1 Multiple Integrals

1.1 Double Integrals over Rectangles

Recall that $\int_a^b f(x)dx$ gives the area under the curve from $x = a$ to $x = b$.

Also, $\int_a^b f(x)dx = \lim_{x\to\infty} \sum_{k=1}^{n} f(x_k^i)\Delta x_k$ (Riemann sums)

Now we have volume.

Volume Problem: Given a function of 2 variables that is continuous and non-negative on a region R in the xy-plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the xy-plane.

Plan:

1. Partition R in rectangles.
2. Choose a point (x_k^*, y_k^*) in each rectangle.
3. Map onto z.
4. Form parallelpiped.

We can then approximate the volume using rectangular parallelpipeds.

Volume $\approx$ area of rectangle $\times$ height $\to$ Volume $\approx \Delta A_k f(x_k^*, y_k^*)$.

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Volume $= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ (this is the formal definition for the volume problem using Riemann sums)

If f has both positive and negative values, then the volume is the difference in volumes between R and the surface above and below the xy-plane.

Definition

If $f(x, y) \geq 0$, then the volume of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is:

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

$$= \iint_R f(x, y)dA$$

Example

Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below $z = 16 - x^2 - 2y^2$. Divide R into 4 equal squares and choose the sample point to be the upper right corner of each square.

So $V = \iint\limits_R (16 - x^2 - 2y^2) dA$.

This is $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k$. We will use approximately 4 parallelpipeds.

We get $V \approx 1 \cdot f(1,1) + 1 \cdot f(2,1) + 1 \cdot f(1,2) + 1 \cdot f(2,2)$.

This is equal to $V \approx 34$ u^3. This is an estimate.

Example

If $R = \{(x,y) | -1 \leq x \leq 1, -2 \leq y \leq 2\}$, evaluate $\iint\limits_R \sqrt{1 - x^2} dA$.

The graph will look like half of a cylinder.

The volume of a cylinder is $\pi r^2 h$.

$V = \frac{1}{2} \pi r^2 h = \frac{1}{2} \pi (1)^2 (4)$, so $\iint\limits_R dA = 2\pi$.

Midpoint Rule - This tells us to evaluate $\iint\limits_R f(x,y) dA$ using Riemann sums and midpoints.

Evaluating Double Integrals $\rightarrow$ uses iterated integration.

$$\int_c^d \int_a^b f(x,y) dx dy = \int_c^d \left[\int_a^b f(x,y) dx \right] dy$$

This integral can be $dxdy$ or $dydx$.

Example

Integrate $\int_0^3 \int_1^2 x^2 y dy dx$.

First integrate $\int_1^2 x^2 y dy$.

This is $x^2 \cdot \frac{1}{2} y^2$ from $y = 1$ to $y = 2$.

And then we have $\int_0^3 \left(\frac{1}{2} x^2 \cdot 2^2 - \frac{1}{2} x^2 \cdot 1^2 \right) dx$.

This is $\int_0^3 \frac{3}{2} x^2 dx = \frac{1}{2} x^3$ from $x = 0$ to $x = 3$, and the answer is $\frac{27}{2}$.

Example

Evaluate $\int_1^2 \int_0^3 x^2 y dx dy$.

First evaluate $\int_0^3 x^2 y dx$ to get $y \cdot \frac{1}{3} x^3$ from $x = 0$ to $x = 3$.

Then we have $\int_1^2 9y dy$ and this is $\frac{9}{2} y^2$ from $y = 1$ to $y = 2$, so the result of the integral is $\frac{27}{2}$.

$\frac{27}{2}$ in both cases is the area underneath $f(x,y) = x^2 y$ and above $R : [0,3] \times [1,2]$. Note that dx and dy are not always interchangable.

Theorem 1.1: Fubini's Theorem

Let R be a rectangular region defined by $a \leq x \leq b$, $c \leq y \leq d$. If $f(x,y)$ is continuous on this rectangle, then:

$$\iint\limits_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

Example

Evaluate $\iint\limits_R (x-3y^2)dA$ with $R = [0,2] \times [1,2]$.

