2, 0)$ and $(0, -3)$.

So we have $t = 0$ and $t = \frac{3\pi}{2}$.

For $(2, 0) \to \kappa(0) = \frac{2}{9}$. so circle with radius $\frac{9}{2}$ and diamater 9.

For $(0, -3)$, $\kappa\left(\frac{3\pi}{2}\right) = \frac{3}{4}$ so radius $r = \frac{3}{4}$ and diameter $\frac{8}{3}$.

For the point $(2, 0)$, we also have the point $(-7, 0)$, so the center is $\left(-\frac{5}{2}, 0\right)$.

So the equation for that is $\left(x + \frac{5}{2}\right)^2 + y^2 = \frac{81}{4}$.

2.4 Motion in Space - Velocity and Acceleration

1. Direction of motion time t is in the direction of $\vec{T}$.
2. speed $= \frac{ds}{dt}$ (instantaneous rate of change of the arc length traveled). This is a scalar
3. velocity vector $\vec{v}(t) = \frac{ds}{dt}\vec{T}(t)$

$\frac{ds}{dt}$ is the magnitude of $\vec{v}(t)$.

$\vec{T}(t)$ denotes direction.

$\vec{v}(t)$ points in direction of motion and has magnitude = speed

If $\vec{r}(t)$ is a position function, then $\vec{v}(t) = \frac{d\vec{r}}{dt}(t)$ and $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt}$.

Speed is $\|\vec{v}(t)\| = \frac{ds}{dt}$

Example

A particle moves along C: $\vec{r}(t) = \langle 2\sin\left(\frac{t}{2}\right), 2\cos\left(\frac{t}{2}\right)\rangle$.

(a) Find its velocity, acceleration, and speed at time t.

$\vec{v}(t) = \vec{r'}(t) = \langle \cos\frac{t}{2}, -\sin\frac{t}{2}\rangle = \vec{v}(t)$.

$\vec{a}(t) = \vec{v'}(t) = \langle -\frac{1}{2}\sin\frac{t}{2}, -\frac{1}{2}\cos\frac{t}{2}\rangle = \vec{a}(t)$

speed $= \|\vec{v}(t)\| = 1$

(b) Show that $\vec{a}(t)$ is orthogonal to $\vec{v}(t)$ for this path only.

$\vec{a}(t)\cdot\vec{v}(t) = -\frac{1}{2}\cos\frac{t}{2}\sin\frac{t}{2} + \frac{1}{2}\sin\frac{t}{2}\cos\frac{t}{2} = 0$.

This implies that $\vec{a}(t)$ is orthogonal to $\vec{v}(t)$.

Example

An object moves in 3-space so that $\vec{v}(t) = \langle 1, t, t^2\rangle$. Find the coordinates of the particle at time $t = 1$ given that at $t = 0$, the particle is at $(-1, 2, 4)$.

$\vec{r}(t) = \int \vec{v}(t)dt = \langle t, \frac{1}{2}t^2, \frac{1}{3}t^3\rangle + \vec{c}$

We know that $\vec{r}(0) = \langle -1, 2, 4\rangle$. This means that $\vec{c} = \langle -1, 2, 4\rangle$.

So, $\vec{r}(t) = \langle t-1, \frac{1}{2}t^2, \frac{1}{3}t^3 + 4\rangle$.

$\vec{r}(1) = \langle 0, \frac{5}{2}, \frac{13}{3}\rangle$.

So this becomes the point $\left(0, \frac{5}{2}, \frac{13}{3}\right)$ at $t = 1$.

Example

An object with mass m that moves in a circular pattern with constant angular speed ω has position vector $\vec{r}(t) = a\cos\omega t\vec{i} + a\sin\omega t\vec{j}$. Find the force acting on the object and show that it is directed toward the origin.

We have a circle toward the origin with radius a and we have points on the circle P at an angle θ.

Newton's 2nd law states that $\vec{F}(t) = m\vec{a}(t)$.

We have the position vector.

$\vec{v}(t) = \langle -a\omega\sin\omega t, a\omega\cos\omega t\rangle$

$\vec{a}(t) = \langle -a\omega^2\cos\omega t, -a\omega^2\sin\omega t\rangle$

$\vec{F}(t) = m\vec{a}(t) = m\langle -a\omega^2\cos\omega t, -a\omega^2\sin\omega t\rangle$.

This can be simplified to $-m\omega^2\langle a\cos\omega t, a\sin\omega t\rangle$. As you can see the vector is just $\vec{r}(t)$.

So $\vec{F}(t) = -m\omega^2\vec{r}(t)$.

The force acts in direction opposite to radius vector $\vec{r}(t)$. It points towards the origin.

Newton's Second Law is $\vec{F} = m\vec{a}$ as we talked about earlier.

Assumptions:

- Mass is constant
- Only force acting on the object after launch is Earth's gravity
- Assume the force of gravity is constant because the object is sufficiently close to the earth

$\vec{F} = m\vec{a}$. m is mass, g is the acceleration due to gravity.

We can find $\vec{a}$ by letting $\vec{F} = -mg\vec{j}$, and we can rewrite as $m\vec{a} = -mg\vec{j}$.

This gives $\vec{a} = -g\vec{j}$.

$\vec{v}(t) = \int \vec{a}(t) = \int -g\vec{j}dt = -gt\vec{j} + c$ at $t = 0, v(0) = v_0$.

This leads us to $\vec{v}(t) = -gt\vec{j} + \vec{v_0}$.

To find position, we need to integrate once more.

$\vec{r}(t) = -\frac{1}{2}gt^2\vec{j} + \vec{v_0}t + \vec{c_2}$, we have initial conditions $\vec{r}(0) = s_0$ and $\vec{c_2} = s_0\vec{j}$ (up)

We can find that $\vec{r}(t) = -\frac{1}{2}gt^2\vec{j} + \vec{v_0}t + s_0\vec{j}$ or written as $\left(-\frac{1}{2}gt^2 + s_0\right)\vec{j} + t\vec{v_0}$

We can express $\vec{v_0}$ in two components, with the x component being $\vec{v_0}\cos\alpha$ and the y component being $\vec{v_0}\sin\alpha$.

So $\vec{v_0} = v_0\cos\alpha\vec{i} + v_0\sin\alpha\vec{j}$.

So $\vec{r}(t) = \left(-\frac{1}{2}gt^2 + s_0\right)\vec{j} + t(v_0\cos\alpha\vec{i} + v_0\sin\alpha\vec{j})$.

This simplifies to $\vec{r}(t) = (v_0\cos\alpha t)\vec{i} + (s_0 + v_0\sin\alpha t - \frac{1}{2}gt^2)\vec{j}$

So $x(t) = v_0\cos\alpha \cdot t$ and $y(t) = s_0 + v_0\sin\alpha \cdot t - \frac{1}{2}gt^2$.

Velocity in each direction is $v_x = v_0\cos\alpha$ and $v_y = v_0\sin\alpha - gt$

Example

A basketball is hit with an initial speed of 80 ft/sec at an angle of $30°$ and an initial height of 3 feet.

(a) Find parametric equations for the trajectory of the ball.

$x = 80\cos(30)t$ so $x(t) = 40\sqrt{3}t$

$y = 3 + 80\sin 30t - \frac{1}{2}(32)t^2$ so $y(t) = 3 + 40t - 16t^2$

(b) How high does the ball get?

We need to find the maximum of y so $\frac{dy}{dt} = 40 - 32t$. $0 = 40 - 32t$ and that gives $t = \frac{5}{4}$ seconds.

Substituting that back in gives $y\left(\frac{5}{4}\right) = 28$ ft.

Before $t = \frac{5}{4}$ the value is positive and after this time it is negative, so it is a maximum by the first derivative test.

(c) How far does it travel horizontally?

$0 = 3 + 40t - 16t^2$ and $t \approx 2.57$ sec. $x(2.57) \approx 178.25$ ft.

(d) What is the speed of the ball when it lands?

It lands at $t \approx 2.57$ sec and speed is $\|\vec{v}(t)\|$.

$\vec{v}(t) = \vec{r'}(t) = \langle 40\sqrt{3}, 40 - 32t\rangle$.

Speed is $\sqrt{(40\sqrt{3})^2 + (40 - 32(2.57))^2} \approx 81.19$ ft/sec

It if often useful to break acceleration into 2 components - one that is in the direction of the tangent vector and one in the direction of the normal vector.

We will define $\|\vec{v}(t)\| = v$. Then $\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\vec{v}(t)}{v}$

So, $\vec{v}(t) = v \cdot \vec{T}(t) = v \cdot \vec{T}$.

Differentiating this gives $\vec{v'} = v'\vec{T} + v\vec{T'}$.

To get $\vec{T'}$, use κ (curvature).

$\kappa = \frac{|\vec{T'}(t)|}{|\vec{r'}(t)|} = \frac{|\vec{T'}(t)|}{v} \implies |\vec{T'}(t)| = \kappa \cdot v$

Also $\vec{N} = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} \implies \vec{T}'(t) = |\vec{T}'(t)|\vec{N}$

Substituting in gives $\vec{T}'(t) = \kappa v \cdot \vec{N}$.

$\vec{v'} = \vec{a}(t) = v'\vec{T} + v(\kappa v\vec{N}) = v'\vec{T} + \kappa v^2\vec{N}$

We can write $\vec{a}(t) = a_T\vec{T} + a_N\vec{N}$.

This tells us that the object always moves according to the direction of motion ($\vec{T}$) and direction the curve is turning ($\vec{N}$)

We can dot $\vec{a}(t)$ with v to get $\vec{v} \cdot \vec{a} = (v\vec{T}) \cdot (v'\vec{T} + \kappa v^2\vec{N})$

This gives us $\vec{v} \cdot \vec{a} = vv'\vec{T} \cdot \vec{T} + \kappa v^3\vec{T} \cdot \vec{N}$. Hence, $\vec{v} \cdot \vec{a} = vv'$.

We know that $v' = a_T$, so $a_T = \frac{\vec{v} \cdot \vec{a}}{v} = \frac{\vec{r'}(t) \cdot \vec{r''}(t)}{|\vec{r'}(t)|}$.

We also know that $a_N = \kappa v^2 = \frac{|\vec{r'}(t)| \times \vec{r''}(t)|}{|\vec{r'}(t)|^3} |r'\vec{(t)}^2$.

This gives $a_N = \frac{|\vec{r'}(t) \times \vec{r''}(t)|}{|\vec{r'}(t)|}$

In summary:

Scalar Tangential component of acceleration

$$a_T = \frac{\vec{r'}(t) \cdot \vec{r''}(t)}{\|\vec{r'}(t)\|}$$

Scalar Normal component of acceleration

$$a_N = \frac{\|\vec{r'}(t) \times \vec{r''}(t)\|}{\|\vec{r'}(t)\|}$$

Example

Suppose a particle moves along $C : \vec{r}(t) = \langle t, t^2, t^3 \rangle$.

(a) Find the scalar tangential and normal components of $\vec{a}$.

The first derivative is $\vec{r'}(t) = \langle 1, 2t, 3t^2 \rangle$ and $\vec{r''}(t) = \langle 0, 2, 6t \rangle$.

$\vec{r'}(t) \cdot \vec{r''}(t) = 4t + 18t^3$.

$|\vec{r'}(t)| = \sqrt{9t^4 + 4t^2 + 1}$.

So, $a_T = \frac{18t^3+4t}{\sqrt{9t^4+4t^2+1}}$.

The cross product of $\vec{r'}(t)$ and $\vec{r''}(t) = \langle 6t^2, -6t, 2 \rangle$.

The magnitude of this vector is $\sqrt{36t^4 + 36t^2 + 4}$.

The scalar normal component $a_N = \sqrt{\frac{36t^4+36t^2+4}{9t^4+4t^2+1}}$.

(b) Find the scalar tangential and normal components of $\vec{a}$ at $(1, 1, 1)$

Plug in to get $a_T = \frac{22}{\sqrt{14}}$ and $a_N = \sqrt{\frac{28}{7}}$.

(c) Find the vector tangential and normal components at $t = 1$.

$\vec{a} = a_T\vec{T} + a_n\vec{N}$.

$\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}$. So $\vec{T}(1) = \frac{\langle 1,2,3 \rangle}{\sqrt{14}}$.

So $a_T\vec{T} = \frac{22}{\sqrt{14}}\frac{\langle 1,2,3 \rangle}{\sqrt{14}} = \langle \frac{11}{7}, \frac{22}{7}, \frac{33}{7} \rangle$.

Now to find the normal one, we can either find $\vec{N}$ or we can use that $\vec{a} = a_T\vec{T} + a_N\vec{N}$.

We know that $\vec{a}(1) = \langle 0, 2, 6 \rangle$ and we can substitute this to find $a_N\vec{N}$.

$\langle 0, 2, 6 \rangle - \langle \frac{11}{7}, \frac{22}{7}, \frac{33}{7} \rangle = a_N\vec{N} = \langle -\frac{11}{7}, -\frac{8}{7}, \frac{9}{7} \rangle$.

(d) Find the curvature of the path at the point $(1, 1, 1)$.

Remember $\kappa(t) = \frac{|\vec{r'} \times \vec{r''}|}{|\vec{r'}(t)|^3}$

Using what we previously found, $\vec{r'}(1) = \langle 1, 2, 3 \rangle$ and $\vec{r''}(1) = \langle 0, 2, 6 \rangle$.

The cross product of these gives $\langle 6, -6, 2 \rangle$.

$\kappa(1) = \frac{\sqrt{76}}{\sqrt{14}^3} = \frac{1}{14}\sqrt{\frac{38}{7}}$

Example

The position particle of a function is given by $\vec{r}(t) = \langle -5t^2, -t, t^2 + t \rangle$. At what time is the speed at a minimum?

speed is $\|\vec{v}(t)\|$.

$\vec{v}(t) = \langle -10t, -1, 2t + 1 \rangle$

speed $= \sqrt{100t^2 + 1 + 4t^2 + 4t + 1} = \sqrt{104t^2 + 4t + 2}$

$\frac{d\text{speed}}{dt} = \frac{1}{2}(104t^2 + 4t + 2)^{-1/2}(208t + 4)$

$0 = \frac{1}{2}(104t^2 + 4t + 2)^{-1/2}(208t + 4)$

The first factor is never 0, the second factor is 0 when $t = -\frac{1}{52}$ sec

Now using the first derivative test, we see values before $-\frac{1}{52}$ are decreasing and after this point are positive, so $t = -\frac{1}{52}$ is a minimum.

3 Partial Derivatives

3.1 Functions of Several Variables

Before: $f(x)$ is a function in terms of x (one variable). An example of this is $y = 4x^2$.

Now: $z = f(x, y)$ (a function of 2 variables). An example is $A = \frac{1}{2}bh$. This is a function $f(b, h) = \frac{1}{2}bh$.

For $z = (x, y)$, z is the dependent variable and x and y are the independent variables.

In 3-space, this becomes $w = f(x, y, z)$.

Domain: The restrictions of the independent variables determine the domain of f.

Example

Find the domain of $f(x, y) = \ln(x, y)$.

$xy > 0$.

If both x and y are positive and x and y are negative, the product will be greater than zero.

This can be expressed as D : all ordered pairs in quadrants I and III. (not on axis)

Or written mathematically as $D : (x, y) : xy > 0$.

Example

Find the domain of $f(x, y, z) = \frac{x}{\sqrt{9-x^2-y^2-z^2}}$.

We know the quantity $9 - x^2 - y^2 - z^2 > 0$.

We can rewrite this as $x^2 + y^2 + z^2 < 9$.

The domain is D : all $(x, y, z) : x^2 + y^2 + z^2 < 9$.

