1 Partial Derivatives

1.1 Functions of Several Variables

Before: $f(x)$ is a function in terms of x (one variable). An example of this is $y = 4x^2$.

Now: $z = f(x, y)$ (a function of 2 variables). An example is $A = \frac{1}{2}bh$. This is a function $f(b, h) = \frac{1}{2}bh$.

For $z = (x, y)$, z is the dependent variable and x and y are the independent variables.

In 3-space, this becomes $w = f(x, y, z)$.

Domain: The restrictions of the independent variables determine the domain of f.

Example

Find the domain of $f(x, y) = \ln(x, y)$.

$xy > 0$.

If both x and y are positive and x and y are negative, the product will be greater than zero.

This can be expressed as D : all ordered pairs in quadrants I and III. (not on axis)

Or written mathematically as $D : (x, y) : xy > 0$.

Example

Find the domain of $f(x, y, z) = \frac{x}{\sqrt{9-x^2-y^2-z^2}}$.

We know the quantity $9 - x^2 - y^2 - z^2 > 0$.

We can rewrite this as $x^2 + y^2 + z^2 < 9$.

The domain is D : all $(x, y, z) : x^2 + y^2 + z^2 < 9$.

This is also known as the set of all (x, y, z) inside sphere of $r = 3$ centered at the origin.

$z = f(x, y)$ is a surface in 3-space.

Example

$f(x, y) = \sqrt{4 - x^2 - y^2}$ graphed.

$f(x, y)$ is essentially z and $z \geq 0$.

Simplifying will give you $x^2 + y^2 + z^2 = 4$, which is a sphere $r = 2$ centered at the origin.

Exercise Graph $f(x, y) = 1 - x - \frac{1}{2}y$.

Definition

The level curves of a function f of two variables are the curves with equations $f(x, y) = k$ where k is a constant (in the range of f.)

Example

Describe the level curves of $z = x^2 + y^2$.

Remember from the first chapter, this will end up being a paraboloid.

Passing various planes through or making z a number gives us an idea that level curves are circles centered at the origin.

Contour plots contain sets of level curves for a function.

If we were to graph the set of level curves (the contour plot) of this equation, we would have circles of varying radii in the xy-plane.

If z was getting bigger, it would be going up, if z was getting smaller, it would be going downwards.

Exercise Sketch the contour plot of $f(x, y) = x^2 - 4y^2$.

Functions of 2 variables are surfaces we can project as level curves.

Functions of 3 variables are 4-D graphs that we can project as level surfaces.

Example

Describe the level surfaces of $f(s, y, z) = x^2 + y^2 + z^2$.

Let $w = x^2 + y^2 + z^2$.

We can write like $1 = x^2 + y^2 + z^2$, $4 = w$, $9 = w$, etc.

This is a graph in 3-space and the level surfaces are spheres centered at the origing. As distance from the origin increases, so will the value of f.

Exercise Graph the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Exercise Graph the hyperbola $y^2 - \frac{x^2}{4} = 1$

1.2 Limits and Continuity

In 2-space, $\lim\limits_{x \to x_0^+ f(x)} f(x) = \lim\limits_{x \to x_0^- f(x)} f(x)$.

In 3-space, we evaluate $\lim\limits_{(x,y) \to (x_0, y_0)} f(x, y)$.

Questions to consider:

1. Is there a point there or are we approaching a point?
2. How many directions/paths of approach do we have? Infinite.

Example

Consider $f(x, y) = \frac{-xy}{x^2+y^2}$. Find $\lim\limits_{(x,y) \to (2,1)} f(x, y)$.

$f(2, 1) = -\frac{2}{5}$.

This is the limit, since there is no funky behavior with this limit.

Example

Consider $f(x,y) = \frac{-xy}{x^2+y^2}$. Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

We can consider $\lim\limits_{x\to 0} = 0$ and this is along the x-axis.

0 is the limit from one direction.

Along the y-axis, we get the same limit $\lim\limits_{y\to 0} = 0$.

Along the line $y = x$, the limit is $\lim\limits_{(x,y)\to(0,0)} \frac{-x\cdot x}{x^2+x^2} = -\frac{1}{2}$.

It seems that the limit does not exist.

This process is inefficient and impractical.

Definition

Let f be a function of 2 variables and assume f is defined at all points of some open disk centered at (x_0, y_0) (except maybe at (x_0, y_0)). Then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if given $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x,y) - L| < \epsilon$ when the distance between (x,y) and (x_0, y_0) satisfies $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

Theorem 1.1

If $f(x,y) \to L$ as $f(x,y) \to (x_0, y_0)$, then $f(x,y) \to L$ as $(x,y) \to (x_0, y_0)$ along any smooth curve.

If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist along some smooth curve or if $f(x,y)$ has different values along different curves, the limit does not exist.

Options:

1. Plug in values.
2. Limit DNE $\to$ need to show that there is a path where DNE.
3. One other option with discontinuity "trick".

Example

Find $\lim\limits_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2}$.

Plug in numbers to get 2.

Example

Find $\lim\limits_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$.