The integral is $\int_0^2 \int_1^2 (x-3y^2)dydx$.

We start with the inner integral and get $xy - y^3$ from $y = 1$ to $y = 2$.

Then we evaluate $\int_0^2 (2x-8) - (1x-1)dx$ to get $\int_0^2 (x-7)dx$.

The result of this integral is -12.

A few properties:

1. $\iint\limits_R cf(x,y)dA = c\iint\limits_R f(x,y)dA$
2. $\iint\limits_R [f(x,y) \pm g(x,y)]dA = \iint\limits_R f(x,y)dA \pm \iint\limits_R g(x,y)dA$
3. $\iint\limits_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$
4. $\iint\limits_R g(x)h(y)dA = \int_a^b g(x) \int_c^d h(y)dy$ where $R = [a,b] \times [c,d]$.

Example

Evaluate $\iint\limits_R \sin x \cos y dA$ where $R = \left[0, \frac{\pi}{2}\right] \times [0,\pi]$.

We can use the fourth property above to get $\int_0^{\pi/2} \sin x dx \cdot \int_0^{\pi} \cos y dy$.

We get $\cos x$ from 0 to $\pi/2$ and subtract $\sin y$ from 0 to π from this.

The answer is 0.

Example

Find the volume of the solid that is bounded above by $f(x,y) = y\sin(xy)$ and below by $R = [1,2] \times [0,\pi]$.

$V = \int_0^{\pi} \int_1^2 y\sin(xy)dxdy = \int_1^2 \int_0^{\pi} y\sin(y)dydx$.

The first option is better.

So we start with $y \cdot -\frac{1}{y}\cos(xy)$ from $x = 1$ to $x = 2$

Then, $\int_0^{\pi} (-\cos 2y + \cos y)dy = 0$.

Average Value: $f_{avg} = \frac{1}{A(R)} \iint\limits_R f(x,y)dA$

Example

Find the average value of $f(x, y) = x^2 y$ over R with vertices $(-1, 0)$, $(-1, 5)$, $(1, 5)$, and $(1, 0)$.

The area of the rectangle is $A_R = 10$.

Set up the integral to get $f_{avg} = \frac{1}{10} \int_{-1}^{1} \int_{0}^{5} x^2 y dy dx/$

1.2 Double Integrals over General Regions

There are 2 Types of regions:

The biggest question is finding the limits of integration.

So if we have y in terms of x, then we use $\iint\limits_D f(x, y) dA = \iint_{g_1(x)}^{g_2(x)} f(x, y) dy dx$.

If we have x in terms of y, then $\iint\limits_D f(x, y) dA = \iint_{h_1(y)}^{h_2(y)} f(x, y) dx dy$

Example

Evaluate $\iint\limits_D (x + 2y) dA$ where D is the region bounded by $y = 2x^2$ and $y = 1 + x^2$.

We have both equations being $y =$ something, so we are integrating with respect to y first in this case. Though the reason for this is mostly because we are integrating vertically.

So the integration is $\int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y) dy dx$.

We then get $xy + y^2$ with limits of integration $y = 2x^2$ to $y = 1 + x^2$.

So this results in $\int_{-1}^{1} [x(1+x^2)+(1+x^2)^2]-[x(2x^2)+(2x^2)^2] dx = \int_{-1}^{1} (-3x^4-x^3+2x^2+x+1) dx = \frac{32}{15}$.

Example

Set up only! Evaluate $\iint\limits_R xy dA$ where R is the region bounded by $y = -x + 1, y = x + 1$, and $y = 3$.

Draw to see the region. We can see from the graph that integrating by x first is probably the better idea, because y would require 2 double integrals.

So setting up the integral gives $\int_{1}^{3} \int_{1-y}^{y-1} xy dx dy$. (Setting the equations in terms of $x =$)

Example

Find the volume of the solid that lies under $z = xy$ and above D where D is the region bounded by $y = x - 1$ and $y^2 = 2x + 6$.

By drawing this, we see we will go by dx first.