This is also known as the set of all (x, y, z) inside sphere of $r = 3$ centered at the origin.

$z = f(x, y)$ is a surface in 3-space.

Example

$f(x, y) = \sqrt{4 - x^2 - y^2}$ graphed.

$f(x, y)$ is essentially z and $z \geq 0$.

Simplifying will give you $x^2 + y^2 + z^2 = 4$, which is a sphere $r = 2$ centered at the origin.

Exercise Graph $f(x, y) = 1 - x - \frac{1}{2}y$.

Definition

The level curves of a function f of two variables are the curves with equations $f(x, y) = k$ where k is a constant (in the range of f.)

Example

Describe the level curves of $z = x^2 + y^2$.

Remember from the first chapter, this will end up being a paraboloid.

Passing various planes through or making z a number gives us an idea that level curves are circles centered at the origin.

Contour plots contain sets of level curves for a function.

If we were to graph the set of level curves (the contour plot) of this equation, we would have circles of varying radii in the xy-plane.

If z was getting bigger, it would be going up, if z was getting smaller, it would be going downwards.

Exercise Sketch the contour plot of $f(x, y) = x^2 - 4y^2$.

Functions of 2 variables are surfaces we can project as level curves.

Functions of 3 variables are 4-D graphs that we can project as level surfaces.

Example

Describe the level surfaces of $f(s, y, z) = x^2 + y^2 + z^2$.

Let $w = x^2 + y^2 + z^2$.

We can write like $1 = x^2 + y^2 + z^2$, $4 = w$, $9 = w$, etc.

This is a graph in 3-space and the level surfaces are spheres centered at the origing. As distance from the origin increases, so will the value of f.

Exercise Graph the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Exercise Graph the hyperbola $y^2 - \frac{x^2}{4} = 1$

3.2 Limits and Continuity

In 2-space, $\lim_{x \to x_0^+ f(x)} f(x) = \lim_{x \to x_0^- f(x)} f(x)$.

In 3-space, we evaluate $\lim_{(x,y) \to (x_0, y_0)} f(x, y)$.

Questions to consider:

1. Is there a point there or are we approaching a point?
2. How many directions/paths of approach do we have? Infinite.

Example

Consider $f(x, y) = \frac{-xy}{x^2+y^2}$. Find $\lim_{(x,y) \to (2,1)} f(x, y)$.

$f(2, 1) = -\frac{2}{5}$.

This is the limit, since there is no funky behavior with this limit.

Example

Consider $f(x,y) = \frac{-xy}{x^2+y^2}$. Find $\lim_{(x,y)\to(0,0)} f(x,y)$.

We can consider $\lim_{x\to 0} = 0$ and this is along the x-axis.

0 is the limit from one direction.

Along the y-axis, we get the same limit $\lim_{y\to 0} = 0$.

Along the line $y = x$, the limit is $\lim_{(x,y)\to(0,0)} \frac{-x\cdot x}{x^2+x^2} = -\frac{1}{2}$.

It seems that the limit does not exist.

This process is inefficient and impractical.

Definition

Let f be a function of 2 variables and assume f is defined at all points of some open disk centered at (x_0, y_0) (except maybe at (x_0, y_0)). Then $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if given $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x,y) - L| < \epsilon$ when the distance between (x,y) and (x_0, y_0) satisfies $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

Theorem 3.1

If $f(x,y) \to L$ as $f(x,y) \to (x_0, y_0)$, then $f(x,y) \to L$ as $(x,y) \to (x_0, y_0)$ along any smooth curve.

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist along some smooth curve or if $f(x,y)$ has different values along different curves, the limit does not exist.

Options:

1. Plug in values.
2. Limit DNE $\to$ need to show that there is a path where DNE.
3. One other option with discontinuity "trick".

Example

Find $\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2}$.

Plug in numbers to get 2.

Example

Find $\lim_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$.

The limit may not exist by plugging in $(0,0)$.

Start with the path along $x = 0$ and we get the limit $\lim_{y\to 0} = 1$.

Along $y = x$, we get $\lim_{x\to 0} = 0$.

Since these are two different limits, then the limit does not exist.

To prove that the limit does exist, you essentially have to be able to calculate it.

Definition

A function $f(x, y)$ is said to be continuous at (x_0, y_0) if $f(x_0, y_0)$ is defined and $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

1. If f is continuous on D, this means f is continuous at every point in an open set.
2. If f is continuous everywhere, this means f is continuous at every point in the xy-plane.

Theorem 3.2

$f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0) if $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0.

Compositions are continuous $(f(x, y) = g(h(x, y)))$ if $h(x, y)$ is continuous at (x_0, y_0) and $g(u)$ is continuous at $u = h(x_0, y_0)$.

Example

(a) Determine continuity for $f(x, y) = 3x^2y^5$.

$g(x) = 3x^2$ and $h(y) = y^5$.

$g(x)$ and $h(y)$ are continuous everywhere, therefore $f(x, y)$ is continuous everywhere.

(b) Determine continuity for $f(x, y) = \sin(3x^2y^5)$.

We already know the function inside is continuous everywhere.

Therefore $\sin(u)$ is continuous everywhere, therefore $f(x, y)$ is continuous everywhere.

3.3 Partial Derivatives

$f(x, y)$ is a function of 2 variables x and y. What happens to x when we hold y constant?

Definition

If $z = f(x, y)$, then the first partial derivatives of f with respect to x and y are f_x and f_y, defined by $f_x(x, y) = \lim_{h\to 0} \frac{f(x+h),y-f(x,y)}{h}$ and $f_y(x, y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$ provided the limits exist.

Example

Find the first partial derivatives f_x and f_y for $f(x, y) = 3x - x^2y^2 + 2x^3y$.

For $f_x(x, y)$ we just treat y as a constant.

So we get $f_x(x, y) = 3 - 2xy^2 + 6x^2y$.

For $f_y(x, y)$ we treat x as a constant.

So $f_y(x, y) = -2x^2y + 2x^3$.

Different notations for partial derivatives are like $f_x = \frac{\partial f}{\partial x}$. If $z = f(x, y)$ then this is also equal to $\frac{\partial z}{\partial x} = z_x$.

Example

Consider $f(x,y) = xe^{x^2y}$. Find f_x and f_y and evaluate both at $(1, \ln 2)$.

$f_x = e^{x^2y} + 2x^2ye^{x^2y}$

$f_y = x^3e^{x^2y}$

Evaluating both of these at the point $(1, \ln 2)$:

$f_x(1, \ln 2) = e^{\ln 2} + 2(\ln 2)e^{\ln 2} = 2 + 4\ln 2$

$f_y(1, \ln 2) = 1 \cdot e^{\ln 2} = 2$

Geometrically $\frac{\partial f}{\partial x}$ is the slope in the x-direction. It tells us how f (or z) changes with respect to x when y is constant.

Example

Find slopes in x and y directions at $(1/2, 1)$ for $z = f(x,y) = -\frac{1}{2}x^2 - y^2 + \frac{25}{8}$. Then interpret these slopes.

$f_x = -x$

$f_y = -2y$

$f_x(1/2, 1) = -\frac{1}{2}$

$f_y(1/2, 1) = -2$

f_x is $\frac{\Delta z}{\Delta x}$ so this tells us z decreases 1 unit for every 2 unit increase in x at this point.

f_y is $\frac{\Delta z}{\Delta y}$ so this tells us that z decreases 2 units for every unit increase in y at this point.

Functions of 3 or more variables have a similar idea. You just hold the other variables constant.

Notation for higher-order partial derivatives. If we did the partial derivative of x and then the partial derivative of x once more, we would get $\frac{\partial^2 f}{\partial x^2} = f_{xx}$. If we did the partial derivative of y after the partial derivative of x we would get f_{xy}.

Example

Find f_{xx}, f_{yy}, f_{xy} and f_{yx} for $f(x,y) = 3xy^2 - 2y + 5x^2y^2$.

$f_x = 3y^2 + 10xy^2$, so $f_{xx} = 10y^2$ and $f_{xy} = 6y + 20xy$

$f_y = 6xy - 2 + 10x^2y$ so $f_{yy} = 6x + 10x^2$ and $f_{yx} = 6y + 20xy$

You will notice that $f_{xy} = f_{yx}$ in this case.

Theorem 3.3

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then for every (x,y) in r, $f_{xy}(x,y) = f_{yx}(x,y)$.

Example

Suppose $f(x, y, z) = ye^x + x \ln z$. Show that $f_{xzz} = f_{zxz} = f_{zzx}$.

$f_x = ye^x + \ln z$, so $f_{xz} = 1/z$ and $f_{xzz} = -\frac{1}{z^2}$.

$f_z = \frac{x}{z}$, so $f_{zx} = \frac{1}{z}$ and $f_{zxz} = -\frac{1}{z^2}$.

$f_z = \frac{x}{z}$ and $f_{zz} = -\frac{x}{z^2}$ and $f_{zzx} = -\frac{1}{z}^2$.

These three are equal to each other so this shows that order does not matter.

Example

Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the y-direction at $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

The goal is to find $\frac{\partial z}{\partial y} = z_y$ at the point given.

Let's start by differentiating by y.

We get $\frac{\partial}{\partial y} = \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(z^2) = \frac{\partial}{\partial y}(1)$.

This is $2y + 2z\frac{\partial z}{\partial y} = 0$.

So $\frac{\partial z}{\partial y} = -\frac{y}{z}$.

At the point given, this evaluates to $-\frac{1}{2}$.

3.4 Tangent Planes and Linear Approximations

Previously, if we consider the graph of 2-D differentiable function, when we zoom in on the graph a lot, the graph begins to look like its tangent line. We can approximate the value of the function at a specific point using this tangent line.

If we consider the graph of a 3-D differentiable function, when we zoom in on the graph a lot, the graph begins to look like its tangent plane. How do we find the equation of this tangent plane? If we find 2 tangent lines, then this tangent plane will contain both lines.

We choose 2 lines that result when they intersect $z = f(x, y)$ with the planes $y = y_0$ and $x = x_0$.

If we recall the equation of a plane: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ where (x_0, y_0, z_0) is a point on this plane, then $z - z_0 = -\frac{A}{C}(x - x_0) - \frac{B}{C}(y - y_0)$.

Let's call $-\frac{A}{C} = a$ and $-\frac{B}{C} = b$. So we get $z - z_0 = a(x - x_0) + b(y - y_0)$. This is the general form of a plane.

Now we use the 2 tangent lines that result from the planes $y = y_0$ and $x = x_0$.

The first plane results $T_1 : y = y_0$. We get $z - z_0 = a(x - x_0)$. We are looking at a line in point slope form with slope a. a is f_x.

Likewise, the second tangent line is $T_2 : x = x_0$. We get $z - z_0 = b(y - y_0)$. Similar idea, the slope b is f_y.

This leads us to the equation of the tangent plane as

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example

Find an equation of the tangent plane $z = 2x^2 + y^2$ at $(1, 1, 3)$.

Recall this equation makes an elliptic paraboloid.

$f_x = 4x$, so $f_x(1, 1) = 4$.

$f_y = 2y$ so $f_y(1, 1) = 2$.

What this gives us is $z - 3 = 4(x - 1) + 2(y - 1)$.

So $z = 4x + 2y - 3$.

In the above example, the linear function $L(x, y) = 4x + 2y - 3$ is a good approximation of $f(x, y)$ when (x, y) is near $(1, 1)$.

For example, $L(1.1, 0.95) = 3.3$. If we find $f(1.1, 0.95)$ this is equal to 3.3225.

L is called the linearization of f at $(1, 1)$.

The approximation is called the linear approximation or tangent line approximation of f at $(1, 1)$.

Recalling the equation of a tangent plane: $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

So the linearization of f at (a, b) is $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

And the linear approximation $f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.

But only if $f(x, y)$ is differentiable at (a, b).

Definition

If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form $\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ where $\epsilon_1, \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Theorem 3.4

If f_x, f_y exist near (a, b) are are continuous at (a, b), then f is differentiable at (a, b). For functions of $3+$ variables it is similar.

Example

Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization. Then use it to approximate $f(1.1, -0.1)$.

$f_x(x, y) = e^{xy} + xye^{xy}$

$f_y(x, y) = x^2 e^{xy}$

These are both continuous at $(1, 0)$ and exist near $(1, 0)$. Therefore $f(x, y)$ is differentiable at $(1, 0)$.

$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$

$L(x, y) = 1 + 1(x - 1) + 1(y - 0) = x + y$.

$f(x, y) = xe^{xy} \approx x + y$.

$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$

In 2-space, $dy = f'(x)dx$.

Δy is the actual change in y from $x = a$ to $x = a + \Delta x$ and dy is the change in y when using tangent line.

When Δx is small, dy can be used to approximate Δy.

This is a similar idea in 3-space.

Total differential: $dz = f_x(x, y)dx + f_y(x, y)dy$ OR $dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$.

dx and dy are independent variables so these can be any numbers.

We take $dx = \Delta x = x - a$ so the linear approximation becomes $f(x, y) \approx f(a, b) + dz$

Example

Consider $z = xy^2$.

(a) Find the total differential.

$f_x = y^2$ and $f_y = 2xy$ so $dz = y^2 dx + 2xydy$.

(b) Approximate the change in z from $(0.5, 1.0)$ to $(0.503, 1.004)$.

$dz = (1)^2(0.503 - 0.5) + 2(0.5)(1)(1.004 - 1) = 0.007$

(c) Compare the approximation to the actual change.

$\Delta z = f(0.503, 1.004) - f(0.5, 1.0) = (0.503)(1.004)^2 - (0.5)(1)^2 = 0.007032\ldots$

Example

The base radius and height of a right circular cone are 10 cm and 25 cm with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

$V = \frac{1}{3}\pi r^2 h$.

So $V_r = \frac{2}{3}\pi rh$ and $V_h = \frac{1}{3}\pi r^2$

$dv = \frac{2}{3}\pi rhdr + \frac{1}{3}\pi r^2 dh$

The $|\Delta r| \leq 0.1$ and $|\Delta h| \leq 0.1$.

$dv = 20\pi \approx 63$ cm^3 from the formula.

This may over/underestimate the volume by as much as 63 cm^3.

Example

The dimensions of a rectangular box are 75 cm, 60 cm, and 40 cm. Each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated.

$V = xyz$, so $v_x = yz$, $v_y = xz$, and $v_z = xy$.

$dv = (yz)dx + (xz)dy + (xy)dz$, so $dv = 1980$ cm^3.

The volume of the box is 180000 cm^3, so the error is $\frac{1980}{180000} = 1.1\%$.

3.5 The Chain Rule

In previous the past, you were given $y(x)$ and $x(t)$. If you wanted to find $\frac{dy}{dx}$ for $y(x(t))$, you used the formula $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$.

Now we are looking at two variables, such as $z = f(x, y)$ where $x = x(t)$ and $y = y(t)$. Then the composition $z = f(x(t), y(t))$ expresses z as a single variable.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example

Consider $z = x^2y - y^2$ where $x = \sin t$ and $y = e^t$. Find $\frac{dz}{dt}$ at $t = 0$.

$\frac{dz}{dt} = (2xy)(\cos t) + (x^2 - 2y)(e^t)$.

Replacing with what was defined gives $\frac{dz}{t} = 2\sin t \cos t e^t + (\sin^2 t - 2e^t)e^t$.

At $t = 0$, this is -2.

Without the chain rule, we would have to do $z(t) = (\sin^2 t)e^t - e^{2t}$. Then the derivative of this would be $2\sin t \cos t e^t + e^t \sin^2 t - 2e^{2t}$.