The limit may not exist by plugging in $(0,0)$.

Start with the path along $x = 0$ and we get the limit $\lim\limits_{y\to 0} = 1$.

Along $y = x$, we get $\lim\limits_{x\to 0} = 0$.

Since these are two different limits, then the limit does not exist.

To prove that the limit does exist, you essentially have to be able to calculate it.

Definition

A function $f(x, y)$ is said to be continuous at (x_0, y_0) if $f(x_0, y_0)$ is defined and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

1. If f is continuous on D, this means f is continuous at every point in an open set.
2. If f is continuous everywhere, this means f is continuous at every point in the xy-plane.

Theorem 1.2

$f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0) if $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0.

Compositions are continuous $(f(x, y) = g(h(x, y)))$ if $h(x, y)$ is continuous at (x_0, y_0) and $g(u)$ is continuous at $u = h(x_0, y_0)$.

Example

(a) Determine continuity for $f(x, y) = 3x^2y^5$.

$g(x) = 3x^2$ and $h(y) = y^5$.

$g(x)$ and $h(y)$ are continuous everywhere, therefore $f(x, y)$ is continuous everywhere.

(b) Determine continuity for $f(x, y) = \sin(3x^2y^5)$.

We already know the function inside is continuous everywhere.

Therefore $\sin(u)$ is continuous everywhere, therefore $f(x, y)$ is continuous everywhere.

1.3 Partial Derivatives

$f(x, y)$ is a function of 2 variables x and y. What happens to x when we hold y constant?

Definition

If $z = f(x, y)$, then the first partial derivatives of f with respect to x and y are f_x and f_y, defined by $f_x(x, y) = \lim\limits_{h\to 0} \frac{f(x+h),y-f(x,y)}{h}$ and $f_y(x, y) = \lim\limits_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$ provided the limits exist.

Example

Find the first partial derivatives f_x and f_y for $f(x, y) = 3x - x^2y^2 + 2x^3y$.

For $f_x(x, y)$ we just treat y as a constant.

So we get $f_x(x, y) = 3 - 2xy^2 + 6x^2y$.

For $f_y(x, y)$ we treat x as a constant.

So $f_y(x, y) = -2x^2y + 2x^3$.

Different notations for partial derivatives are like $f_x = \frac{\partial f}{\partial x}$. If $z = f(x, y)$ then this is also equal to $\frac{\partial z}{\partial x} = z_x$.

Example

Consider $f(x, y) = xe^{x^2y}$. Find f_x and f_y and evaluate both at $(1, \ln 2)$.

$f_x = e^{x^2y} + 2x^2ye^{x^2y}$

$f_y = x^3e^{x^2y}$

Evaluating both of these at the point $(1, \ln 2)$:

$f_x(1, \ln 2) = e^{\ln 2} + 2(\ln 2)e^{\ln 2} = 2 + 4\ln 2$

$f_y(1, \ln 2) = 1 \cdot e^{\ln 2} = 2$

Geometrically $\frac{\partial f}{\partial x}$ is the slope in the x-direction. It tells us how f (or z) changes with respect to x when y is constant.

Example

Find slopes in x and y directions at $(1/2, 1)$ for $z = f(x, y) = -\frac{1}{2}x^2 - y^2 + \frac{25}{8}$. Then interpret these slopes.

$f_x = -x$

$f_y = -2y$

$f_x(1/2, 1) = -\frac{1}{2}$

$f_y(1/2, 1) = -2$

f_x is $\frac{\Delta z}{\Delta x}$ so this tells us z decreases 1 unit for every 2 unit increase in x at this point.

f_y is $\frac{\Delta z}{\Delta y}$ so this tells us that z decreases 2 units for every unit increase in y at this point.

Functions of 3 or more variables have a similar idea. You just hold the other variables constant.

Notation for higher-order partial derivatives. If we did the partial derivative of x and then the partial derivative of x once more, we would get $\frac{\partial^2 f}{\partial x^2} = f_{xx}$. If we did the partial derivative of y after the partial derivative of x we would get f_{xy}.

Example

Find f_{xx}, f_{yy}, f_{xy} and f_{yx} for $f(x, y) = 3xy^2 - 2y + 5x^2y^2$.

$f_x = 3y^2 + 10xy^2$, so $f_{xx} = 10y^2$ and $f_{xy} = 6y + 20xy$

$f_y = 6xy - 2 + 10x^2y$ so $f_{yy} = 6x + 10x^2$ and $f_{yx} = 6y + 20xy$

You will notice that $f_{xy} = f_{yx}$ in this case.

Theorem 1.3

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then for every (x, y) in r, $f_{xy}(x, y) = f_{yx}(x, y)$.

Example

Suppose $f(x, y, z) = ye^x + x \ln z$. Show that $f_{xzz} = f_{zxz} = f_{zzx}$.

$f_x = ye^x + \ln z$, so $f_{xz} = 1/z$ and $f_{xzz} = -\frac{1}{z^2}$.