Setting up the integral is $\int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy dx dy$.

Starting the integration, we get $\frac{1}{2}x^2 y$ with the limits of the first integral.

This gives $\frac{1}{2} \int_{-2}^{4} (y + 1)^2 \cdot y - \left(\frac{1}{2}y^2 - 3\right)^2 \cdot y dy$.

Simplifying some more gives $\frac{1}{2} \int_{-2}^{4} -\frac{1}{4}y^5 + 4y^3 + 2y^2 - 8y dy$ and this gives 36 as an answer.

Example

$\int_0^1 \int_x^1 \sin(y^2)dydx$

If we draw the region, we have $y = 1$ and $y = x$.

We go from $x = 0$ to $x = y$ and for the y limits, we go from 0 to 1.

Therefore the integral is $\int_0^1 \int_0^y \sin(y^2)dxdy$.

Solving this integral gives $-\frac{1}{2}\cos(1) + \frac{1}{2}$.

Example

Use a double integral to find the area of the region enclosed between $y = x^3$ and $y = 2x$ in the first quadrant.

Set up the integral $A = \int_0^{\sqrt{2}} \int_{x^3}^2 x$ to get the area.

Exercise Evaluate by reversing the order of integration: $\int_0^2 \int_{y/2}^1 e^{x^2} dxdy$.

1.3 Double Integrals in Polar Coordinates

Recall that polar coordinates are in form (r, θ) and rectangular coordinates are in (x, y).

In polar, for the unit circle, we can write $0 \leq r \leq 1$ or $0 \leq \theta \leq 2\pi$.

We have 3 equations for converting:

- $r^2 = x^2 + y^2$
- $x = r\cos\theta$
- $y = r\sin\theta$

To find the volume, the process is similar to the Riemann sum process for double integrals.

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k = \iint f(r, \theta)dA$$

In the above, ΔA_k represents the area of the polar rectangle.

Now we need to find the area of a polar rectangle.

First we know that the area of a sector is $\frac{1}{2}r^2\theta$ and that ΔA_k is the large sector minus the little sector.

Therefore we have $\frac{1}{2}\left(r_k^* + \frac{1}{2}\Delta r_k\right)^2 \Delta\theta_k - \frac{1}{2}\left(r_k^* - \frac{1}{2}\Delta r_k\right)^2 \Delta\theta k$.

And this simplifies to $r_k^*\Delta r_k \Delta\theta_k$, so $\Delta A_k = r_k^*\Delta r_k\Delta\theta_k$.

So, $V = \iint\limits_R f(r, \theta)dA = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(r_k^*, \theta_k^*)r_k^*\Delta r_k\Delta\theta_k$.

So we have

$$V = \iint\limits_R f(r, \theta)rdrd\theta$$

Example

Find $\iint\limits_R \sin\theta dA$ where R is the region outside the circle $r = 2$ and inside $r = 2 + 2\cos\theta$ in the 1st quadrant.

We see that the circle will be hit first, then the other polar curve.

So the integral is $\int_0^{\pi/2}\int_2^{2+2\cos\theta} \sin\theta \cdot rdrd\theta$.

Note the outer integral goes to $\frac{\pi}{2}$ because that is the first quadrant.

So the inner integral becomes $\sin\theta \cdot \frac{1}{2}r^2$ from the bounds $r = 2$ to $r = 2 + 2\cos\theta$.

We then integrate $\frac{1}{2}\int_0^{\pi/2} \sin\theta[(2 + 2\cos\theta)^2 - 2^2]d\theta$.

Solving this gives $8/3$.

Example

Find the volume of the solid bounded by $z = 0$ and $z = 1 - x^2 - y^2$.

The graph of $z = 1 - x^2 - y^2$ will be a paraboloid.

So R is a circle with radius 1 when we draw this paraboloid.

We could integrate as $V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1 - x^2 - y^2)dydx$, or we could convert to polar.

In polar, we know R is a circle and then we can convert to polar to get $\int_0^{2\pi}\int_0^1(1 - r^2) \cdot rdrd\theta$, which is equal to $V = \frac{\pi}{2}$.