Suppose that u is a differentiable function of n variables, $x_1, x_2, \ldots, x_n$, and each x_i is a differentiable function of m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

If $z = f(x(t), y(t))$, then we can see that we can do partial derivatives to x and y, but non partial derivatives to get to t. So $\frac{dz}{dt} = \frac{\partial t}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$.

Example

Consider $z = f(x, y)$ where $x = x(u, v)$ and $y = y(u, v)$. Write the chain rule for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

So we need partial derviatives to go from z to x to u and z to x to v, and likewise for y, we see we get that $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$.

Similarly, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$.

Example

Consider $z = 2xy$ where $x = u^2 + v^2$ and $y = \frac{u}{v}$. Find $\frac{\partial z}{\partial u}$.

So $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$.

This is equal to $(2y)(2u) + (2x)\left(\frac{1}{v}\right)$.

We want this in terms of u.

Substitute to get $\frac{\partial z}{\partial u} = \left(2 \cdot \frac{u}{v}\right)(2u) + (2 \cdot (u^2 + v^2))\left(\frac{1}{v}\right) = \frac{4u^2}{v} + \frac{2u^2+2v^2}{v} = \frac{6u^2+2v^2}{v}$.

Example

Find a rule for $\frac{dw}{dx}$ if $w = xy + yz$, $y = \sin x$ and $z = e^x$.

$\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial w}{\partial t} \cdot \frac{\partial z}{\partial x}$.

This is $\frac{dw}{dx} = (y) + (x + z)(\cos x) + (y)(e^x)$.

Now everything needs to be in terms of x.

$\frac{dw}{dx} = \sin x + (x + e^x)\cos x + \sin x e^x$

Example

Scenario: Consider $y^3 + y^2 - 5y - x^2 + 4 = 0$.

Previously, we would use implicit differentiation.

$3y^2 \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$.

Get $\frac{dy}{dx} = \frac{2x}{3y^2+2y-5}$.

With partial derivatives: $f(x, y) = C$.

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$.

Isolate $\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$

$\frac{dy}{dx} = \frac{-(-2x)}{3y^2+2y-5}$.

Example

Consider $x^2 + y^2 + z^2 = 1$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$f(x, y, z) = x^2 + y^2 + z^2 = C = 1$

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$

$\frac{\partial z}{\partial x} = \frac{-\partial f/\partial x}{\partial f/\partial z}$

Likewise, $\frac{\partial z}{\partial y} = \frac{-\partial f/\partial y}{\partial f/\partial z}$

And we get $\frac{\partial z}{\partial x} = -\frac{x}{z}$.

And $\frac{\partial z}{\partial y} = -\frac{y}{z}$.

Implicitly we could use $2x\frac{\partial x}{\partial y} + 2y\frac{\partial y}{\partial y} + 2z\frac{\partial z}{\partial y} = 0$ and get $2y + 2z\frac{\partial z}{\partial y} = 0$ to get $\frac{\partial z}{\partial y} = -\frac{y}{z}$.

3.6 Directional Derivatives and the Gradient Vector

Here's what we can do with partial derivatives:

1. Find instantaneous rates of change in 2 directions.
2. Find the slope of a tangent line parallel to the x- or y-axis.

What if we want to go in other directions?

Plan: Have some point $P(x_0, y_0, z_0)$ and $\vec{u} = \langle a, b \rangle$.

- Pass plane through surface and that plane is vertically passing through $\vec{u}$
- The slope of the tangent line is rate of change of z in the direction of $\vec{u}$
- Choose another point Q that is on the surface and plane. $Q(x, y, z)$
- Project P and Q onto the xy-plane. $P'(x_0, y_0, 0)$ and $Q'(x, y, 0)$
- $\vec{P'Q'}$ will be parallel to $\vec{u}$, so $\vec{P'Q'} = h\vec{u} = \langle ha, hb \rangle$.
- $x = x_0 + ha$, $y = y_0 + hb$ so $\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0+ha, y_0+hb) - f(x_0, y_0)}{h}$

Take $\lim\limits_{h \to 0}$ to obtain rate of change of z in direction of $\vec{u}$.

Definition: Directional Derivative

The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b\rangle$ is:

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

The directional derivative tells rate of change of z in direction of $\vec{u}$.

Example

Use the weather map in the below figure to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.

The change in temperature over the change in miles is $\frac{60-50}{75} = \frac{2}{15}$ °F/mi.

Theorem 3.5

The directional derivative of f in the direction of a unit vector is: $D_{\vec{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$. (Similar idea in 3-space)

Example

Find $D_{\vec{u}}(1, -2, 0)$ for $f(x, y, z) = x^2y - yz^3 + z$ in the direction of $\vec{a} = \langle 2, 1, -2\rangle$.

$|\vec{a}| = \sqrt{4+1+4} = 3$ so $\vec{u} = \frac{\vec{a}}{|\vec{a}|} = \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\rangle$

$f_x = 2xy$, $f_y = x^2 - z^3$ and $f_z = -3yz^2 + 1$

$f_x(1, -2, 0) = -4$

$f_y(1, -2, 0) = 1$

$f_z(1, -2, 0) = 1$

$D_{\vec{u}} f(1, -2, 0) = -4\left(\frac{2}{3}\right) + 1\left(\frac{1}{3}\right) + 1\left(-\frac{2}{3}\right) = -3$

What does this value mean?

For every unit travelled in direction of $\vec{a}$, $f(x, y, z)$ decreases by 3 units.

Example

Find $D_{\vec{u}}f(x,y)$ if $f(x,y) = x^2e^{-y}$ and $\vec{u}$ is in the unit vector given by $\theta = 2\pi/3$. What is $D_{\vec{u}}f(1,0)$?

$f_x = 2xe^{-y}$ and $f_y = -x^2e^{-y}$

$\vec{u} = \langle \cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}\rangle = \langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$.

$D_{\vec{u}}f(x,y) = 2xe^{-y} \cdot -\frac{1}{2} + (-x^2e^{-y})\left(\frac{\sqrt{3}}{2}\right)$

$D_{\vec{u}}f(x,y) = -xe^{-y} - \frac{\sqrt{3}x^2e^{-y}}{2}$

$D_{\vec{u}}f(1,0) = -1 - \frac{\sqrt{3}}{2} \approx -1.866$

Earlier: $D_{\vec{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$

This can be reexpressed as $D_{\vec{u}}f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a,b\rangle = \langle f_x(x,y), f_y(x,y)\rangle \cdot \vec{u}$.

The gradient is $\langle f_x(x,y), f_y(x,y)\rangle$.

Definition

If f is a function of x and y, then the gradient is defined by:

$$\nabla f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j} = \langle f_x(x,y), f_y(x,y)\rangle$$

So $D_{\vec{u}}f(x,y) = \vec{\nabla}f(x,y) \cdot \vec{u}$.

Example

If $f(x,y,z) = x\sin yz$ find:

(a) the gradient of f

$\vec{\nabla}f = \langle \sin yz, xz\cos yz, xy\cos yz\rangle$

(b) $D_{\vec{u}}f(1,3,0)$ if $\vec{u} = \langle 1,2,-1\rangle$

$|\vec{u}| = \sqrt{6}$ so $\frac{\vec{u}}{|\vec{u}|} = \langle -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\rangle$

$D_{\vec{u}}f(1,3,0) = \vec{\nabla}f(1,3,0) \cdot \langle -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\rangle = -\frac{3}{\sqrt{6}} = -\frac{\sqrt{6}}{2}$.

Theorem 3.6

Suppose f is a differentiable function of 2 or 3 variables. The maximum value (slope) of $D_{\vec{u}}f$ is $\|\nabla f\|$ and it occurs in the direction of the gradient. The minimum slope is $-\|\nabla f\|$.

Another fun fact: If $\nabla f(x,y) = \vec{0}$, then $D_{\vec{u}}f(x,y) = 0$ in all directions at the point (x,y).

Example

Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = \frac{80}{1+x^2+2y^2+3z^2}$ where T is measured in °C and x, y, and z are measured in meters. In wh ich direction does the temperature increase the fastest at the point $(1, 1, -2)$. What is the maximum rate of increase?

We have $\vec{\nabla} T = \langle T_x, T_y, T_z \rangle$.

This is $\vec{\nabla} T = \frac{160}{(1+x^2+2y^2+3z^2)^2} \langle -x, -2y, -3z \rangle$.

$\vec{\nabla} T(1, 1, -2) = \frac{160}{256} \langle -1, -2, 6 \rangle$.

Direction: $\frac{5}{8}(-\vec{i} - 2\vec{j} + 6\vec{k})$

The maximum rate of increase is the length of the gradient.

This is $|\vec{\nabla} T = \frac{5}{8}\sqrt{1 + 4 + 36} = \frac{5}{8}\sqrt{41} \approx 4$°C/m

Let S be a surface with equation $F(x, y, z) = k$ (is a level surface function F of 3 variables).

Let $P(x_0, y_0, z_0)$ be a point on S. C is a curve on S passing through P.

$C : \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Let P correspond to t_0.

Because C lies on S, then $F(x(t), y(t), z(t)) = k$. Differentiate with respect to t.

We get $\frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z} \cdot \frac{dz}{dt} = 0$.

We get $\vec{\nabla} F \cdot \vec{r'}(t) = 0$. We are interested in point P.

$\vec{\nabla} F(x_0, y_0, z_0) \cdot \vec{r'}(t_0) = 0$.

This tells us that the gradient vector at P is perpendicular to the tangent vector passing through P.

So, the tangent plane to a level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is given by

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Alternatively, $\vec{\nabla} F(x_0, y_0, z_0) \cdot (\vec{r} - \vec{r}_0) = 0$.

Normal line passes through P and is perpendicular to a tangent plane.

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Example

Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$

Recall this is an ellipsoid.

We have $F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9} \quad (k = 3)$.

$\vec{\nabla} F = \langle \frac{2x}{4}, 2y, \frac{2z}{9} \rangle$.

$\vec{\nabla} F(-2, 1, -3) = \langle -1, 2, -\frac{2}{3} \rangle$.

Tangent plane: $\langle -1, 2, -\frac{2}{3} \rangle \cdot \langle x + 2, y - 1, z + 3 \rangle = 0 = -(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0$

OR expressed as $3x - 6y + 2z + 18 = 0$.

The normal line is $\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-\frac{2}{3}}$.

Special case:

What if s is in the form $z = f(x, y)$?

Then, define $F(x, y, z)$ as $f(x, y) - z$. So $\nabla F(x, y, z) = \langle f_x(x, y), f_y(x, y), -1\rangle$.

So the equation of the tangent plane becomes $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$.

3.7 Maximum and Minimum Values

How do you know if a point is a relative max or min? By using the first derivative test where $f'(x) = 0$ and see if it is increasing/decreasing.

Now we are using $f(x, y)$. Peaks in this graph would be relative maxima and valleys are relative minima. Highest maximum is the absolute maximum and lowest minimum is the absolute minimum.

Definition

A function f has a relative (local) maximum at (x_0, y_0) if there exists a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in the disk.

Relative (local) minimum $= f(x_0, y_0) \leq f(x, y)$ in the disk

Absolute maximum$= f(x_0, y_0) \geq f(x, y)$ for all (x, y) in the domain

Absolute minimum$= f(x_0, y_0) \leq f(x, y)$ f or all (x, y) in the domain

Why a disk? In previous classes we used an interval, but now we can travel in more than one direction.

Bounded Sets - In 2-space, a set of points is bounded if we can draw a rectangle around the points.

In 3-space, we use a prism/cube/box.

Goal: To determine if there are relative extrema and to find their location

Theorem 3.7: Extreme Value Theorem

If $f(x, y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum.

Theorem 3.8

If f has a relative extremum at a point (x_0, y_0), and if the first order partials of f exist at (x_0, y_0), then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

If we find a point where $f_x, f_y = 0$, but that point is not a maximum or a minimum, then this is a saddle point.

2nd Partials Test

Let f be a function of 2 variables with continuous 2nd order partials in some disk centered around a critical point (x_0, y_0) and let $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$.

If $D > 0 : f_{xx}(x_0, y_0) > 0 \to (x_0, y_0)$ is a relative (local) minimum. $f_{xx(x_0,y_0)<0} \to (x_0, y_0)$ is a relative maximum. (Can also use f_{yy} instead)

If $D < 0, (x_0, y_0)$ is a saddle point.

If $D = 0$ then inconclusive.

Example

Find all relative extrema for $f(x, y) = -x^3 + 4xy - 2y^2 + 1$.

First find all critical points. This is where $f_x, f_y = 0$.

$f_x = -3x^2 + 4y = 0$

$f_y = 4x - 4y = 0$

We have a system to solve.

Solving this equation gives $x = 0, 4/3$ and $y = 0, 4/3$.

We get points $(0, 0), (4/3, 4/3)$.

Now for the 2nd partials test. We know that $f_{xx} = -6x$, $f_{yy} = -4$ and $f_{xy} = 4$.

critical points	$D = f_{xx}f_{yy} - (f_{xy})^2$	f_{xx}	conclusion
$(0, 0)$	$(0)(-4) - (4)^2 = -16$	No need to do this	saddle point
$(4/3, 4/3)$	$(-8)(-4) - (4)^2 = 16$	$-8 < 0$	relative maximum

$(0, 0)$ is a saddle point, and $(4/3, 4/3)$ is a relative maximum.

Note: Pay attention to whether you are asked for the input or output.

Example

Find all relative extrema for $f(x, y) = 4xy - x^4 - y^4$.

$f_x = 4y - 4x^3 = 0$ and $f_y = 4x - 4y^3 = 0$.

Setting these equation to each other, we get $y = x^3$.

Substituting this into $4x - 4y^3 = 0$ gives $4x(1 - x^2)(1 + x^2)(1 + x^4) = 0$, so $x = 0, \pm 1$.

From these we get the points $(0, 0)$, $(1, 1)$, $(-1, -1)$ as critical points.

We also have $f_{xx} = -12x^2$, $f_{yy} = -12y^2$ and $f_{xy} = 4$.

critical points	$D = f_{xx}f_{yy} - (f_{xy})^2$	f_{xx} (or f_{yy})	conclusion
$(0, 0)$	$(0)(0) - 4^2 < 0$	no need	saddle point
$(1, 1)$	$(-12)(-12) - 4^2 > 0$	< 0	relative maximum
$(-1, -1)$	$(-12)(-12) - 4^2 \geq 0$	< 0	relative maximum

$(0, 0)$ is a saddle point, $(1, 1)$ and $(-1, -1)$ are relative maximum.

Finding Absolute Extrema on a Closed and Bounded Set

1. Find all critical points in R (region)
2. Find all boundary points at which absolute extrema can occur.
3. Evaluate $f(x, y)$ at all points.

Example

Consider $f(x, y) = xy - 2x$. Find the absolute maximum and absolute minimum on R: a triangular region formed by $(0, 0), (0, 4)$, and $(4, 0)$.

The critical points:

$f_x = y - 2 = 0$ and $f_y = x = 0$, so critical point is $(0, 2)$. This is not in R.

The boundary lines are $x = 0$, $y = 0$ and $y = 4 - x$.

We can see $x = 0$. $f(x, y) = 0$, so $f_x = 0$ and $f_y = 0$, so there are no critical points on the boundary $x = 0$.

For $y = 0$, then $f(x, y) = -2x$. $f_x = -2 \neq 0$ so there are no critical points on this boundary.

$y = 4 - x$ gives $f(x, y) = x(4 - x) - 2x = -x^2 + 2x$. $f_x = -2x + 2$ and $f_y = 0$. We get $x = 1$. This gives us $y = 3$. The critical point is $(1, 3)$.