$f_z = \frac{x}{z}$, so $f_{zx} = \frac{1}{z}$ and $f_{zxz} = -\frac{1}{z^2}$.

$f_z = \frac{x}{z}$ and $f_{zz} = -\frac{x}{z^2}$ and $f_{zzx} = -\frac{1}{z}^2$.

These three are equal to each other so this shows that order does not matter.

Example

Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the y-direction at $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

The goal is to find $\frac{\partial z}{\partial y} = z_y$ at the point given.

Let's start by differentiating by y.

We get $\frac{\partial}{\partial y} = \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(z^2) = \frac{\partial}{\partial y}(1)$.

This is $2y + 2z\frac{\partial z}{\partial y} = 0$.

So $\frac{\partial z}{\partial y} = -\frac{y}{z}$.

At the point given, this evaluates to $-\frac{1}{2}$.

1.4 Tangent Planes and Linear Approximations

Previously, if we consider the graph of 2-D differentiable function, when we zoom in on the graph a lot, the graph begins to look like its tangent line. We can approximate the value of the function at a specific point using this tangent line.

If we consider the graph of a 3-D differentiable function, when we zoom in on the graph a lot, the graph begins to look like its tangent plane. How do we find the equation of this tangent plane? If we find 2 tangent lines, then this tangent plane will contain both lines.

We choose 2 lines that result when they intersect $z = f(x, y)$ with the planes $y = y_0$ and $x = x_0$.

If we recall the equation of a plane: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ where (x_0, y_0, z_0) is a point on this plane, then $z - z_0 = -\frac{A}{C}(x - x_0) - \frac{B}{C}(y - y_0)$.

Let's call $-\frac{A}{C} = a$ and $-\frac{B}{C} = b$. So we get $z - z_0 = a(x - x_0) + b(y - y_0)$. This is the general form of a plane.

Now we use the 2 tangent lines that result from the planes $y = y_0$ and $x = x_0$.

The first plane results $T_1 : y = y_0$. We get $z - z_0 = a(x - x_0)$. We are looking at a line in point slope form with slope a. a is f_x.

Likewise, the second tangent line is $T_2 : x = x_0$. We get $z - z_0 = b(y - y_0)$. Similar idea, the slope b is f_y.

This leads us to the equation of the tangent plane as

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example

Find an equation of the tangent plane $z = 2x^2 + y^2$ at $(1, 1, 3)$.

Recall this equation makes an elliptic paraboloid.

$f_x = 4x$, so $f_x(1, 1) = 4$.

$f_y = 2y$ so $f_y(1, 1) = 2$.

What this gives us is $z - 3 = 4(x - 1) + 2(y - 1)$.

So $z = 4x + 2y - 3$.

In the above example, the linear function $L(x, y) = 4x + 2y - 3$ is a good approximation of $f(x, y)$ when (x, y) is near $(1, 1)$.

For example, $L(1.1, 0.95) = 3.3$. If we find $f(1.1, 0.95)$ this is equal to 3.3225.

L is called the linearization of f at $(1, 1)$.

The approximation is called the linear approximation or tangent line approximation of f at $(1, 1)$.

Recalling the equation of a tangent plane: $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

So the linearization of f at (a, b) is $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

And the linear approximation $f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.

But only if $f(x, y)$ is differentiable at (a, b).

Definition

If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form $\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ where $\epsilon_1, \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Theorem 1.4

If f_x, f_y exist near (a, b) are are continuous at (a, b), then f is differentiable at (a, b). For functions of $3+$ variables it is similar.

Example

Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization. Then use it to approximate $f(1.1, -0.1)$.

$f_x(x, y) = e^{xy} + xye^{xy}$

$f_y(x, y) = x^2 e^{xy}$

These are both continuous at $(1, 0)$ and exist near $(1, 0)$. Therefore $f(x, y)$ is differentiable at $(1, 0)$.

$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$

$L(x, y) = 1 + 1(x - 1) + 1(y - 0) = x + y$.

$f(x, y) = xe^{xy} \approx x + y$.

$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$

In 2-space, $dy = f'(x)dx$.

Δy is the actual change in y from $x = a$ to $x = a + \Delta x$ and dy is the change in y when using tangent line.

When Δx is small, dy can be used to approximate Δy.

This is a similar idea in 3-space.

Total differential: $dz = f_x(x,y)dx + f_y(x,y)dy$ OR $dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)$.

dx and dy are independent variables so these can be any numbers.

We take $dx = \Delta x = x - a$ so the linear approximation becomes $f(x,y) \approx f(a,b) + dz$

Example

Consider $z = xy^2$.

(a) Find the total differential.

$f_x = y^2$ and $f_y = 2xy$ so $dz = y^2dx + 2xydy$.

(b) Approximate the change in z from $(0.5, 1.0)$ to $(0.503, 1.004)$.

$dz = (1)^2(0.503 - 0.5) + 2(0.5)(1)(1.004 - 1) = 0.007$

(c) Compare the approximation to the actual change.