Area is the same as before, recall this was $A = \iint\limits_R 1 \cdot dA$, and now it is $\iint\limits_R rdrd\theta$.

Example

Use a double integral to find the area enclosed by one loop of the four-leaf rose of $r = \cos 2\theta$.

If you know how to draw this, then we can find the area $A = \int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} rdrd\theta$, and this integral is simple to solve, the answer is $\pi/8$.

Example

Find the volume of the solid that lies under $z = x^2 + y^2$, above the xy-plane, and inside $x^2 + y^2 = 2x$.

The graph $x^2 + y^2 = 2x$ can be rearranged to complete the square. We have then $(x - 1)^2 + y^2 = 1$.

So if we were to convert $x^2 + y^2$ to polar, we get r^2.

We have $r^2 = 2r\cos\theta$ and $r = 2\cos\theta$.

The integral then we get $\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}(r^2) \cdot rdrd\theta$.

The integral is $V = \frac{3\pi}{2}$

1.4 Surface Area

The formula for surface area is

$$A(S) = \lim_{n\to 0}\sum_{k=1}^{n}\sqrt{(z_x)^2 + (z_y)^2 + 1}\Delta A$$

which becomes

$$A(S) = \iint_R \sqrt{(z_x)^2 + (z_y)^2 + 1} dA$$

Example

Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region with vertices $(0, 0), (1, 0)$, and $(1, 1)$.

$z = x^2 + 2y$, $z_x = 2x$ and $z_y = 2$.

So $A(S) = \int_0^1 \int_0^x \sqrt{(2x)^2 + (2)^2 + 1} dydx$.

This integral gives $\frac{1}{12}(27 - 5\sqrt{5})$.

Example

Find the surface area of the portion of $z = x^2 + y^2$ below the plane $z = 9$.

Paraboloid!

The integral is $A(S) = \iint_R \sqrt{(2x)^2 + (2y)^2 + 1} dA$.

We might be able to see that simplifying this gives $4x^2 + 4y^2 + 1$ inside the square root, and we have an $x^2 + y^2 = 9$ in the paraboloid.

We should use polar.

So we now have $\int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1} \cdot r dr d\theta$.

This gives you $\frac{\pi}{6}(37^{3/2} - 1)$.

Example

Find the surface area of $z = \sqrt{4 - x^2}$ above $R : [0, 1] \times [0, 4]$.

Setting up the integral gives $\iint_R \sqrt{\left(-\frac{x}{\sqrt{4-x^2}}\right)^2 + 0^2 + 1}$.

It doesn't really matter the way we integrate, so we get $\int_0^1 \int_0^4 \sqrt{\frac{x^2}{4-x^2} + \frac{4-x^2}{4-x^2}} dydx$.

This simplifies to $\int_0^1 \frac{8}{\sqrt{4-x^2}} dx$.

We notice that this becomes $8 \sin^{-1}\left(\frac{x}{2}\right)$ with bounds 0 to 1, and this gives $\frac{4\pi}{3}$.

1.5 Triple Integrals

So far we have

- D is closed (can be contained in a rectangle)
- Taking limit as $n \to \infty$ gave us the volume under $z = f(x, y)$.

Now, triple integrals.

- Closed solid B (can be contained in a box).
- Divide B into n sub-boxes.
- Volume of each box is ΔV and a point in the box is $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$.

Then $\lim_{n\to\infty} \sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right) \Delta V = \iiint_B f(x,y,z)dV$.

This gives us hypervolume.

- Same properties and evaluation as before (double integrals)
- If B is a box defined by $a \le x \le b$, $c \le y \le d$, $e \le z \le f$, then $\iiint_B f(x,y,z)dV = \int_e^f \int_c^d \int_a^b f(x,y,z)dxdydz$.

Example

Evaluate $\iiint_G xyz^2 dV$ where $G : \{(x,y,z) | 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$.

Convention is to do $\int_0^1 \int_{-1}^2 \int_0^3 xyz^2 dzdydx$.