Critical point	$(0, 0)$	$(0, 4)$	$(4, 0)$	$(1, 3)$
$f(x, y) = xy - 2x$	0	0	-8	1

Absolute maximum is 1 and absolute minimum is -8.

Example

Find the dimensions of a box (open at the top) having volume 32 ft^3 requiring the least amoung of material.

We know that $xyz = 32$.

The surface area is $S = xy + 2yz + 2xz$.

The constraint is $z = \frac{32}{xy}$. Plugging this in gives $S(x, y) = xy + \frac{64}{x} + \frac{64}{y}$.

$S_x = y - \frac{64}{x^2} = 0$ and $S_y = x - \frac{64}{y^2} = 0$.

The system gives $y = \frac{64}{x^2}$ and substituting this in gives $x = 0, x = 4$. We only need to consider $x = 4$, so substituting this in gives $y = 4$.

Then going back, $z = \frac{32}{4 \cdot 4} = 2$.

The dimensions of the box seem to be 4 ft $\times$ 4 ft $\times$ 2 ft.

But we need to prove that this is the minimum amount of material.

We do this by the 2nd partials test.

$S_{xx} = \frac{128}{x^3}$, $S_{yy} = \frac{128}{y^3}$ and $S_{xy} = 1$.

$D : \left(\frac{128}{4^3}\right)\left(\frac{128}{4^3}\right) - 1^2 > 0$, and $S_{xx} > 0$, so $(4, 4, 2)$ is a relative minimum.

3.8 Lagrange Multipliers

Previously, we minimized $S = xy + 2xz + 2yz$ subject to the constraint $xyz = 32$ (volume).

We solved for z, substituted into S, and minimized. What if we can't get a function of two variables?

Example

Suppose we want to find a rectangle with the largest area that can be inscribed in the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$.

We have $g(x,y)=\frac{x^2}{9}+\frac{y^2}{16}$ with $(g(x,y)=1)$.

$A(x,y)=4xy$

Consider $g(x,y)$ as a set of level curves, same for $A(x,y)=f(x,y)$.

Let $4xy=k$, then $y=\frac{k/4}{x}$.

We are interested in the level curve that "barely" satisfies the constraint (the ellipse).

Remember 2 curves are tangent to a point if and only if their gradient vectors are parallel. There are two level curves remember, the ones for $g(x,y)$ and $A(x,y)$.

The gradients are perpendicular to the curve.

$\vec{\nabla} f=\lambda\vec{\nabla} g(x,y)$. This says they are scalar multiples. λ is the lagrange multiplier.

Solving system of equations that result from this is called "Method of Lagrange Multipliers"

Now to answer the question.

We are finding $f(x,y)=4xy$ subject to $\frac{x^2}{9}+\frac{y^2}{16}=1$.

We need the gradient of f: $\vec{\nabla} f(x,y)=\langle 4y,4x\rangle$.

$\vec{\nabla} g(x,y)=\langle\frac{2}{9}x,\frac{1}{8}y\rangle$.

Therefore $\langle 4y,4x\rangle=\lambda\langle\frac{2}{9}x,\frac{1}{8}y\rangle$.

Now we have $4y=\frac{2\lambda x}{9}$ and $4x=\frac{\lambda y}{8}$.

We have $\lambda=\frac{18y}{x}$ and substituting this gives $4x=\frac{9y^2}{4x}$.

Using the constraint: $\frac{x^2}{9}+\frac{y^2}{16}=1$.

If we solve for x^2 we can use it: $x^2=\frac{9y^2}{16}$.

So we get $\frac{y^2}{16}+\frac{y^2}{16}=1$, so $y=\pm 2\sqrt{2}$.

Plugging this in gives $x^2=\frac{9}{2}$, so $x=\pm\frac{3}{\sqrt{2}}$.

Our area function is $4xy$, so the maximum area is $4\left(\frac{3}{\sqrt{2}}\right)(2\sqrt{2})=24$ u^2.

Example

Minimum $S=xy+2xz+2yz$ subject to the constraint $xyz=32$.

$f(x,y,z)=xy+2xz+2yz$ and $g(x,y,z)=xyz$ is the constraint.

$\vec{\nabla} f=\langle y+2z,x+2z,2x+2y\rangle$ and $\vec{\nabla} g=\langle yz,xz,xy\rangle$.

So we have $y+2z=\lambda yz$, $x+2z=\lambda xz$ and $2x+2y=\lambda xy$.

We have $\lambda=\frac{1}{z}+\frac{2}{y}$, $\lambda=\frac{1}{z}+\frac{2}{x}$, and $\lambda=\frac{2}{y}+\frac{2}{x}$.

We can see that $\frac{1}{z}+\frac{2}{y}=\frac{1}{z}+\frac{2}{x}$, so $x=y$.

We also see that $\frac{1}{z}+\frac{2}{y}=\frac{2}{y}+\frac{2}{x}$, so $z=\frac{1}{2}x$.

The constraint is $xyz=32$, so $x\cdot x\cdot\frac{1}{2}x=32$, so $x=4$.

Recall dimensions are $4'\times 4'\times 2'$.

Be careful! When solving systems, you need to consider if $x=0,y=0,z=0$ or $\lambda=0$ is possible.

Example

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

$g(x, y) = x^2 + y^2$.

$\vec{\nabla} f = \langle 2x, 4y \rangle$ and $\vec{\nabla} g = \langle 2x, 2y \rangle$.

$\langle 2x, 4y \rangle = \lambda \langle 2x, 2y \rangle$.

We have $2x = 2x\lambda$ and $4y = 2y\lambda$.

Note $\lambda = 1$ or $x = 0$.

This gives us $4y = 2y$, so $y = 0$ and gives $y = \pm 1$, so $(0, 1), (0, -1)$.

$x = \pm 1$ gives the points $(1, 0), (-1, 0)$.

$f(1, 0) = 1$

$f(0, 1) = 2$

$f(-1, 0) = 1$

$f(0, -1)2$

The maximum is 2 and the minimum is 1.

Exercise Find the extreme values of $f(x, y) = 3x + y$ subject to $x^2 + y^2 = 10$.

4 Multiple Integrals

4.1 Double Integrals over Rectangles

Recall that $\int_a^b f(x)dx$ gives the area under the curve from $x = a$ to $x = b$.

Also, $\int_a^b f(x)dx = \lim_{x\to\infty} \sum_{k=1}^{n} f(x_k^i)\Delta x_k$ (Riemann sums)

Now we have volume.

Volume Problem: Given a function of 2 variables that is continuous and non-negative on a region R in the xy-plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the xy-plane.

Plan:

1. Partition R in rectangles.
2. Choose a point (x_k^*, y_k^*) in each rectangle.
3. Map onto z.
4. Form parallelpiped.

We can then approximate the volume using rectangular parallelpipeds.

Volume $\approx$ area of rectangle $\times$ height $\to$ Volume $\approx \Delta A_k f(x_k^*, y_k^*)$.

Volume $\approx \Delta A_k f(x_k^*, y_k^*)$.

Volume $= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ (this is the formal definition for the volume problem using Riemann sums)

If f has both positive and negative values, then the volume is the difference in volumes between R and the surface above and below the xy-plane.

Definition

If $f(x, y) \geq 0$, then the volume of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is:

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

$$= \iint_R f(x, y)dA$$

Example

Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below $z = 16 - x^2 - 2y^2$. Divide R into 4 equal squares and choose the sample point to be the upper right corner of each square.

So $V = \iint\limits_R (16 - x^2 - 2y^2)dA$.

This is $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$. We will use approximately 4 parallelpipeds.

We get $V \approx 1 \cdot f(1,1) + 1 \cdot f(2,1) + 1 \cdot f(1,2) + 1 \cdot f(2,2)$.

This is equal to $V \approx 34$ u^3. This is an estimate.

Example

If $R = \{(x,y) | -1 \le x \le 1, -2 \le y \le 2\}$, evaluate $\iint\limits_R \sqrt{1-x^2}dA$.

The graph will look like half of a cylinder.

The volume of a cylinder is $\pi r^2 h$.

$V = \frac{1}{2}\pi r^2 h = \frac{1}{2}\pi(1)^2(4)$, so $\iint\limits_R dA = 2\pi$.

Midpoint Rule - This tells us to evaluate $\iint\limits_R f(x,y)dA$ using Riemann sums and midpoints.

Evaluating Double Integrals $\to$ uses iterated integration.

$$\int_c^d \int_a^b f(x,y)dxdy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy$$

This integral can be $dxdy$ or $dydx$.

Example

Integrate $\int_0^3 \int_1^2 x^2 y dy dx$.

First integrate $\int_1^2 x^2 y dy$.

This is $x^2 \cdot \frac{1}{2}y^2$ from $y = 1$ to $y = 2$.

And then we have $\int_0^3 \left(\frac{1}{2}x^2 \cdot 2^2 - \frac{1}{2}x^2 \cdot 1^2\right) dx$.

This is $\int_0^3 \frac{3}{2}x^2 dx = \frac{1}{2}x^3$ from $x = 0$ to $x = 3$, and the answer is $\frac{27}{2}$.

Example

Evaluate $\int_1^2 \int_0^3 x^2 y dx dy$.

First evaluate $\int_0^3 x^2 y dx$ to get $y \cdot \frac{1}{3}x^3$ from $x = 0$ to $x = 3$.

Then we have $\int_1^2 9y dy$ and this is $\frac{9}{2}y^2$ from $y = 1$ to $y = 2$, so the result of the integral is $\frac{27}{2}$.

$\frac{27}{2}$ in both cases is the area underneath $f(x,y) = x^2 y$ and above $R : [0,3] \times [1,2]$. Note that dx and dy are not always interchangable.

Theorem 4.1: Fubini's Theorem

Let R be a rectangular region defined by $a \leq x \leq b$, $c \leq y \leq d$. If $f(x, y)$ is continuous on this rectangle, then:

$$\iint\limits_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

Example

Evaluate $\iint\limits_R (x - 3y^2)dA$ with $R = [0, 2] \times [1, 2]$.

The integral is $\int_0^2 \int_1^2 (x - 3y^2)dydx$.

We start with the inner integral and get $xy - y^3$ from $y = 1$ to $y = 2$.

Then we evaluate $\int_0^2 (2x - 8) - (1x - 1)dx$ to get $\int_0^2 (x - 7)dx$.

The result of this integral is -12.

A few properties:

1. $\iint\limits_R cf(x,y)dA = c\iint\limits_R f(x,y)dA$
2. $\iint\limits_R [f(x,y) \pm g(x,y)]dA = \iint\limits_R f(x,y)dA \pm \iint\limits_R g(x,y)dA$
3. $\iint\limits_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$
4. $\iint\limits_R g(x)h(y)dA = \int_a^b g(x) \int_c^d h(y)dy$ where $R = [a, b] \times [c, d]$.

Example

Evaluate $\iint\limits_R \sin x \cos y dA$ where $R = \left[0, \frac{\pi}{2}\right] \times [0, \pi]$.

We can use the fourth property above to get $\int_0^{\pi/2} \sin x dx \cdot \int_0^{\pi} \cos y dy$.

We get $\cos x$ from 0 to $\pi/2$ and subtract $\sin y$ from 0 to π from this.

The answer is 0.

Example

Find the volume of the solid that is bounded above by $f(x, y) = y \sin(xy)$ and below by $R = [1, 2]\times[0, \pi]$.

$V = \int_0^{\pi} \int_1^2 y \sin(xy)dxdy = \int_1^2 \int_0^{\pi} y \sin(y)dydx$.

The first option is better.

So we start with $y \cdot -\frac{1}{y} \cos(xy)$ from $x = 1$ to $x = 2$

Then, $\int_0^{\pi} (-\cos 2y + \cos y)dy = 0$.

Average Value: $f_{avg} = \frac{1}{A(R)} \iint\limits_R f(x,y)dA$

Example

Find the average value of $f(x,y) = x^2y$ over R with vertices $(-1,0)$, $(-1,5)$, $(1,5)$, and $(1,0)$.

The area of the rectangle is $A_R = 10$.

Set up the integral to get $f_{avg} = \frac{1}{10}\int_{-1}^{1}\int_{0}^{5} x^2ydydx/$

4.2 Double Integrals over General Regions

There are 2 Types of regions:

The biggest question is finding the limits of integration.

So if we have y in terms of x, then we use $\iint\limits_D f(x,y)dA = \iint_{g_1(x)}^{g_2(x)} f(x,y)dydx$.

If we have x in terms of y, then $\iint\limits_D f(x,y)dA = \iint_{h_1(y)}^{h_2(y)} f(x,y)dxdy$

Example

Evaluate $\iint\limits_D (x+2y)dA$ where D is the region bounded by $y = 2x^2$ and $y = 1 + x^2$.

We have both equations being $y =$ something, so we are integrating with respect to y first in this case. Though the reason for this is mostly because we are integrating vertically.

So the integration is $\int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)dydx$.

We then get $xy + y^2$ with limits of integration $y = 2x^2$ to $y = 1 + x^2$.

So this results in $\int_{-1}^{1}[x(1+x^2)+(1+x^2)^2]-[x(2x^2)+(2x^2)^2]dx = \int_{-1}^{1}(-3x^4-x^3+2x^2+x+1)dx = \frac{32}{15}$.

Example

Set up only! Evaluate $\iint\limits_R xydA$ where R is the region bounded by $y = -x+1, y = x+1$, and $y = 3$.

Draw to see the region. We can see from the graph that integrating by x first is probably the better idea, because y would require 2 double integrals.

So setting up the integral gives $\int_1^3\int_{1-y}^{y-1} xydxdy$. (Setting the equations in terms of $x =$)

Example

Find the volume of the solid that lies under $z = xy$ and above D where D is the region bounded by $y = x - 1$ and $y^2 = 2x + 6$.

By drawing this, we see we will go by dx first.

Setting up the integral is $\int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xydxdy$.

Starting the integration, we get $\frac{1}{2}x^2y$ with the limits of the first integral.

This gives $\frac{1}{2}\int_{-2}^{4}(y+1)^2\cdot y - \left(\frac{1}{2}y^2-3\right)^2\cdot ydy$.

Simplifying some more gives $\frac{1}{2}\int_{-2}^{4} -\frac{1}{4}y^5 + 4y^3 + 2y^2 - 8ydy$ and this gives 36 as an answer.

Example

$\int_0^1 \int_x^1 \sin(y^2) dy dx$

If we draw the region, we have $y = 1$ and $y = x$.

We go from $x = 0$ to $x = y$ and for the y limits, we go from 0 to 1.

Therefore the integral is $\int_0^1 \int_0^y \sin(y^2) dx dy$.

Solving this integral gives $-\frac{1}{2}\cos(1) + \frac{1}{2}$.

Example

Use a double integral to find the area of the region enclosed between $y = x^3$ and $y = 2x$ in the first quadrant.

Set up the integral $A = \int_0^{\sqrt{2}} \int_{x^3}^{2} x$ to get the area.

Exercise Evaluate by reversing the order of integration: $\int_0^2 \int_{y/2}^1 e^{x^2} dx dy$.

4.3 Double Integrals in Polar Coordinates

Recall that polar coordinates are in form (r, θ) and rectangular coordinates are in (x, y).

In polar, for the unit circle, we can write $0 \leq r \leq 1$ or $0 \leq \theta \leq 2\pi$.

We have 3 equations for converting:

- $r^2 = x^2 + y^2$
- $x = r\cos\theta$
- $y = r\sin\theta$

To find the volume, the process is similar to the Riemann sum process for double integrals.

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) \Delta A_k = \iint f(r, \theta) dA$$

In the above, ΔA_k represents the area of the polar rectangle.