$\Delta z = f(0.503, 1.004) - f(0.5, 1.0) = (0.503)(1.004)^2 - (0.5)(1)^2 = 0.007032\ldots$

Example

The base radius and height of a right circular cone are 10 cm and 25 cm with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

$V = \frac{1}{3}\pi r^2 h$.

So $V_r = \frac{2}{3}\pi rh$ and $V_h = \frac{1}{3}\pi r^2$

$dv = \frac{2}{3}\pi rhdr + \frac{1}{3}\pi r^2 dh$

The $|\Delta r| \leq 0.1$ and $|\Delta h| \leq 0.1$.

$dv = 20\pi \approx 63$ cm^3 from the formula.

This may over/underestimate the volume by as much as 63 cm^3.

Example

The dimensions of a rectangular box are 75 cm, 60 cm, and 40 cm. Each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated.

$V = xyz$, so $v_x = yz$, $v_y = xz$, and $v_z = xy$.

$dv = (yz)dx + (xz)dy + (xy)dz$, so $dv = 1980$ cm^3.

The volume of the box is 180000 cm^3, so the error is $\frac{1980}{180000} = 1.1\%$.

1.5 The Chain Rule

In previous the past, you were given $y(x)$ and $x(t)$. If you wanted to find $\frac{dy}{dx}$ for $y(x(t))$, you used the formula $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$.

Now we are looking at two variables, such as $z = f(x,y)$ where $x = x(t)$ and $y = y(t)$. Then the composition $z = f(x(t), y(t))$ expresses z as a single variable.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example

Consider $z = x^2y - y^2$ where $x = \sin t$ and $y = e^t$. Find $\frac{dz}{dt}$ at $t = 0$.

$\frac{dz}{dt} = (2xy)(\cos t) + (x^2 - 2y)(e^t)$.

Replacing with what was defined gives $\frac{dz}{t} = 2\sin t\cos te^t + (\sin^2 t - 2e^t)e^t$.

At $t = 0$, this is -2.

Without the chain rule, we would have to do $z(t) = (\sin^2 t)e^t - e^{2t}$. Then the derivative of this would be $2\sin t\cos te^t + e^t\sin^2 t - 2e^{2t}$.

Suppose that u is a differentiable function of n variables, $x_1, x_2, \ldots, x_n$, and each x_i is a differentiable function of m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

If $z = f(x(t), y(t))$, then we can see that we can do partial derivatives to x and y, but non partial derivatives to get to t. So $\frac{dz}{dt} = \frac{\partial t}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$.

Example

Consider $z = f(x, y)$ where $x = x(u, v)$ and $y = y(u, v)$. Write the chain rule for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

So we need partial derviatives to go from z to x to u and z to x to v, and likewise for y, we see we get that $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$.

Similarly, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$.

Example

Consider $z = 2xy$ where $x = u^2 + v^2$ and $y = \frac{u}{v}$. Find $\frac{\partial z}{\partial u}$.

So $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$.

This is equal to $(2y)(2u) + (2x)\left(\frac{1}{v}\right)$.

We want this in terms of u.

Substitute to get $\frac{\partial z}{\partial u} = \left(2 \cdot \frac{u}{v}\right)(2u) + (2 \cdot (u^2 + v^2))\left(\frac{1}{v}\right) = \frac{4u^2}{v} + \frac{2u^2+2v^2}{v} = \frac{6u^2+2v^2}{v}$.

Example

Find a rule for $\frac{dw}{dx}$ if $w = xy + yz$, $y = \sin x$ and $z = e^x$.

$\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial w}{\partial t} \cdot \frac{\partial z}{\partial x}$.

This is $\frac{dw}{dx} = (y) + (x + z)(\cos x) + (y)(e^x)$.

Now everything needs to be in terms of x.

$\frac{dw}{dx} = \sin x + (x + e^x)\cos x + \sin xe^x$

Example

Scenario: Consider $y^3 + y^2 - 5y - x^2 + 4 = 0$.

Previously, we would use implicit differentiation.

$3y^2 \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$.

Get $\frac{dy}{dx} = \frac{2x}{3y^2+2y-5}$.

With partial derivatives: $f(x, y) = C$.

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$.

Isolate $\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$

$\frac{dy}{dx} = \frac{-(-2x)}{3y^2+2y-5}$.

Example

Consider $x^2 + y^2 + z^2 = 1$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$f(x, y, z) = x^2 + y^2 + z^2 = C = 1$

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$

$\frac{\partial z}{\partial x} = \frac{-\partial f/\partial x}{\partial f/\partial z}$

Likewise, $\frac{\partial z}{\partial y} = \frac{-\partial f/\partial y}{\partial f/\partial z}$

And we get $\frac{\partial z}{\partial x} = -\frac{x}{z}$.

And $\frac{\partial z}{\partial y} = -\frac{y}{z}$.

Implicitly we could use $2x\frac{\partial x}{\partial y} + 2y\frac{\partial y}{\partial y} + 2z\frac{\partial z}{\partial y} = 0$ and get $2y + 2z\frac{\partial z}{\partial y} = 0$ to get $\frac{\partial z}{\partial y} = -\frac{y}{z}$.