Let us start with the z part to get $\int_0^1 \int_{-1}^2 \frac{1}{3}xyz^3$ from $z = 0$ to $z = 3$.

Then we get $\int_0^1 \frac{9}{2}xy^2$ from $y = -1$ to $y = 2$.

And then we get $\frac{9}{2}\int_0^1 3xdx = \frac{27}{4}$.

If the region B is rectangular and the function is a product (such as $f(x,y,z) = g(x) \cdot h(y) \cdot j(z)$), we can split up the integral.

The above integral will become: $\int_0^1 xdx \cdot \int_{-1}^2 ydy \cdot \int_0^3 z^2 dz$.

Type I Solid: E is a solid with upper surface $z = u_2(x,y)$ and lower surface $z = u_1(x,y)$. D is the projection of E onto the xy-plane.

Then $\iiint_E f(x,y,z)dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdA$.

To find limits of integration:

1. Find upper and lower surfaces bounding E. These are limits for z.
2. Make a 2-d sketch of the projection D on xy-plane.
3. Treat like usual.

Example

Evaluate $\iiint_T zdV$ where T is the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

This is a tetrahedral in the first octant and $x + y + z = 1$ is a plane.

We have $z = 1 - x - y$, so $\int \int \int_0^{1-x-y} z \cdot dzdydz$.

And from the other bounds we get $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \cdot dzdydx$.

The outer integral note should always have constants.

The answer of this integral becomes $\frac{1}{24}$.

Sometimes we have lateral surfaces bounding, not the top or bottom.

Type II Solid:

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx\right] dA$$

Type III Solid:

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy\right] dA$$

Example

Evaluate $\iiint\limits_R \sqrt{x^2+z^2}dV$ where E is bounded by $y = x^2 + z^2$ and $y = 4$.

$y = x^2 + z^2$ is a paraboloid and $y = 4$ is a plane.

We want to start integrating from y since the paraboloid goes towards the plane always.

We project the figure now on the xz-plane and now get a circle with equation $x^2 + z^2 = 4$.

So we can write the integral now as $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4}\sqrt{x^2+z^2}dydzdx$.

First we have the first part of the integral be equal to $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-x^2-z^2)\sqrt{x^2+z^2}dzdx$.

In this we see that $r^2 = x^2 + z^2$.

When we switch to polar we end up getting $\int_0^{2\pi}\int_0^2(4-r^2)r\cdot rdrd\theta$.

This gives you $\frac{128\pi}{15}$.

Volume as a triple integral: $V = \iiint\limits_E 1\cdot dV$.

Example

Setup an integral to find the volume of the wedge in the 1st octant that is cut from the solid cylinder $y^2 + z^2 \leq 1$ by the planes $y = x$ and $x = 0$.

We start with z then we can see $y = x$ from the projection.

So $V = \int_0^1\int_0^x\int_0^{\sqrt{1-y^2}} 1\cdot dzdxdy$.

Note it is not always the case that the order of integration can be changed without changing the bounds.

Example

Find the volume of the solid enclosed between the paraboloids $z = 5x^2 + 5y^2$ and $z = 6 - 7x^2 - y^2$.

To find projection D we have $5x^2 + 5y^2 = 6 - 7x^2 - y^2$. Solving gives us $y = \pm\sqrt{1-2x^2}$ (or a circle).

The integral becomes $V = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}}\int_{5x^2+5y^2}^{6-7x^2-y^2} 1\cdot dzdydx$.

1.6 Triple Integrals in Cylindrical Coordinates

Cylindrical coordinates are like polar, but in 3-D.

To convert cylindrical to rectangular, it is the same as polar. $x = r\cos\theta$, $y = r\sin\theta$, and the additional part is $z = z$.

We also get $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$ and $z = z$ once again to convert rectangular to cylindrical.

Exercise Plot $\left(2, \frac{2\pi}{3}, 1\right)$ and find rectangular coordinates.

Example

Find cylindrical coordinates for $(3, -3, -7)$.

$r^2 = 3^2 + (-3)^2$ gives $r = 3\sqrt{2}$.