Now we need to find the area of a polar rectangle.

First we know that the area of a sector is $\frac{1}{2}r^2\theta$ and that ΔA_k is the large sector minus the little sector.

Therefore we have $\frac{1}{2}\left(r_k^* + \frac{1}{2}\Delta r_k\right)^2 \Delta\theta_k - \frac{1}{2}\left(r_k^* - \frac{1}{2}\Delta r_k\right)^2 \Delta\theta k$.

And this simplifies to $r_k^* \Delta r_k \Delta\theta_k$, so $\Delta A_k = r_k^* \Delta r_k \Delta\theta_k$.

So, $V = \iint\limits_R f(r, \theta) dA = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(r_k^*, \theta_k^*) r_k^* \Delta r_k \Delta\theta_k$.

So we have

$$V = \iint\limits_R f(r, \theta) r dr d\theta$$

Example

Find $\iint\limits_R \sin\theta dA$ where R is the region outside the circle $r = 2$ and inside $r = 2 + 2\cos\theta$ in the 1st quadrant.

We see that the circle will be hit first, then the other polar curve.

So the integral is $\int_0^{\pi/2} \int_2^{2+2\cos\theta} \sin\theta \cdot rdrd\theta$.

Note the outer integral goes to $\frac{\pi}{2}$ because that is the first quadrant.

So the inner integral becomes $\sin\theta \cdot \frac{1}{2}r^2$ from the bounds $r = 2$ to $r = 2 + 2\cos\theta$.

We then integrate $\frac{1}{2}\int_0^{\pi/2} \sin\theta[(2 + 2\cos\theta)^2 - 2^2]d\theta$.

Solving this gives $8/3$.

Example

Find the volume of the solid bounded by $z = 0$ and $z = 1 - x^2 - y^2$.

The graph of $z = 1 - x^2 - y^2$ will be a paraboloid.

So R is a circle with radius 1 when we draw this paraboloid.

We could integrate as $V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)dydx$, or we could convert to polar.

In polar, we know R is a circle and then we can convert to polar to get $\int_0^{2\pi} \int_0^1 (1 - r^2) \cdot rdrd\theta$, which is equal to $V = \frac{\pi}{2}$.

Area is the same as before, recall this was $A = \iint\limits_R 1 \cdot dA$, and now it is $\iint\limits_R rdrd\theta$.

Example

Use a double integral to find the area enclosed by one loop of the four-leaf rose of $r = \cos 2\theta$.

If you know how to draw this, then we can find the area $A = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} rdrd\theta$, and this integral is simple to solve, the answer is $\pi/8$.

Example

Find the volume of the solid that lies under $z = x^2 + y^2$, above the xy-plane, and inside $x^2 + y^2 = 2x$.

The graph $x^2 + y^2 = 2x$ can be rearranged to complete the square. We have then $(x - 1)^2 + y^2 = 1$.

So if we were to convert $x^2 + y^2$ to polar, we get r^2.

We have $r^2 = 2r\cos\theta$ and $r = 2\cos\theta$.

The integral then we get $\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} (r^2) \cdot rdrd\theta$.

The integral is $V = \frac{3\pi}{2}$

4.4 Surface Area

The formula for surface area is

$$A(S) = \lim_{n\to 0} \sum_{k=1}^{n} \sqrt{(z_x)^2 + (z_y)^2 + 1}\Delta A$$

which becomes

$$A(S) = \iint_R \sqrt{(z_x)^2 + (z_y)^2 + 1} dA$$

Example

Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region with vertices $(0, 0), (1, 0)$, and $(1, 1)$.

$z = x^2 + 2y$, $z_x = 2x$ and $z_y = 2$.

So $A(S) = \int_0^1 \int_0^x \sqrt{(2x)^2 + (2)^2 + 1} dydx$.

This integral gives $\frac{1}{12}(27 - 5\sqrt{5})$.

Example

Find the surface area of the portion of $z = x^2 + y^2$ below the plane $z = 9$.

Paraboloid!

The integral is $A(S) = \iint_R \sqrt{(2x)^2 + (2y)^2 + 1} dA$.

We might be able to see that simplifying this gives $4x^2 + 4y^2 + 1$ inside the square root, and we have an $x^2 + y^2 = 9$ in the paraboloid.

We should use polar.

So we now have $\int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1} \cdot rdrd\theta$.

This gives you $\frac{\pi}{6}(37^{3/2} - 1)$.

Example

Find the surface area of $z = \sqrt{4 - x^2}$ above $R : [0, 1] \times [0, 4]$.

Setting up the integral gives $\iint_R \sqrt{\left(-\frac{x}{\sqrt{4-x^2}}\right)^2 + 0^2 + 1}$.

It doesn't really matter the way we integrate, so we get $\int_0^1 \int_0^4 \sqrt{\frac{x^2}{4-x^2} + \frac{4-x^2}{4-x^2}} dydx$.

This simplifies to $\int_0^1 \frac{8}{\sqrt{4-x^2}} dx$.

We notice that this becomes $8 \sin^{-1}\left(\frac{x}{2}\right)$ with bounds 0 to 1, and this gives $\frac{4\pi}{3}$.

4.5 Triple Integrals

So far we have

- D is closed (can be contained in a rectangle)
- Taking limit as $n \to \infty$ gave us the volume under $z = f(x, y)$.

Now, triple integrals.

- Closed solid B (can be contained in a box).
- Divide B into n sub-boxes.
- Volume of each box is ΔV and a point in the box is $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$.

Then $\lim_{n\to\infty} \sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right) \Delta V = \iiint\limits_B f(x,y,z)dV$.

This gives us hypervolume.

- Same properties and evaluation as before (double integrals)
- If B is a box defined by $a \le x \le b$, $c \le y \le d$, $e \le z \le f$, then $\iiint\limits_B f(x,y,z)dV = \int_e^f \int_c^d \int_a^b f(x,y,z)dxdydz$.

Example

Evaluate $\iiint\limits_G xyz^2 dV$ where $G : \{(x,y,z)|0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$.

Convention is to do $\int_0^1 \int_{-1}^2 \int_0^3 xyz^2 dzdydx$.

Let us start with the z part to get $\int_0^1 \int_{-1}^2 \frac{1}{3}xyz^3$ from $z = 0$ to $z = 3$.

Then we get $\int_0^1 \frac{9}{2}xy^2$ from $y = -1$ to $y = 2$.

And then we get $\frac{9}{2}\int_0^1 3xdx = \frac{27}{4}$.

If the region B is rectangular and the function is a product (such as $f(x,y,z) = g(x)\cdot h(y)\cdot j(z)$), we can split up the integral.

The above integral will become: $\int_0^1 xdx \cdot \int_{-1}^2 ydy \cdot \int_0^3 z^2 dz$.

Type I Solid: E is a solid with upper surface $z = u_2(x,y)$ and lower surface $z = u_1(x,y)$. D is the projection of E onto the xy-plane.

Then $\iiint\limits_E f(x,y,z)dV = \iint\limits_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdA$.

To find limits of integration:

1. Find upper and lower surfaces bounding E. These are limits for z.
2. Make a 2-d sketch of the projection D on xy-plane.
3. Treat like usual.

Example

Evaluate $\iiint\limits_T zdV$ where T is the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

This is a tetrahedral in the first octant and $x + y + z = 1$ is a plane.

We have $z = 1 - x - y$, so $\int\int\int_0^{1-x-y} z \cdot dzdydz$.

And from the other bounds we get $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \cdot dzdydx$.

The outer integral note should always have constants.

The answer of this integral becomes $\frac{1}{24}$.

Sometimes we have lateral surfaces bounding, not the top or bottom.

Type II Solid:

$$\iiint\limits_E f(x,y,z)dV = \iint\limits_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx\right] dA$$

Type III Solid:

$$\iiint\limits_E f(x,y,z)dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy\right] dA$$

Example

Evaluate $\iiint\limits_R \sqrt{x^2+z^2}dV$ where E is bounded by $y = x^2+z^2$ and $y = 4$.

$y = x^2 + z^2$ is a paraboloid and $y = 4$ is a plane.

We want to start integrating from y since the paraboloid goes towards the plane always.

We project the figure now on the xz-plane and now get a circle with equation $x^2 + z^2 = 4$.

So we can write the integral now as $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4}\sqrt{x^2+z^2}dydzdx$.

First we have the first part of the integral be equal to $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-x^2-z^2)\sqrt{x^2+z^2}dzdx$.

In this we see that $r^2 = x^2 + z^2$.

When we switch to polar we end up getting $\int_0^{2\pi}\int_0^2(4-r^2)r\cdot rdrd\theta$.

This gives you $\frac{128\pi}{15}$.

Volume as a triple integral: $V = \iiint\limits_E 1\cdot dV$.

Example

Setup an integral to find the volume of the wedge in the 1st octant that is cut from the solid cylinder $y^2+z^2 \leq 1$ by the planes $y = x$ and $x = 0$.

We start with z then we can see $y = x$ from the projection.

So $V = \int_0^1\int_0^x\int_0^{\sqrt{1-y^2}} 1\cdot dzdxdy$.

Note it is not always the case that the order of integration can be changed without changing the bounds.

Example

Find the volume of the solid enclosed between the paraboloids $z = 5x^2+5y^2$ and $z = 6-7x^2-y^2$.

To find projection D we have $5x^2+5y^2 = 6-7x^2-y^2$. Solving gives us $y = \pm\sqrt{1-2x^2}$ (or a circle).

The integral becomes $V = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}}\int_{5x^2+5y^2}^{6-7x^2-y^2} 1\cdot dzdydx$.

4.6 Triple Integrals in Cylindrical Coordinates

Cylindrical coordinates are like polar, but in 3-D.

To convert cylindrical to rectangular, it is the same as polar. $x = r\cos\theta$, $y = r\sin\theta$, and the additional part is $z = z$.

We also get $r^2 = x^2+y^2$, $\tan\theta = \frac{y}{x}$ and $z = z$ once again to convert rectangular to cylindrical.

Exercise Plot $\left(2, \frac{2\pi}{3}, 1\right)$ and find rectangular coordinates.

Example

Find cylindrical coordinates for $(3,-3,-7)$.

$r^2 = 3^2 + (-3)^2$ gives $r = 3\sqrt{2}$.

$\tan\theta = -1$, so $\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$.

We pick $\frac{7\pi}{4}$ for this, and then we get coordinates $(3\sqrt{2}, \frac{7\pi}{4}, -7)$.

Example

Describe the surface $z = r$.

We can rewrite this was $z = \sqrt{x^2+y^2}$ and therefore $z^2 = x^2 + y^2$, which describes a cone.

Recall from earlier: $\iiint\limits_E f(x,y,z)dV = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right)\Delta V_k$.

ΔV_k is the area of base times the height. From earlier, $\Delta V_k = r_k^* \Delta r_k \Delta\theta_k \cdot \Delta z_k$.

So, $\iiint\limits_E f(x,y,z)dV = \iiint f(r,\theta,z)\cdot r dr d\theta dz$.

To find limits of integration:

1. Identify upper surface $z = g_2(r,\theta)$ and lower surface $z = g_1(r,\theta)$.
2. Make a 2-D sketch of projection onto xy-plane to determine bounds for r and θ.

Example

Evaluate $\iiint\limits_G dV$ where G lies within $x^2+y^2 = 1$, below $z = 4$, and above $z = 1 - x^2 - y^2$.

We are finding a volume.

The graph gives a cylinder, with a hemisphere at the bottom removing a part of the cylinder (bad explanation but it's ok).

So the integral is $V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{1-x^2-y^2}^{4} 1\cdot dz dy dx$.

As we can see, the projection on the xy-plane is a circle, so cylindrical coordinates are best.

Now we have $\int_0^{2\pi}\int_0^1\int_{1-r^2}^{4} 1\cdot r\cdot dz dr d\theta$

Integrating this gives you $V = \frac{7}{2}\pi$.

Example

Convert from rectangular to cylindrical: $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)dz dy dx$.

R is a circle with radius $r = 2$, so the integral becomes $\int_0^{2\pi}\int_0^2\int_r^2 r^2\cdot r dz dr d\theta$.

The integral evaluates to $\frac{16\pi}{5}$.

Example

Let E be the region inside the sphere of radius 2 centered at the origin and above the plane $z = 1$. Find the volume of E.

THe equation of the sphere is $x^2 + y^2 + z^2 = 4$, so we have $z^2 = 4 - r^2$.

The integral can be $V = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} 1 \cdot r dz dr d\theta$.

We have figure $D : x^2 + y^2 + z^2 = 4$ and $z = 1$, so $x^2 + y^2 = 3$.

4.7 Triple Integrals in Spherical Coordinates

Spherical Coordinates: (ρ, θ, ϕ), where ρ is the distance from the origin to the point, θ represents the same as before (the angle on the xy-plane from the x-axis), and ϕ represents the angle between the positive z-axis and point.

Bounds for ρ, θ, and ϕ:

- $\rho \geq 0$
- $0 \leq \theta \leq 2\pi$
- $0 \leq \phi \leq \pi$

A few common graphs:

- $\rho = c$ gives a sphere
- $\theta = c$ gives a "half" plane
- $\phi = c$ gives a cone ($0 < c < \pi/2$ will give the top part)

Converting:

- $x = \rho \sin \phi \cos \theta$
- $y = \rho \sin \phi \sin \theta$
- $z = \rho \cos \phi$

Also, $\rho^2 = x^2 + y^2 + z^2$.

Example

Convert $\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$ to rectangular.

We know $x = \rho \sin \phi \cos \theta$, so $x = 2 \sin \frac{\pi}{3} \cos \frac{\pi}{4} = \sqrt{\frac{3}{2}}$.

$y = \rho \sin \phi \sin \theta$, so pluggin in gives $\sqrt{\frac{3}{2}}$.

$z = \rho \cos \phi = 2 \cos \frac{\pi}{3} = 1$.

The coordinates are $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right)$.

Example

Convert $(0, 2\sqrt{3}, -2)$ to spherical.

$\rho^2 = x^2 + y^2 + z^2$, so $\rho = 4$.

$z = \rho\cos\phi$, and we can see that $\cos\phi = \frac{z}{\rho}$, so $\phi = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$, but knowing the bounds for ϕ gives $\phi = \frac{2\pi}{3}$.

We know that $\cos\theta = \frac{x}{\rho\sin\phi}$, so $\cos\theta = 0$, and $\theta = \frac{\pi}{2}$ is the only θ that works in the xy-plane for this.

The point is $\left(4, \frac{\pi}{2}, \frac{2\pi}{3}\right)$

Recall: $\iiint\limits_E f(x,y,z)dV = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} f\left(x^*_{ijk}, y^*_{ijk}.z^*_{ijk}\right)\Delta V_k = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi^2}\int_{\rho_1}^{\rho_2} f(\rho,\theta,\phi)\rho^2\sin\phi d\rho d\phi d\theta$.

Example

Evaluate $\iiint\limits_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dzdydx$ were B is the unit sphere.

If try rectangular, we would find the limits of integration are messy.

So we use spherical.

The integral is $\int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{(\rho^2)^{3/2}}\rho^2\sin\phi d\rho d\phi d\theta$.

This is equal to $\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2 e^{\rho^3}\sin\phi d\rho d\phi d\theta$.

Integrating this fully gives $\frac{4\pi(e-1)}{3}$.

Example

Convert to spherical: $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}dzdydx$.

We can see that the whole inside part gives $(\rho\cos\phi)^2\cdot\rho\cdot\rho^2\sin\phi d\rho d\phi d\theta$.

The most inner integrand is a hemisphere so bounds go from 0 to 2.