1.6 Directional Derivatives and the Gradient Vector

Here's what we can do with partial derivatives:

1. Find instantaneous rates of change in 2 directions.
2. Find the slope of a tangent line parallel to the x- or y-axis.

What if we want to go in other directions?

Plan: Have some point $P(x_0, y_0, z_0)$ and $\vec{u} = \langle a, b \rangle$.

- Pass plane through surface and that plane is vertically passing through $\vec{u}$
- The slope of the tangent line is rate of change of z in the direction of $\vec{u}$
- Choose another point Q that is on the surface and plane. $Q(x, y, z)$
- Project P and Q onto the xy-plane. $P'(x_0, y_0, 0)$ and $Q'(x, y, 0)$
- $\vec{P'Q'}$ will be parallel to $\vec{u}$, so $\vec{P'Q'} = h\vec{u} = \langle ha, hb \rangle$.
- $x = x_0 + ha$, $y = y_0 + hb$ so $\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0+ha, y_0+hb) - f(x_0, y_0)}{h}$

Take $\lim\limits_{h \to 0}$ to obtain rate of change of z in direction of $\vec{u}$.

Definition: Directional Derivative

The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b\rangle$ is:

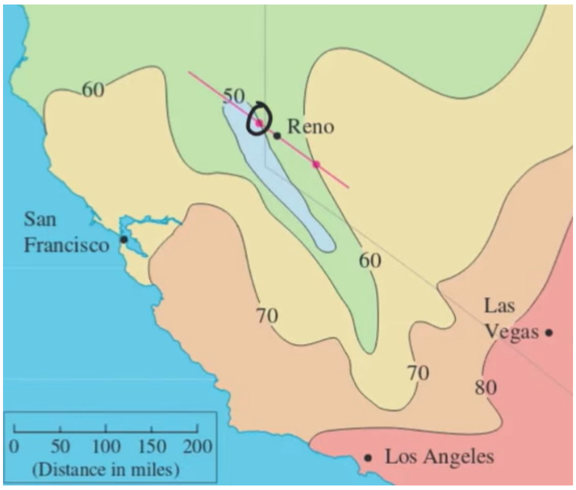
$$D_{\vec{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

The directional derivative tells rate of change of z in direction of $\vec{u}$.

Example

Use the weather map in the below figure to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.



The change in temperature over the change in miles is $\frac{60-50}{75} = \frac{2}{15}$ °F/mi.

Theorem 1.5

The directional derivative of f in the direction of a unit vector is: $D_{\vec{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$. (Similar idea in 3-space)

Example

Find $D_{\vec{u}}(1, -2, 0)$ for $f(x, y, z) = x^2y - yz^3 + z$ in the direction of $\vec{a} = \langle 2, 1, -2\rangle$.

$|\vec{a}| = \sqrt{4 + 1 + 4} = 3$ so $\vec{u} = \frac{\vec{a}}{|\vec{a}|} = \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\rangle$

$f_x = 2xy$, $f_y = x^2 - z^3$ and $f_z = -3yz^2 + 1$

$f_x(1, -2, 0) = -4$

$f_y(1, -2, 0) = 1$

$f_z(1, -2, 0) = 1$

$D_{\vec{u}} f(1, -2, 0) = -4\left(\frac{2}{3}\right) + 1\left(\frac{1}{3}\right) + 1\left(-\frac{2}{3}\right) = -3$

What does this value mean?

For every unit travelled in direction of $\vec{a}$, $f(x, y, z)$ decreases by 3 units.

Example

Find $D_{\vec{u}}f(x,y)$ if $f(x,y) = x^2e^{-y}$ and $\vec{u}$ is in the unit vector given by $\theta = 2\pi/3$. What is $D_{\vec{u}}f(1,0)$?

$f_x = 2xe^{-y}$ and $f_y = -x^2e^{-y}$

$\vec{u} = \langle \cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}\rangle = \langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$.

$D_{\vec{u}}f(x,y) = 2xe^{-y} \cdot -\frac{1}{2} + (-x^2e^{-y})\left(\frac{\sqrt{3}}{2}\right)$

$D_{\vec{u}}f(x,y) = -xe^{-y} - \frac{\sqrt{3}x^2e^{-y}}{2}$

$D_{\vec{u}}f(1,0) = -1 - \frac{\sqrt{3}}{2} \approx -1.866$

Earlier: $D_{\vec{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$

This can be reexpressed as $D_{\vec{u}}f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a,b\rangle = \langle f_x(x,y), f_y(x,y)\rangle \cdot \vec{u}$.

The gradient is $\langle f_x(x,y), f_y(x,y)\rangle$.

Definition

If f is a function of x and y, then the gradient is defined by:

$$\nabla f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j} = \langle f_x(x,y), f_y(x,y)\rangle$$

So $D_{\vec{u}}f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{u}$.