$\tan\theta = -1$, so $\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$.

We pick $\frac{7\pi}{4}$ for this, and then we get coordinates $(3\sqrt{2}, \frac{7\pi}{4}, -7)$.

Example

Describe the surface $z = r$.

We can rewrite this was $z = \sqrt{x^2 + y^2}$ and therefore $z^2 = x^2 + y^2$, which describes a cone.

Recall from earlier: $\iiint\limits_E f(x,y,z)dV = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f\left(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\right)\Delta V_k$.

ΔV_k is the area of base times the height. From earlier, $\Delta V_k = r^*_k \Delta r_k \Delta\theta_k \cdot \Delta z_k$.

So, $\iiint\limits_E f(x,y,z)dV = \iiint f(r,\theta,z) \cdot r dr d\theta dz$.

To find limits of integration:

1. Identify upper surface $z = g_2(r,\theta)$ and lower surface $z = g_1(r,\theta)$.
2. Make a 2-D sketch of projection onto xy-plane to determine bounds for r and θ.

Example

Evaluate $\iiint\limits_G dV$ where G lies within $x^2 + y^2 = 1$, below $z = 4$, and above $z = 1 - x^2 - y^2$.

We are finding a volume.

The graph gives a cylinder, with a hemisphere at the bottom removing a part of the cylinder (bad explanation but it's ok).

So the integral is $V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{1-x^2-y^2}^{4} 1 \cdot dz dy dx$.

As we can see, the projection on the xy-plane is a circle, so cylindrical coordinates are best.

Now we have $\int_0^{2\pi}\int_0^1\int_{1-r^2}^4 1 \cdot r \cdot dz dr d\theta$

Integrating this gives you $V = \frac{7}{2}\pi$.

Example

Convert from rectangular to cylindrical: $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)dz dy dx$.

R is a circle with radius $r = 2$, so the integral becomes $\int_0^{2\pi}\int_0^2\int_r^2 r^2 \cdot r dz dr d\theta$.

The integral evaluates to $\frac{16\pi}{5}$.

Example

Let E be the region inside the sphere of radius 2 centered at the origin and above the plane $z = 1$. Find the volume of E.

THe equation of the sphere is $x^2 + y^2 + z^2 = 4$, so we have $z^2 = 4 - r^2$.

The integral can be $V = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} 1 \cdot r dz dr d\theta$.

We have figure $D : x^2 + y^2 + z^2 = 4$ and $z = 1$, so $x^2 + y^2 = 3$.

1.7 Triple Integrals in Spherical Coordinates

Spherical Coordinates: (ρ, θ, ϕ), where ρ is the distance from the origin to the point, θ represents the same as before (the angle on the xy-plane from the x-axis), and ϕ represents the angle between the positive z-axis and point.

Bounds for ρ, θ, and ϕ:

- $\rho \geq 0$
- $0 \leq \theta \leq 2\pi$
- $0 \leq \phi \leq \pi$

A few common graphs:

- $\rho = c$ gives a sphere
- $\theta = c$ gives a "half" plane
- $\phi = c$ gives a cone ($0 < c < \pi/2$ will give the top part)

Converting:

- $x = \rho \sin \phi \cos \theta$
- $y = \rho \sin \phi \sin \theta$
- $z = \rho \cos \phi$

Also, $\rho^2 = x^2 + y^2 + z^2$.

Example

Convert $\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$ to rectangular.

We know $x = \rho \sin \phi \cos \theta$, so $x = 2 \sin \frac{\pi}{3} \cos \frac{\pi}{4} = \sqrt{\frac{3}{2}}$.

$y = \rho \sin \phi \sin \theta$, so pluggin in gives $\sqrt{\frac{3}{2}}$.

$z = \rho \cos \phi = 2 \cos \frac{\pi}{3} = 1$.

The coordinates are $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right)$.

Example

Convert $(0, 2\sqrt{3}, -2)$ to spherical.

$\rho^2 = x^2 + y^2 + z^2$, so $\rho = 4$.