The second integrand is a circle so bounds go from 0 to $\pi/2$.

The integral is $\int_0^{2\pi}\int_0^{\frac{\pi}{2}},\int_0^2 \rho^5\cos^2\phi\sin\phi d\rho d\phi d\theta$.

The answer of this integral is $\frac{64}{9}\pi$.

Example

Find the volume of the ice cream cone bounded by $x^2+y^2+z^2 = z$ and $z = \sqrt{x^2+y^2}$.

Remember $V = \iiint\limits_E 1\cdot dV$.

So the sphere equation can be rewritten as $x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \frac{1}{4}$.

So the center of the sphere is $\left(0,0,\frac{1}{2}\right)$ with $r = \frac{1}{2}$.

$\rho : x^2+y^2+z^2 = z,\ \rho^2 = \rho\cos\phi,\ \rho = \cos\phi$.

$\phi : z = \sqrt{x^2+y^2},\ \rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$, so $\cos\phi = \sin\phi$, gives $\phi = \frac{\pi}{4}$.

So the integral becomes $V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} 1\cdot\rho^2\sin\phi d\rho d\phi d\theta$.

Solving this integral gives $V = \frac{\pi}{8}$.

We can split a triple integral into 3 separate integrals when all the bounds are numbers, and functions are as products of functions.

4.8 Change of Variables in Multiple Integrals

Example

Let T be the transformation from uv-plane to xy-plane defined by $x = \frac{1}{4}(u+v)$ and $y = \frac{1}{2}(u-v)$.

(a) Find $T(1,3)$.

$x = \frac{1}{4}(1+3) = 1$ and doing the same gives $y = -1$, so $(1,-1)$.

(b) Sketch the image under T bounded by $-2 \leq u \leq 2$ and $-2 \leq v \leq 2$.

The figure for uv-plane is a square centered at the origin with side lengths of 4.

The plan to draw the image on the xy-plane is to sketch several u, v curves and then you need to know u, v in terms of x and y.

$4x = u+v$ and $2y = u-v$, and this gives $4x+2y = 2u$ and $u = 2x+y$ as a result.

We can see $u = -2$ gives $y = -2x-2$, then $y = -2x-1$ when $u = -1$, and when $u = 2$, $y = -2x+2$.

Similarly, $v = 2x-y$, and we see a pattern here as well.

In a way, it is easier to integrate over a different region like mapped above.

Definition: Jacobian

If $x = g(u,v)$ and $y = h(u,v)$, then the Jacobian of x and y with respect to u and v is:

$$\frac{\partial(x,y)}{\partial(u,v)} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

We use this to give us the extra factor when converting integrals.

It is similar to converting in polar (or cylindrical), where you added r, or simliar to spherical when you added $\rho^2 \sin\phi$.

So

$$\iint\limits_R f(x,y)dA = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

Example

Consider $x = r\cos\theta$ and $y = r\sin\theta$. Find the Jacobian.

The determinant of this will be $\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r^2\cos^2\theta + r\sin^2\theta = r$

Example

Evaluate $\iint\limits_R 3xydA$ where R is bounded by $x-2y=0$, $x-2y=-4$, $x+y=4$, and $x+y=1$.

We let $u=x-2y$, $v=x+y$, so $u=0,-4$ and $v=4,1$. The region of this is rectangular, much easier to solve than if you were to do it based on y.

For change of variables, first you need to find the Jacobian.

We need $x(u,v)$ and $y(u,v)$.

So we have $y=\frac{1}{3}(v-u)$ and $x=\frac{1}{3}(u+2v)$ from the values of u and v.

The jacobian is then $\begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}$.

Then we have to do change of variables in the integral.

So the integral is $\int_1^4 \int_{-4}^0 3\left(\frac{1}{3}(u+2v)\right)\left(\frac{1}{3}(v-u)\right)\cdot\left|\frac{1}{3}\right| dudv$.

This integral results in $\frac{164}{9}$.

Now for 3 variables.

Example

Find the Jacobian of $x=\rho\sin\phi\cos\theta$, $y=x=\rho\sin\phi\sin\theta$, and $z=\rho\cos\phi$.

The Jacobian becomes $\begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$

From this, we get and doing some simplifications gives you the determinant of this, which is $\rho^2\sin\phi$.

5 Vector Calculus

5.1 Vector Fields

We will focus on the mathematical description for flow.

Examples are air flow (wind), fluid flow, and electromagnetic fields.

Definition: Vector Field

Written as $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j} \rightarrow$ at each point (x,y), there is a vector that depends on x and y.

Example

Sketch $\vec{F} = x\vec{i}$.

Basically at a point $x = 1$, then the vector is $\vec{i}$ in the positive direction, the same applies for all x.

Exercise Sketch $\vec{F} = \langle y, x \rangle$.

Of course, vector fields can be drawn in 3-space.

Other notations are $\vec{F}$ =vector field or if we associate x, y with a radius vector, $\vec{F}(\vec{r})$.

Example

We will look at inverse square fields.

Newton's Law of Gravitation is defined $\left|\vec{F}\right| = \frac{mMG}{r^2}$, essentially particles of mass m and M attract each other with a force $\left|\vec{F}\right|$.

Force exerted by particle of mass M to particle of mass m in the direction of a unit vector.

The direction of this is $-\frac{\vec{r}}{\|\vec{r}\|}$.

So we have $\left|\vec{F}(\vec{r})\right| = -\frac{mMG}{r^2} \cdot \frac{\vec{r}}{\|\vec{r}\|}$ and this is called the gravitational field.

As a vector this is written as $\vec{F}(\vec{r}) = \frac{c}{|\vec{r}|^3}\vec{r}$.

Recall the gradient is $\nabla f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$.

$\nabla f(x,y)$ is really a vector field in 2-space and is called the gradient vector field.

Example

Sketch the gradient field os $\phi(x,y) = x^2 + y^2$.

$\vec{\nabla}\phi = \langle 2x, 2y \rangle$.

This can be used to draw the gradient field.

At each point, the vector points in the direction of maximum increase.

Gradient vectors are long where level curves are close to each other. This is because the length of the gradient vector is equal to the value of the directional derivative.

Definition

A vector field $\vec{F}$ is conservative in a region if it is the gradient field for some function f in that region.

$\vec{F}$ is conservative if $\vec{F} = \vec{\nabla} f$.

f is called a potential function of $\vec{F}$.

Example

Consider $\vec{F} = \langle 2x, y \rangle$. Is $\vec{F}$ conservative?

If we can write $\vec{F} = \langle 2x, y \rangle = \vec{\nabla} f$, then it is conservative.

If $f(x) = x^2 + \frac{1}{2}y^2$, then $f_x = 2x$ and $f_y = y$

Therefore, $\vec{F}$ is conservative.

5.2 Line Integrals

What we know:

- $\int$ is integrated over an interval
- $\iint$ is integrated over an area in 2-space
- $\iiint$ is integrated over an area in 3-space

Now, we are integrating along a curve in order to find mass, fluid flow, force.

Let us have a curve $C : x = x(t), y = y(t), a \leq t \leq b$.

We can integrate then as $\sum\limits_{i=1}^{n} (x_i^*, y_i^*) \Delta s_i$, where Δs_i is the length of subarc.

Definition

If f is defined on a smooth curve $C : x = x(t), y = y(t)$, then the line integral of f along C is:

$$\int\limits_C f(x,y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i$$

Recall, arc length is $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$.

So, $\int\limits_C f(x,y)ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

Remember, $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$, so $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

Example

Evaluate $\int_C (2+x^2y)ds$, where C is the upper half of the unit circle.

We need a parameterization of this curve. So $C: x=\cos t, y=\sin t$.

The integral is $\int_C (2+x^2y)ds = \int_0^\pi (2+\cos^2 t+\sin t)\sqrt{(-\sin t)^2+(\cos t)^2}dt$.

This is equal to $\int_0^\pi 2+\cos^2 t \sin t dt = 2\pi + 2/3$.

Example

Evaluate $\int_C xds$ where C is the portions of $y=x$ and $y=x^2$ from $(0,0)$ to $(1,1)$.

Our first curve is $y=x$ and second curve is $y=x^2$.

Let $c_1 : y=x$, so the parameterization is $x=t, y=t, 0\le t\le 1$.

So integrate $\int_0^1 t\cdot\sqrt{1^2+1^2}dt = \int_0^1 t\sqrt{2}dt = \frac{\sqrt{2}}{2}$.

Let $c_2 : y=x^2$. If we look at parameterization $x=t$ and $y=t^2$, we see that this will not work. So we have to parameterize as $x=1-t, y=(1-t)^2, 0\le t\le 1$.

So we can integrate now $\int_0^1 (1-t)\sqrt{1+4(1-t)^2}dt$.

Integrating this gives $\frac{1}{12}(5^{3/2}-1)$.

Therefore, $\int_C xdS = \frac{\sqrt{2}}{2}+\frac{1}{12}(5^{3/2}-1)$.

Right now, we have arc length parameterizations, so orientation does not matter. (For example, for c_2 above, we could have used parameterization $x=t, y=t^2$.)

What does this actually mean? It depends on the interpretation of the function f.

1. If $f(x,y,z)$ represents density $\int_C f(x,y,z)ds$ is the mass of a thin wire shaped like C.
2. If $f(x,y)$ is a curve, then the area underneath the curve is $A=\int_C f(x,y)ds$.

So far we have calculated line integrals with respect to arc length: $\int_C f(x,y)ds$.

We can also calculate line integrals with respect to x and y:

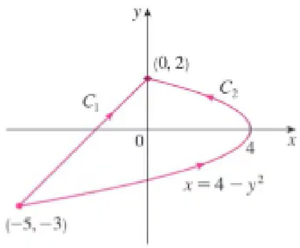
$$\int_C f(x,y)dx = \int_a^b f(x(t),y(t))\cdot x'(t)dt$$

$$\int_C f(x,y)dy = \int_a^b f(x(t),y(t))\cdot y'(t)dt$$

Sometimes these occur together: $\int_C P(x,y)dx + Q(x,y)dy$

Example

Evaluate $\int\limits_C y^2 dx + x dy$ along (a) C_1 and (b) C_2 where C is as shown.



The curve c_1 is a line from $(-5,-3)$ to $(0,2)$.

So $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1, 0 \leq t \leq 1$, $\vec{r}_0 = \langle -5,-3\rangle, \vec{r}_1 = \langle 0,2\rangle$.

$\vec{r}(t) = \langle -5+5t, 5t-3\rangle 0 \leq t \leq 1$.

So, $x = 5t-5, y = 5t-3$ and $dx = 5dt$ and $dy = 5$ with $0 \leq t \leq 1$.

So we can integrate $\int\limits_C y^2 dx + x dy$.

This is $\int_0^1 [(5t-3)^2 \cdot 5dt + (5t-5) \cdot 5dt] = -\frac{5}{6}$.

For c_2, let $y = t$ and $x = 4 - t^2$ for bounds $-3 \leq t \leq 2$.

What this gives us is $dy = 1dt$ and $dx = -2tdt$.

The integral for this is $\int_{-3}^2 [(t)^2(-2tdt) + (4-t^2) \cdot 1dt]$, which is equal to $\frac{245}{6}$.

This tells us that path matters. Line integrals in 3-space are almost the same as 2-space.

Example

Evaluate $\int\limits_C y \sin z ds$ where $C : x = \cos t, y = \sin t, z = t$ for $0 \leq t \leq 2\pi$.

$\int\limits_C y \sin z ds = \int_0^{2\pi} \sin t \cdot \sin t \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} dt$.

This is $\int_0^{2\pi} \sin^2 t \sqrt{2} dt$.

This can be rewritten as $\sqrt{2} \int_0^{2\pi} \frac{1}{2} - \frac{1}{2} \cos 2t dt$.

This is equal to $\sqrt{2}\pi$.

Example

Evaluate $\int\limits_C y dx + z dy + x dz$ where C consists of the line segment from $(2,0,0)$ to $(3,4,5)$, then the vertical line segment from $(3,4,5)$ to $(3,4,0)$.

For the first curve we have $c_1 : \vec{r} = (1-t)\langle 2,0,0\rangle + t\langle 3,4,5\rangle$ and $\vec{r}_1(t) = \langle 2+t, 4t, 5t\rangle 0 \leq t \leq 1$.

This gives $\int_0^1 4t \cdot 1dt + 5t \cdot 4dt + (2+t) \cdot 5dt = 24.5$.

$c_2 : \vec{r} = (1-t)\langle 3,4,5\rangle + t\langle 3,4,0\rangle$ and get $\vec{r}_2(t) = \langle 3,4,5-5t\rangle$.

The integral is $\int_0^1 4(0dt) + (5-5t)(0dt) + 3(-5dt) = -15$.

The final answer is $24.5 + -15 = 9.5$.

Recall: $W = \vec{F} \cdot \vec{d}$.

Now, suppose $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a continuous force field (gravitational force field or electric force field). We may want to calculate the work done in moving the particle among a smooth curve C.

Definition

Let $\vec{F}$ be a continuous vector field defined on a smooth curve C given by $\vec{r}(t)$.

Then the line integral of $\vec{F}$ along C is: $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt = \int_C \vec{F} \cdot \vec{T}ds$.

This calculates work.

Example

Find the work done by $\vec{F}(x, y, z) = \langle -\frac{1}{2}x, -\frac{1}{2}y, \frac{1}{4}\rangle$ on a particle moving along the helix $\vec{r}(t) = \langle \cos t, \sin t, t\rangle, 0 \leq t \leq 3\pi$.

We have $\vec{r}'(t) = \langle -\sin t, \cos t, 1\rangle$ and $\vec{F}(\vec{r}(t)) = \langle -\frac{1}{2}\cos t, -\frac{1}{2}\sin t, \frac{1}{4}\rangle$.

So $W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{3\pi} \langle -\frac{1}{2}\cos t, -\frac{1}{2}\sin t, \frac{1}{4}\rangle \cdot \langle -\sin t, \cos t, 1\rangle dt$.

To simplify this, we have $\int_0^{3\pi} (\frac{1}{2}\cos t \sin t - \frac{1}{2}\sin t \cos t + \frac{1}{4})dt$.

This is equal to $W = \frac{3}{4}\pi$.

Remark: $\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r}$. If we reverse the path, work will be the opposite value.

Example

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ and $C : x = t, y = t^2, z - t^3, 0 \leq t \leq 1$.

$\vec{r} = \langle t, t^2, t^3\rangle$, $\vec{r}' = \langle 1, 2t, 3t^2\rangle$, so $\vec{F}(\vec{r}(t)) = \langle t^3, t^5, t^4\rangle$.

So we end up getting $\int_C \vec{F} \cdot d\vec{r} = \int_0^1 t^3 + 5t^6 dt = \frac{27}{28}$.

Consider $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}, \vec{r}(t) = \langle x(t), y(t), z(t)\rangle$.

Then $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \langle P, Q, R\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle dt = \int_a^b (P \cdot x'(t) + Qy'(t) + R \cdot z'(t))dt$.

So $\int_C \vec{F} \cdot d\vec{r} = \int_C +Pdx + Qdy + Rdz$.

Example

Calculate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle \cos x, \sin x\rangle$ and $C : \vec{r}(t) = t\vec{i} + t^2\vec{j}, -1 \leq t \leq 2$.

$\vec{F} = \langle \cos t, \sin t\rangle$

$\int_C \vec{F} \cdot d\vec{r} = \int_{-1}^2 \langle \cos t, \sin t\rangle \cdot \langle 1, 2t\rangle dt = \int_{-1}^2 \cos t + 2t \sin t dt$.