Example

If $f(x,y,z) = x\sin yz$ find:

(a) the gradient of f

$\vec{\nabla} f = \langle \sin yz, xz\cos yz, xy\cos yz\rangle$

(b) $D_{\vec{u}}f(1,3,0)$ if $\vec{u} = \langle 1,2,-1\rangle$

$|\vec{u}| = \sqrt{6}$ so $\frac{\vec{u}}{|\vec{u}|} = \langle -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\rangle$

$D_{\vec{u}}f(1,3,0) = \vec{\nabla} f(1,3,0) \cdot \langle -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\rangle = -\frac{3}{\sqrt{6}} = -\frac{\sqrt{6}}{2}$.

Theorem 1.6

Suppose f is a differentiable function of 2 or 3 variables. The maximum value (slope) of $D_{\vec{u}}f$ is $\|\nabla f\|$ and it occurs in the direction of the gradient. The minimum slope is $-\|\nabla f\|$.

Another fun fact: If $\nabla f(x,y) = \vec{0}$, then $D_{\vec{u}}f(x,y) = 0$ in all directions at the point (x,y).

Example

Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = \frac{80}{1+x^2+2y^2+3z^2}$ where T is measured in °C and x, y, and z are measured in meters. In wh ich direction does the temperature increase the fastest at the point $(1, 1, -2)$. What is the maximum rate of increase?

We have $\vec{\nabla}T = \langle T_x, T_y, T_z\rangle$.

This is $\vec{\nabla}T = \frac{160}{(1+x^2+2y^2+3z^2)^2}\langle -x, -2y, -3z\rangle$.

$\vec{\nabla}T(1, 1, -2) = \frac{160}{256}\langle -1, -2, 6\rangle$.

Direction: $\frac{5}{8}(-\vec{i} - 2\vec{j} + 6\vec{k})$

The maximum rate of increase is the length of the gradient.

This is $|\vec{\nabla}T = \frac{5}{8}\sqrt{1 + 4 + 36} = \frac{5}{8}\sqrt{41} \approx 4$°C/m

Let S be a surface with equation $F(x, y, z) = k$ (is a level surface function F of 3 variables).

Let $P(x_0, y_0, z_0)$ be a point on S. C is a curve on S passing through P.

$C : \vec{r}(t) = \langle x(t), y(t), z(t)\rangle$

Let P correspond to t_0.

Because C lies on S, then $F(x(t), y(t), z(t)) = k$. Differentiate with respect to t.

We get $\frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z} \cdot \frac{dz}{dt} = 0$.

We get $\vec{\nabla}F \cdot \vec{r'}(t) = 0$. We are interested in point P.

$\vec{\nabla}F(x_0, y_0, z_0) \cdot \vec{r'}(t_0) = 0$.

This tells us that the gradient vector at P is perpendicular to the tangent vector passing through P.

So, the tangent plane to a level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is given by

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Alternatively, $\vec{\nabla}F(x_0, y_0, z_0) \cdot (\vec{r} - \vec{r}_0) = 0$.

Normal line passes through P and is perpendicular to a tangent plane.

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Example

Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$

Recall this is an ellipsoid.

We have $F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9} \quad (k = 3)$.

$\vec{\nabla}F = \langle \frac{2x}{4}, 2y, \frac{2z}{9}\rangle$.

$\vec{\nabla}F(-2, 1, -3) = \langle -1, 2, -\frac{2}{3}\rangle$.

Tangent plane: $\langle -1, 2, -\frac{2}{3}\rangle \cdot \langle x + 2, y - 1, z + 3\rangle = 0 = -(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0$

OR expressed as $3x - 6y + 2z + 18 = 0$.

The normal line is $\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-\frac{2}{3}}$.

Special case:

What if s is in the form $z = f(x, y)$?

Then, define $F(x, y, z)$ as $f(x, y) - z$. So $\nabla F(x, y, z) = \langle f_x(x, y), f_y(x, y), -1\rangle$.

So the equation of the tangent plane becomes $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$.

1.7 Maximum and Minimum Values

How do you know if a point is a relative max or min? By using the first derivative test where $f'(x) = 0$ and see if it is increasing/decreasing.

Now we are using $f(x, y)$. Peaks in this graph would be relative maxima and valleys are relative minima. Highest maximum is the absolute maximum and lowest minimum is the absolute minimum.

Definition

A function f has a relative (local) maximum at (x_0, y_0) if there exists a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in the disk.

Relative (local) minimum $= f(x_0, y_0) \leq f(x, y)$ in the disk

Absolute maximum$= f(x_0, y_0) \geq f(x, y)$ for all (x, y) in the domain

Absolute minimum$= f(x_0, y_0) \leq f(x, y)$ f or all (x, y) in the domain

Why a disk? In previous classes we used an interval, but now we can travel in more than one direction.

Bounded Sets - In 2-space, a set of points is bounded if we can draw a rectangle around the points.

In 3-space, we use a prism/cube/box.

Goal: To determine if there are relative extrema and to find their location

Theorem 1.7: Extreme Value Theorem

If $f(x, y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum.

Theorem 1.8

If f has a relative extremum at a point (x_0, y_0), and if the first order partials of f exist at (x_0, y_0), then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

If we find a point where $f_x, f_y = 0$, but that point is not a maximum or a minimum, then this is a saddle point.