When we integrate all of this we get $\sin t - 2t\cos t + 2\sin t$ from these bounds and the answer is $3\sin 2 + 3\sin 1 - 4\cos 2 - 2\cos 1$.

5.3 The Fundamental Theorem for Line Integrals

Recall the Fundamental Theorem of Calculus: $\int_a^b F'(x)dx = F(b) - F(a)$

Now, we have the Fundamental Theorem of Line Integrals.

Theorem 5.1: Fundamental Theorem of Line Integrals

Let C be a smooth curve given by $\vec{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function whose gradient ∇f is continuous on C. Then $\int\limits_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$, where f is a potential function.

Example

Confirm that $f(x,y) = x^2y - \frac{1}{2}y^2$ is a potential function for $\vec{F}(x,y) = 2xy\vec{i} + (x^2 - y)\vec{j}$. Then evaluate $\int\limits_C \vec{F} \cdot d\vec{r}$ for C where C is a curve from $(-1,4)$ to $(1,2)$.

The potential function is: $\vec{\nabla} f = \vec{F}$ and $\langle 2xy, x^2 - y\rangle = \langle 2xy, x^2 - y\rangle$.

Then we can just see can integrate $\int\limits_C \vec{\nabla} f \cdot d\vec{r}$ and get $f(1,2) - f(-1,4) = 4$.

This value represents the work done in moving a particle from $(-1,4)$ to $(1,2)$. Previously we would have to use $\vec{r}(t)$ for a curve and parameterize and integrate.

Recall that $\int\limits_{C_1} \vec{F} \cdot d\vec{r} \neq \int\limits_{C_2} \vec{F} \cdot d\vec{r}$ (value of line integral depends on the path)

But: $\int\limits_{C_1} \nabla f \cdot d\vec{r} = \int\limits_{C_2} \nabla f \cdot d\vec{r}$ as long as C_1 and C_2 have the same endpoints (from the theorem above).

This is true for line integrals of a conservative vector field ($\vec{F} = \vec{\nabla} f$). In general, $\int\limits_C \vec{F} \cdot d\vec{r}$ is independent of path if $\int\limits_{C_1} \vec{F} \cdot d\vec{r} = \int\limits_{C_2} \vec{F} \cdot d\vec{r}$ for any 2 paths C_1, C_2, that have the same initial points and endpoints.

Line Integrals along Closed Paths: The following are equivalent

1. $\vec{F}(x,y) = \langle f(x,y), g(x,y)\rangle$ is conservative.
2. $\int\limits_C \vec{F} \cdot d\vec{r} = 0$ (same initial/endpoint)
3. $\int\limits_C \vec{F} \cdot d\vec{r}$ is independent of path from any point P to any point Q.

Any single one of these statements implies the other two statements for closed paths.

Simple Path: a path that goes from one point to another point.

Not Simple: A path that may curve back on itself.

Simply-Connected region: closed, non-intersecting

Multiply-Conected region: region that has holes in the domain

So the theorem only applies when $\vec{F}$ is conservative. Then $\int\limits_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$ where $\nabla f = \vec{F}$.

Conservative Field Test:

Let $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ be a vector field on an open, simply-connected region D. If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D, then $\vec{F}$ is conservative.

Example

Show that $\vec{F}(x,y) = \langle 2xy^3, 1+3x^2y^2 \rangle$ is conservative. Then find the potential function.

Previously, we would have found the potential function.

We have $\frac{\partial P}{\partial y} = 6xy^2$ and $\frac{\partial Q}{\partial x} = 6xy^2$, so it is conservative.

So integrating $\int \frac{\partial F}{\partial x} dx = \int 2xy^3 dx$ and get $f = x^2y^3 + k(y)$.

Now we can use $\frac{\partial f}{\partial y} = 3x^2y^2 + k'(y)$.

So we cna see that this has to be equal to $3x^2y^2 + k'(y) = 1 + 3x^2y^2$.

So we have $\int k'(y) = \int 1dy$, so $k(y) = y + C$.

Therefore, $f(x,y) = x^2y^3 + y + C$.

Example

Find the work done by the vector field $\vec{F}(x,y) = (y^3+1)\vec{i} + (3xy^2+1)\vec{j}$ in moving a particle from $(0,0)$ to $(2,0)$.

Work is $\int\limits_C \vec{F} \cdot d\vec{r}$.

Before, we would have parameterized.

Now we can ask if $\vec{F}$ is conservative.

$\frac{\partial P}{\partial y} = 3y^2 = \frac{\partial Q}{\partial x} = 3y^2$.

Because the vector fields are conservative, then $\int\limits_C \vec{F} \cdot d\vec{r} = f(2,0) - f(0,0)$, where $\vec{\nabla} f = \vec{F}$.

Start $\int \frac{\partial f}{\partial y} dy = \int 3xy^2 + 1dy$ and we get $f = xy^3 + y + k(x)$.

Now $\frac{\partial f}{\partial x} = y^3 + k'(x) = y^3 + 1$.

So $\int k'(x) = \int 1dx$ and $k(x) = x + C$.

Therefore $f(x,y) = xy^3 + y + x + C$.

Work is $f(2,0) - f(0,0) = 2 - 0 = 2$.

If $\vec{F}$ is conservative, then $\vec{F}$ is independent of path. If you cannot remember how to find f, choose any path from $(0,0)$ to $(2,0)$.

For example, we could have done $\vec{r}(t) = \langle 2t, 0 \rangle, 0 \le t \le 1$, so work is $\int_0^1 \langle 1,1 \rangle \cdot \langle 2,0 \rangle dt = \int_0^1 2dt = 2$.

Options for $\int\limits_C \vec{F} \cdot d\vec{r}$ (work)

1. Parameterize Curve

   If $\vec{F}$ is conservative/independent of path

2. Find f (potential function)
3. Use any path from P to Q

Example

$\vec{F} = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$. Find f such that $\nabla f = \vec{F}$.

Let f_x, f_y, f_z be the terms in $\vec{F}$ in order.

So $\int f_x dx = \int y^2 dx$, or $f = xy^2 + k(y, z)$.

Now $f_y = 2xy + k_y(y, z) = 2xy + e^{3z}$.

Integrating gives $\int ky(y, z)dy = \int e^{3z} dy$ so $k(y, z) = ye^{3z} + k(z)$.

So right now we have $f = xy^2 + ye^{3z} + k(z)$.

So $f_z = 3ye^{3z} + k'(z) = 3ye^{3t}$ so $k'(z) = 0$.

Therefore $f(x, y, z) = xy^2 + ye^{3z} + C$

We can check by finding f_x, f_y, and f_z from this.

5.4 Green's Theorem

George Green (1793-1841) was a self-taught british mathematician and physicist.

Theorem 5.2: Green's Theorem

Let R be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve oriented counterclockwise.

If $f(x, y)$ and $g(x, y)$ are continuous and have continuous first partials on some open set containing R, then

$$\int_C f(x, y)dx + g(x, y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

Proof. We need to show $\int_C f(x, y)dx = \iint_R -\frac{\partial f}{\partial y} dA$

Let c_1 and c_2 be boundary curves. Then $\int_C f(X, y)dx = \int_{c_1} f(x, y)dx + \int_{c_2} f(x, y)dx$.

This is equivalent to $\int_C f(X, y)dx = \int_{c_1} f(x, y)dx - \int_{-c_2} f(x, y)dx$.

When we parameterize this, we get $c_1 : x = t, y = g_1(t)$ and $c_2 : x = t, y = g_2(t)$.

So the integral $\int_C f(x, y)dx = \int_a^b f(t, g_1(t))dt - \int_a^b f(t, g_2(t))dt$.

Then we have $-\int_a^b [f(t, g_2(t))dt - f(t, g_1(t))]dt$. This is equal to $-\int_a^b f(t, y)$ between the values $y = g_1(t)$ and $y = g_2(t)$ wrt t.

We can write this now as a double integral $-\int_a^b \int_{g_1(t)}^{g_2(t)} \frac{\partial f}{\partial y} dydx = -\iint_R \frac{\partial f}{\partial y} dA$. $\square$

Example

$\int\limits_C y^3dx + (x^3 + 3xy^2)dy$ where C is below. Also find $\int\limits_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle y^3, x^3 + 3xy^2 \rangle$.

C goes from $(0, 0)$ to $(1, 1)$ and goes from the line $y = x$ then the line $y = x^3$.

This is closed and counterclockwise, so we can use Green's Theorem.

This integral becomes $\iint\limits_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$.

So we have $\iint\limits_R (3x^2 + 3y^2) - 3y^2 dA = \int_0^1 \int_{x^3}^x 3x^2 dydx = \frac{1}{4}$.

For the other part, we need to do the line integral of $\int\limits_C \vec{F} \cdot d\vec{r}$.

$\int\limits_C \vec{F} \cdot d\vec{r} = \frac{1}{4}$, it is basically the same as above.

If we have a closed path, then $\oint\limits_C f(x, y)dx + g(x, y)dy = \iint\limits_R \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} dA$.

Sometimes we get the orientation, such as $\oint\limits_C$ or $\oint\limits_C$.

If $\vec{F}$ is conservative, then $\oint\limits_C \vec{F} \cdot d\vec{r} = 0$.

Another Application of Green's Theorem:

Area $= \iint\limits_R dA = \oint\limits_C xdy = \oint\limits_C -ydx = \frac{1}{2} \oint\limits_C -ydx + xdy$.

Example

Use a line integral to find the area of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

This is an ellipse.

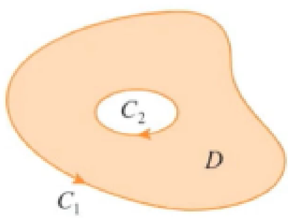
If we parameterize this, we get $x = a \cos t$ and $y = b \sin t$ for $0 \le t \le 2\pi$.

We have $A = \oint xdy = \int_0^{2\pi} a \cos t \cdot b \cos tdt$.

This is equal to $ab \int_0^{2\pi} \cos^2 tdt = ab \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2} \cos 2t \right) dt$.

This is equal to $A = \pi ab$.

For multiply-connected regions:



We want to calculate the line integral for this.

For the outside curve, we are finding $\oint\limits_{c_1} f(x, y)dx + g(x, y)dy$.

We have to go in and exit, so the inner region contributes nothing.

For the second curve, it is oriented clockwise so we have to reverse the orientation.

So we add $\oint\limits_{c_2} f(x,y)dx + g(x,y)dy$.

Using Green's Theorem, this becomes $\iint\limits_R \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} dA + \iint \frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} dA$.

Now we will see is the curve has negative orientation and if we can use Green's Theorem.

Example

Evaluate $\int\limits_C x^4 dx + xydx$ where C is the triangle curve consisting of line segments from $(0,0)$ to $(0,1)$, to $(1,0)$, and then back to $(0,0)$.

We need the curve to have the reverse orientation to use Green's Theorem.

So we can just do $-\iint\limits_{-c}(y-0)dA = -\iint\limits_R ydA$.

5.5 Curl and Divergence

Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$. Then curl $\vec{F} = \vec{\nabla} \times \vec{F}$.

Curl relates to the rotational properties of a fluid at a point.

Example

Find curl $\vec{F}$ for $\vec{F} = \langle xz, xyz, -y^2 \rangle$.

The cross product $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$.

This is equal to $\langle \frac{\partial}{\partial y} - \frac{\partial}{\partial z}(xyz), -[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)], \frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz) \rangle$.

This is equal to curl $\vec{F} = \langle -y(2+x), x, yz \rangle$.

Theorem 5.3

curl$(\nabla f) = \vec{0}$ (Note f must have continuous 2nd order partials)

Proof. curl$(\vec{\nabla} f) = \vec{\nabla} \times \vec{\nabla} f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$

This is equal to $\langle \frac{\partial^2 f}{\partial y} \partial z - \frac{\partial^2 f}{\partial z \partial y} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right), \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \rangle$.

Everything subtracts from each other, so $0\vec{i} + 0\vec{j} + 0\vec{k}$ is the result, or $\vec{0}$. □

We care about this because if $\vec{F}$ is a conservative vector field, then $\vec{\nabla} f = \vec{F}$.

So the theorem also says if $\vec{F}$ is conservative, then curl $\vec{F} = \vec{0}$.

Theorem 5.4

If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\vec{F} = \vec{0}$, then $\vec{F}$ is a conservative vector field.

Example

Show that $\vec{F}(x,y,z) = y^2z^3\vec{i} + 2xyz^3\vec{j} + 3xy^2z^2\vec{k}$ is conservative and find the potential function.

Before, we would use $\vec{F}(x,y) = P\vec{i} + Q\vec{j}$ and find the partials.

Now we can show in 3-space that curl $\vec{F} = \vec{0}$.

Remember that curl $\vec{F} = \vec{\nabla} \times \vec{F}$.

This is equal to $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$

This gives $\langle 6xyz^2 - 6xyz^2, -(3y^2z^2 - 3y^2z^2), 2yz^3 - 2yz^3 \rangle = \vec{0}$. $\vec{F}$ is a conservative vector field.

Start with $\int f_x dx = \int y^2z^3 dx$, so $f = xy^2z^3 + k(y,z)$.

Now we have $f_y = 2xyz^3 + k'(y,z) = 2xyz^3$, so then $\int k'(y,z) = \int 0$ so $k(y,z) = k(z)$.

Now we have $f = xy^2z^3 + k(z)$.

$f_z = 3xy^2z^2 + k'(z)$ and from this we can see that $k'(z) =$ a constant.

So our potential function is $f(x,y,z) = xy^2z^3 + k$.

Divergence: div $\vec{F} = \vec{\nabla} \cdot \vec{F}$.

Divergence relates to the way in which fluid flows away or towards a point.

Example

Find div $\vec{F}$ for $\vec{F} = \langle xz, xyz, -y^2 \rangle$.

So div $\vec{F} = \vec{\nabla} \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle xz, xy^2, -y^2 \rangle$.

This gives us div $\vec{F} = z + xz + 0$, so div $\vec{F} = z + xz$. This is called a scalar field.

Laplacian: ∇^2

$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

If applied to $f(x,y,z) \to \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

So, $\nabla^2 f =$ div$(\vec{\nabla} f)$.

Vector Forms of Green's Theorem:

Consider $\vec{F} = P\vec{i} + Q\vec{j}$. Suppose all conditions for Green's Teorem are satisfied. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy = \iint_D (Q_x - P_y) dA$$

curl $\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial t} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$.

So we can rewrite Green's theorem as:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl}\vec{F}) \cdot \vec{k} dA$$

This is equal to $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k} \cdot \vec{k} dA$ or $\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$.

5.6 Surface Integrals

Recall: $\int$, $\iint$, $\iiint$, $\int\limits_C ds$, or ($\int\limits_C d\vec{r}$ or $\oint\limits_C$ - Green's theorem)

The idea now is integrating over a smooth parametric surface.

Take $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$.

Then form Riemann sum: $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f\left(P_{ij}^*\right)\Delta S_{ij}$.

If we take the limit $\lim\limits_{m,n\to\infty}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f\left(P_{ij}^*\right)\Delta S_{ij}$ this becomes equal to $\iint\limits_S f(x,y,z)dS$.

The above expression represents the surface integral of f over S.

To evaluate:

1. If parameterize: $\iint\limits_S f(x,y,z)dS = \iint\limits_D f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|dA$. (Note $|\vec{r}_u \times \vec{r}_v|$ denotes the surface area of a surface S)
2. If rectangular: $\iint\limits_S f(x,y,z)dS = \iint\limits_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}dA$

Interpretation:

1. Mass
2. $\iint\limits_S dS$ is surface area
3. Mass/center of mass: $m = \iint\limits_S \rho(x,y,z)dS$ and $\overline{x} = \frac{1}{m}\iint\limits_S x\cdot\rho(x,y,z)dS$

Example

Evaluate $\iint\limits_S x^2 dS$ over the sphere $S: x^2+y^2+z^2=1$.

Let $x = \sin\phi cos\theta$, $y = \sin\phi\sin\theta$ and $z = cos\phi$.

So $\vec{r}(\phi,\theta)$.

So cross product $\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} =$

$\langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta\rangle$.

So we get $\langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\rangle$.

So the magnitude of this is $\sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi} = \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} = \sqrt{\sin^2\phi} = \sin\phi$.

Now we have $\iint\limits_S x^2 dS = \int_0^{2\pi}\int_0^{\pi}(\sin\phi\cos\theta)^2\cdot\sin\phi d\phi d\theta$.

This is equal to $\int_0^{2\pi}\cos^2\theta d\theta \cdot \int_0^{\pi}\sin^3\phi d\phi$.

The first integral gives $\int_0^{2\pi}\left[\frac{1}{2} + \frac{1}{2}\cos 2\theta\right]d\theta = \pi$.

The second integral becomes $\int_0^{\pi}\sin\phi(1-\cos^2\phi)d\phi = \int_0^{\pi}(\sin\phi - \sin\phi\cos^2\phi)d\phi = \frac{4}{3}$.

These are being multiplied, so $\pi\cdot\frac{4}{3} = \frac{4}{3}\pi$.

Example

Evaluate $\iint\limits_S (y^2 + 2ys)dS$ where S: first octant of $2x + y + 2z = 6$.

We will use the $\iint\limits_D f(x, y, g(x, y))$ to solve this.

We have $z = \frac{1}{2}(6 - 2x - y)$ and $z_x = -1, z_y = -\frac{1}{2}$. Then we have $\sqrt{1^2 + \frac{1}{2}^2 + 1} = \frac{3}{2}$.

The surface integral becomes $\iint\limits_D [y^2 + 2y(\frac{1}{2}(6 - 2x - y))] \cdot \frac{3}{2} dA$.

We end up getting $\frac{3}{2} \int_0^3 \int_0^{6-2x} (y^2 + 6y - 2xy - y^2)$.

This evaluates to $\frac{243}{2}$.

Surface Integrals of Vector Fields:

Main Applications:

1. Line integrals calculate work.
2. Surface integrals calculate flux.

Flow Fields - think about $\vec{F}(x, y, z)$ representing velocity of a fluid particle at the point (x, y, z).

Oriented Surface:

"orientable": surface has 2 sides (outside and inside)

"non-orientable": one side only

$\vec{n}$ is a unit vector at each point. $-\vec{n}$ is a unit normal vector in opposite direction. Unit normals will never have an abrupt change in direction.

Orientation of a smooth parametric surface is defined by $\vec{n}$.

$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ gives a unit vector with a positive orientation.

Example

Find the natural orientation for $\vec{r}(u, v) = \langle \cos u, v, -\sin u \rangle, 0 \leq u \leq 2\pi, 0 \leq v \leq 3$.

The cross product is $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin u & 0 & -\cos u \\ 0 & 1 & 0 \end{vmatrix} = \langle \cos u, 0, -\sin u \rangle$.

The magnitude $|\vec{r}_u \times \vec{r}_v| = 1$ and $\vec{n} = \langle \cos u, 0, -\sin u \rangle$.

When $u = 0, v = 0$, then $\vec{n} = \langle 1, 0, 0 \rangle$.

Outward is positive orientation at this point, and negative is inward.

Flux conditions:

- We will only deal with incompressible fluids (uniform density)
- We will assume that the velocity of a fluid at a fixed point does not vary with time (steady-state)

Definition: Flux

The volume of a fluid that passes through a surface in one unit of time.

This depends on

1. speed of fluid (greater speed gives greater volume)
2. how surface is positioned relative to flow. Maximum flux is when fluid is perpendicular to surface and no flux happens when nothing is passing through.
3. Area of surace (larger area gives more flux)

How do we calculate flux?

Flux $= \phi = \iint\limits_S \vec{F} \cdot d\vec{S} = \iint\limits_S \vec{F} \cdot \vec{n} dS = \iint\limits_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| dA$

Flux: $\phi = \iint\limits_S \cdot d\vec{S} = \iint\limits_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA$

Example

Find the flux of the vector field $\vec{F}(x,y,z) = z\vec{i} + y\vec{j} + x\vec{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Writing in terms of ϕ and θ, we have $r(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$.

Crossing gives $\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} = \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \rangle$.

So $\phi = \iint\limits_S \vec{F} \cdot d\vec{S} = \iint\limits_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA$

We have that $\vec{F}(\vec{r}(\phi,\theta)) \cdot (\vec{r}_\phi \times \vec{r}_\theta) = \cos\phi\sin^2\phi\cos\theta + \sin^3\phi\sin^2\theta + \sin^2\phi\cos\phi\cos\theta$.

Flux is $\int_0^{2\pi}\int_0^{\pi} (\cos\phi\sin^2\phi\cos\theta + \sin^3\phi\sin^2\theta + \sin^2\phi\cos\phi\cos\theta) d\phi d\theta$.

We can pull out some things and get $2\int_0^{2\pi} \sin^2\phi\cos\phi d\phi \cdot \int_0^{2\pi} \cos\theta + \int_0^{\pi} \sin^3\phi d\phi \cdot \int_0^{2\pi} \sin^2\theta d\theta$.

Simplifying this gives $\int_0^{\pi} \sin^3\phi d\phi \cdot \int_0^{2\pi} \sin^3\theta d\theta = \frac{4\pi}{3}$.

Now for Non-Parametric Surfaces:

Suppose $z = g(x,y)$ is a surface oriented upward. Rewrite this function as $z - g(x,y) = 0$.

For $\vec{n}$, use $\vec{n} = \frac{\langle -z_x - z_y, 1 \rangle}{|\sqrt{z_x^2 + z_y^2 + 1}|}$. If we call $z - g(x,y) = 0$ as $G(x,y,z)$, then $\vec{n} = \frac{\vec{\nabla} G}{|\vec{\nabla} G|}$.

So, if oriented upwards, $\iint\limits_S \vec{F} \cdot \langle -z_x, -z_y, 1 \rangle dA$. Downwards, we get $\iint\limits_S \vec{F} \cdot \langle z_x, z_y, -1 \rangle dA$.

Example

Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle y, x, z \rangle$ and S is the boundary of the solid region E enclosed by $z = 1 - x^2 - y^2$ and $z = 0$. Assume outward orientation.

When we draw this, we can call the upwards orientation s_1 and the downwards orientation s_2.

For s_1: $\iint_{s_1} \vec{F} \cdot d\vec{s}$.

This is $\iint_D \langle y, x, 1 - x^2 - y^2 \rangle \cdot \langle 2x, 2y, 1 \rangle dA = \iint_D (2xy + 2xy + 1 - x^2 - y^2) dA$.

This is equal to $\iint_D (1 + 4xy - x^2 - y^2) dA$. Polar will make this integral easier.

The integral becomes $\int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos\theta \sin\theta - r^2) \cdot dr d\theta = \frac{\pi}{2}$.

For s_2, we have $z = 0$.

So we can rewrite $\iint_{s_2} \vec{F} \cdot d\vec{s} = \iint_D \langle y, x, 0 \rangle \cdot \langle 0, 0, -1 \rangle dA$.

This is $\iint_D 0 dA = 0$.

So, the flux is $\frac{\pi}{2}$.

5.7 Stokes' Theorem

Recall: $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$.

Green's theorem applies for closed curves C on the xy-plane.

We previously talked about writing Green's Theorem as $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\vec{F} \cdot \vec{K} dA$.

We could do this because $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle \cdot \langle 0, 0, 1 \rangle$.

So, Green's theorem can be tied to curl.

Recall the right-hand rule, the surface on the left is the positive orientation.

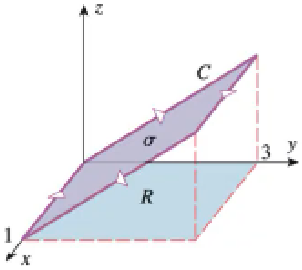
Theorem 5.5: Stokes' Theorem

Let S be an oriented piecewise smooth surface bounded by a curve C with positive orientation.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\vec{F} \cdot d\vec{s}$$

Example

$\vec{F}(x,y,z) = \langle x^2, 4xy^3, xy^2\rangle$. Find the work done by $\vec{F}$ on a partial traversing $z = y$ as shown.



Previously, we could have used Green's theorem if the curve was on the xy-plane.

We could have also parameterized the 4 lines and set up 4 line integrals.

Using Stokes' Theorem, we must ask if the surface is smooth (yes), do we have a curve with positive orientation (yes if $\vec{n}$ is down).

So we have $W = \int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\vec{F} \cdot d\vec{s}$.

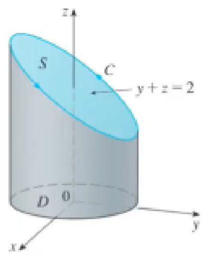
Finding curl$\vec{F} = \langle 2xy, -y^2, 4y^3\rangle$.

To find $d\vec{s}$, we find $\langle z_x, z_y, -1\rangle = \langle 0, 1, -1\rangle$.

So the integral becomes $W = \iint_R \langle 2xy, -y^2, 4y^3\rangle \cdot \langle 0,1,-1\rangle dA = \int_0^1 \int_0^3 (-y^2 - 4y^3) dy dx = -90$.

Example

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y,z) = -y^2\vec{i} + x\vec{j} + z^2\vec{k}$ and C is the curve of intersection of $y + z = 2$ and $x^2 + y^2 = 1$. Orient C to be counterclockwise when viewed from above.



The surface is smooth, and the curve c has positive orientation so Stokes' theorem can be used.

The curl$\vec{F}$ is $\langle 0, 0, 1 + 2y\rangle$.

$y + z = 2$ gives $z = 2 - y$, and since $\vec{n}$ is up, we have $\langle -z_x, -z_y, 1\rangle = \langle 0, 1, 1\rangle$.

So the line integral is $\int_C \vec{F} \cdot d\vec{r} = \iint_D \langle 0, 0, 1 + 2y\rangle \cdot \langle 0, 1, 1\rangle dA$.

This is $\iint_D (1 + 2y) dA$. Since the region is a circle, polar is a good option.

So we have $\int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta) \cdot r dr d\theta = \pi$

$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ is a special case of Stokes' Theorem.

Example

Verify Stokes' Theorem for $\vec{F}(x,y,z) = \langle 2z, 3x, 5y\rangle$ where $S : 4 - x^2 - y^2, z \geq 0$, upward orientation and $C : x^2 + y^2 = 4$ on xy-plane (positively oriented).

We start with $\vec{r} = \langle 2\cos t, 2\sin t, 0\rangle$.

We use $\int_C \vec{F} \cdot d\vec{r} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r'}(t)dt$.

So the integral is $\int_0^{2\pi} 12\cos^2 tdt$ after the dot product and this is equal to 12π.

Using $\iint_S$ curl$\vec{F} \cdot d\vec{s}$ should give us similar answer.

The curl$\vec{F}$ is $\langle 5, 2, 3\rangle$, and S has upward orientation, so $\langle -z_x, -z_y, 1\rangle = \langle 2x, 2y, 1\rangle$.

So the integral is $\iint_R \langle 5, 2, 3\rangle \cdot \langle 2x, 2y, 1\rangle = \iint_R (10x + 4y + 3)dA$.

The integral is $\int_0^{2\pi} \int_0^2 (10r\cos\theta + 4r\sin\theta + 3)rdrd\theta = 12\pi$.

Example

$\iint_S$ curl$\vec{F} \cdot d\vec{S}$ for $\vec{F} = \langle xz, yz, xy\rangle$ where S is the part of sphere $x^2 + y^2 + z^2 = 4$ that lies inside $x^2 + y^2 =$ above the xy-plane.

We can use a line integral for this.

We need to find c first:

$x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$, so $1 + z^2 = 4$, so we can get $z = \sqrt{3}$ from this.

So $C : x^2 + y^2 = 1, z = \sqrt{3}$.

We have $\vec{r} = \langle \cos t, \sin t, \sqrt{3}\rangle$, so $\vec{r'}(t) = \langle -\sin t, \cos t, 0\rangle$.

Therefore $\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \cos t\sin t\rangle \cdot \langle -\sin t, \cos t, 0\rangle dt$.

This is equal to $\int_0^{2\pi} (-\sqrt{3}\cos t\sin t + \sqrt{3}\sin t\cos t)dt = 0$.

5.8 The Divergence Theorem

Recall that given $\vec{F}(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z)\rangle$, that div$\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$ (fluid flow)

Theorem 5.6: Divergence Theorem

Let E be a solid (closed) and let S be the boundary surface of E, given with positive (outward) orientation. Then

$$\phi = \iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\vec{F} dV$$

This is like using the potential function to calculate work when you have a conservative vector field.

Example

Find the flux of the vector field $\vec{F}(x,y,z) = z\vec{i} + y\vec{j} + x\vec{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$. We will assume outward orientation.

It is closed and div$\vec{F} = 0 + 1 + 0 = 1$.

The flux is $\iint\limits_S \vec{F} \cdot d\vec{S} = \iiint\limits_E 1 \cdot dV$.

E is unit sphere $x^2 + y^2 + z^2 \le 1$ and S is boundary $x^2 + y^2 + z^2 = 1$.

The options are set up bounds (spherical coordinates) or to find the volume with a triple integral noting the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

The flux is $\frac{4}{3}\pi$.

Example

Find the (outward) flux of the vector field $\vec{F} = \langle x, y^2, z \rangle$ across S: tetrahedron bounded by $2x+2y+z = 6$ in the 1st octant.

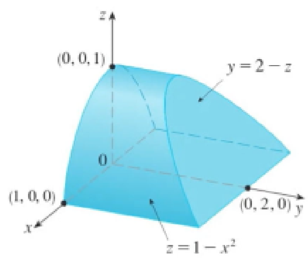
Divergence theorem can be used because the solid is closed. div$\vec{F} = 2 + 2y$.

Flux is $\int_0^3 \int_0^{3-x} \int_0^{6-2x-2y} (2+2y) dz dy dx$.

The answer of the integral is 31.5.

Example

Evaluate $\iint\limits_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle xy, y^2 + e^{xz^2}, \sin(xy) \rangle$ and S is the surface of the region bounded by $z = 1 - x^2, z = 0, y = 0$, and $y + z = 2$.



We get div$\vec{F} = 3y$.

The flux is $\phi = \iint\limits_S \vec{F} \cdot d\vec{S} = \int_{-1}^1 \int_0^{1-x^2} \int_0^{2-z} 3y dy dz dx$.

The answer is $\frac{184}{35}$.

Sources and Sinks

A source is a point where fluid enters flow. (div$\vec{F}(p_0) > 0$)

A sink is a point where fluid leaves flow. (div$\vec{F}(p_0) < 0$)

Example

Determine if $\vec{F} = (y+z)\vec{i} - xz^3\vec{j} + (x^2 \sin y)\vec{k}$ is free of sources and sinks. If not, locate them.

We have div$\vec{F} = 0$, so there are no sources and no sinks.

Example

Find all sources and sinks for $\vec{F}(x, y, z) = \langle xy, -xy, y^2 \rangle$.

The divergence of $\vec{F}$ is div$\vec{F} = y - x$.

Sources will happen when $y - x > 0$ and sinks will happen when $y - x < 0$.

So sources at all points where $x < y$ and there are sinks at all points where $x > y$.