2nd Partials Test

Let f be a function of 2 variables with continuous 2nd order partials in some disk centered around a critical point (x_0, y_0) and let $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$.

If $D > 0 : f_{xx}(x_0, y_0) > 0 \rightarrow (x_0, y_0)$ is a relative (local) minimum. $f_{xx(x_0,y_0)<0} \rightarrow (x_0, y_0)$ is a relative maximum. (Can also use f_{yy} instead)

If $D < 0, (x_0, y_0)$ is a saddle point.

If $D = 0$ then inconclusive.

Example

Find all relative extrema for $f(x,y) = -x^3 + 4xy - 2y^2 + 1$.

First find all critical points. This is where $f_x, f_y = 0$.

$f_x = -3x^2 + 4y = 0$

$f_y = 4x - 4y = 0$

We have a system to solve.

Solving this equation gives $x = 0, 4/3$ and $y = 0, 4/3$.

We get points $(0,0), (4/3, 4/3)$.

Now for the 2nd partials test. We know that $f_{xx} = -6x$, $f_{yy} = -4$ and $f_{xy} = 4$.

critical points	$D = f_{xx}f_{yy} - (f_{xy})^2$	f_{xx}	conclusion
$(0,0)$	$(0)(-4) - (4)^2 = -16$	No need to do this	saddle point
$(4/3, 4/3)$	$(-8)(-4) - (4)^2 = 16$	$-8 < 0$	relative maximum

$(0,0)$ is a saddle point, and $(4/3, 4/3)$ is a relative maximum.

Note: Pay attention to whether you are asked for the input or output.

Example

Find all relative extrema for $f(x,y) = 4xy - x^4 - y^4$.

$f_x = 4y - 4x^3 = 0$ and $f_y = 4x - 4y^3 = 0$.

Setting these equation to each other, we get $y = x^3$.

Substituting this into $4x - 4y^3 = 0$ gives $4x(1 - x^2)(1 + x^2)(1 + x^4) = 0$, so $x = 0, \pm 1$.

From these we get the points $(0,0)$, $(1,1)$, $(-1,-1)$ as critical points.

We also have $f_{xx} = -12x^2$, $f_{yy} = -12y^2$ and $f_{xy} = 4$.

critical points	$D = f_{xx}f_{yy} - (f_{xy})^2$	f_{xx} (or f_{yy})	conclusion
$(0,0)$	$(0)(0) - 4^2 < 0$	no need	saddle point
$(1,1)$	$(-12)(-12) - 4^2 > 0$	< 0	relative maximum
$(-1,-1)$	$(-12)(-12) - 4^2 \geq 0$	< 0	relative maximum

$(0,0)$ is a saddle point, $(1,1)$ and $(-1,-1)$ are relative maximum.

Finding Absolute Extrema on a Closed and Bounded Set

1. Find all critical points in R (region)
2. Find all boundary points at which absolute extrema can occur.
3. Evaluate $f(x,y)$ at all points.

Example

Consider $f(x,y) = xy - 2x$. Find the absolute maximum and absolute minimum on R: a triangular region formed by $(0,0), (0,4)$, and $(4,0)$.

The critical points:

$f_x = y - 2 = 0$ and $f_y = x = 0$, so critical point is $(0,2)$. This is not in R.

The boundary lines are $x = 0$, $y = 0$ and $y = 4 - x$.

We can see $x = 0$. $f(x,y) = 0$, so $f_x = 0$ and $f_y = 0$, so there are no critical points on the boundary $x = 0$.

For $y = 0$, then $f(x,y) = -2x$. $f_x = -2 \neq 0$ so there are no critical points on this boundary.

$y = 4 - x$ gives $f(x,y) = x(4-x) - 2x = -x^2 + 2x$. $f_x = -2x + 2$ and $f_y = 0$. We get $x = 1$. This gives us $y = 3$. The critical point is $(1,3)$.

Critical point	$(0,0)$	$(0,4)$	$(4,0)$	$(1,3)$
$f(x,y) = xy - 2x$	0	0	-8	1

Absolute maximum is 1 and absolute minimum is -8.

Example

Find the dimensions of a box (open at the top) having volume 32 ft^3 requiring the least amoung of material.

We know that $xyz = 32$.

The surface area is $S = xy + 2yz + 2xz$.

The constraint is $z = \frac{32}{xy}$. Plugging this in gives $S(x,y) = xy + \frac{64}{x} + \frac{64}{y}$.

$S_x = y - \frac{64}{x^2} = 0$ and $S_y = x - \frac{64}{y^2} = 0$.

The system gives $y = \frac{64}{x^2}$ and substituting this in gives $x = 0, x = 4$. We only need to consider $x = 4$, so substituting this in gives $y = 4$.

Then going back, $z = \frac{32}{4 \cdot 4} = 2$.

The dimensions of the box seem to be 4 ft $\times$ 4 ft $\times$ 2 ft.

But we need to prove that this is the minimum amount of material.

We do this by the 2nd partials test.

$S_{xx} = \frac{128}{x^3}$, $S_{yy} = \frac{128}{y^3}$ and $S_{xy} = 1$.

$D: \left(\frac{128}{4^3}\right)\left(\frac{128}{4^3}\right) - 1^2 > 0$, and $S_{xx} > 0$, so $(4,4,2)$ is a relative minimum.

1.8 Lagrange Multipliers

Previously, we minimized $S = xy + 2xz + 2yz$ subject to the constraint $xyz = 32$ (volume).

We solved for z, substituted into S, and minimized. What if we can't get a function of two variables?

Example

Suppose we want to find a rectangle with the largest area that can be inscribed in the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$.

We have $g(x,y)=\frac{x^2}{9}+\frac{y^2}{16}$ with $(g(x,y)=1)$.

$A(x,y)=4xy$

Consider $g(x,y)$ as a set of level curves, same for $A(x,y)=f(x,y)$.

Let $4xy=k$, then $y=\frac{k/4}{x}$.

We are interested in the level curve that "barely" satisfies the constraint (the ellipse).

Remember 2 curves are tangent to a point if and only if their gradient vectors are parallel. There are two level curves remember, the ones for $g(x,y)$ and $A(x,y)$.

The gradients are perpendicular to the curve.

$\vec{\nabla} f=\lambda\vec{\nabla} g(x,y)$. This says they are scalar multiples. λ is the lagrange multiplier.

Solving system of equations that result from this is called "Method of Lagrange Multipliers"

Now to answer the question.

We are finding $f(x,y)=4xy$ subject to $\frac{x^2}{9}+\frac{y^2}{16}=1$.

We need the gradient of f: $\vec{\nabla} f(x,y)=\langle 4y,4x\rangle$.

$\vec{\nabla} g(x,y)=\langle \frac{2}{9}x,\frac{1}{8}y\rangle$.

Therefore $\langle 4y,4x\rangle=\lambda\langle \frac{2}{9}x,\frac{1}{8}y\rangle$.

Now we have $4y=\frac{2\lambda x}{9}$ and $4x=\frac{\lambda y}{8}$.

We have $\lambda=\frac{18y}{x}$ and substituting this gives $4x=\frac{9y^2}{4x}$.

Using the constraint: $\frac{x^2}{9}+\frac{y^2}{16}=1$.

If we solve for x^2 we can use it: $x^2=\frac{9y^2}{16}$.

So we get $\frac{y^2}{16}+\frac{y^2}{16}=1$, so $y=\pm 2\sqrt{2}$.

Plugging this in gives $x^2=\frac{9}{2}$, so $x=\pm\frac{3}{\sqrt{2}}$.

Our area function is $4xy$, so the maximum area is $4\left(\frac{3}{\sqrt{2}}\right)(2\sqrt{2})=24$ u^2.

Example

Minimum $S=xy+2xz+2yz$ subject to the constraint $xyz=32$.

$f(x,y,z)=xy+2xz+2yz$ and $g(x,y,z)=xyz$ is the constraint.

$\vec{\nabla} f=\langle y+2z,x+2z,2x+2y\rangle$ and $\vec{\nabla} g=\langle yz,xz,xy\rangle$.

So we have $y+2z=\lambda yz$, $x+2z=\lambda xz$ and $2x+2y=\lambda xy$.

We have $\lambda=\frac{1}{z}+\frac{2}{y}$, $\lambda=\frac{1}{z}+\frac{2}{x}$, and $\lambda=\frac{2}{y}+\frac{2}{x}$.

We can see that $\frac{1}{z}+\frac{2}{y}=\frac{1}{z}+\frac{2}{x}$, so $x=y$.

We also see that $\frac{1}{z}+\frac{2}{y}=\frac{2}{y}+\frac{2}{x}$, so $z=\frac{1}{2}x$.

The constraint is $xyz=32$, so $x\cdot x\cdot\frac{1}{2}x=32$, so $x=4$.

Recall dimensions are $4'\times 4'\times 2'$.

Be careful! When solving systems, you need to consider if $x=0,y=0,z=0$ or $\lambda=0$ is possible.

Example

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

$g(x, y) = x^2 + y^2$.

$\vec{\nabla} f = \langle 2x, 4y \rangle$ and $\vec{\nabla} g = \langle 2x, 2y \rangle$.

$\langle 2x, 4y \rangle = \lambda \langle 2x, 2y \rangle$.

We have $2x = 2x\lambda$ and $4y = 2y\lambda$.

Note $\lambda = 1$ or $x = 0$.

This gives us $4y = 2y$, so $y = 0$ and gives $y = \pm 1$, so $(0, 1), (0, -1)$.

$x = \pm 1$ gives the points $(1, 0), (-1, 0)$.

$f(1, 0) = 1$

$f(0, 1) = 2$

$f(-1, 0) = 1$

$f(0, -1)2$

The maximum is 2 and the minimum is 1.

Exercise Find the extreme values of $f(x, y) = 3x + y$ subject to $x^2 + y^2 = 10$.