$z = \rho\cos\phi$, and we can see that $\cos\phi = \frac{z}{\rho}$, so $\phi = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$, but knowing the bounds for ϕ gives $\phi = \frac{2\pi}{3}$.

We know that $\cos\theta = \frac{x}{\rho\sin\phi}$, so $\cos\theta = 0$, and $\theta = \frac{\pi}{2}$ is the only θ that works in the xy-plane for this.

The point is $\left(4, \frac{\pi}{2}, \frac{2\pi}{3}\right)$

Recall: $\iiint\limits_E f(x,y,z)dV = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} f\left(x^*_{ijk}, y^*_{ijk}.z^*_{ijk}\right)\Delta V_k = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi^2}\int_{\rho_1}^{\rho_2} f(\rho,\theta,\phi)\rho^2\sin\phi d\rho d\phi d\theta$.

Example

Evaluate $\iiint\limits_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dzdydx$ were B is the unit sphere.

If try rectangular, we would find the limits of integration are messy.

So we use spherical.

The integral is $\int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{(\rho^2)^{3/2}}\rho^2\sin\phi d\rho d\phi d\theta$.

This is equal to $\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2 e^{\rho^3}\sin\phi d\rho d\phi d\theta$.

Integrating this fully gives $\frac{4\pi(e-1)}{3}$.

Example

Convert to spherical: $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}dzdydx$.

We can see that the whole inside part gives $(\rho\cos\phi)^2\cdot\rho\cdot\rho^2\sin\phi d\rho d\phi d\theta$.

The most inner integrand is a hemisphere so bounds go from 0 to 2.

The second integrand is a circle so bounds go from 0 to $\pi/2$.

The integral is $\int_0^{2\pi}\int_0^{\frac{\pi}{2}},\int_0^2 \rho^5\cos^2\phi\sin\phi d\rho d\phi d\theta$.

The answer of this integral is $\frac{64}{9}\pi$.

Example

Find the volume of the ice cream cone bounded by $x^2+y^2+z^2 = z$ and $z = \sqrt{x^2+y^2}$.

Remember $V = \iiint\limits_E 1\cdot dV$.

So the sphere equation can be rewritten as $x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \frac{1}{4}$.

So the center of the sphere is $\left(0,0,\frac{1}{2}\right)$ with $r = \frac{1}{2}$.

$\rho : x^2+y^2+z^2 = z,\ \rho^2 = \rho\cos\phi,\ \rho = \cos\phi$.

$\phi : z = \sqrt{x^2+y^2},\ \rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta}$, so $\cos\phi = \sin\phi$, gives $\phi = \frac{\pi}{4}$.

So the integral becomes $V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} 1\cdot\rho^2\sin\phi d\rho d\phi d\theta$.

Solving this integral gives $V = \frac{\pi}{8}$.

We can split a triple integral into 3 separate integrals when all the bounds are numbers, and functions are as products of functions.

1.8 Change of Variables in Multiple Integrals

Example

Let T be the transformation from uv-plane to xy-plane defined by $x = \frac{1}{4}(u+v)$ and $y = \frac{1}{2}(u-v)$.

(a) Find $T(1,3)$.

$x = \frac{1}{4}(1+3) = 1$ and doing the same gives $y = -1$, so $(1,-1)$.

(b) Sketch the image under T bounded by $-2 \le u \le 2$ and $-2 \le v \le 2$.

The figure for uv-plane is a square centered at the origin with side lengths of 4.

The plan to draw the image on the xy-plane is to sketch several u, v curves and then you need to know u, v in terms of x and y.

$4x = u + v$ and $2y = u - v$, and this gives $4x + 2y = 2u$ and $u = 2x + y$ as a result.

We can see $u = -2$ gives $y = -2x - 2$, then $y = -2x - 1$ when $u = -1$, and when $u = 2$, $y = -2x + 2$.

Similarly, $v = 2x - y$, and we see a pattern here as well.

In a way, it is easier to integrate over a different region like mapped above.

Definition: Jacobian

If $x = g(u,v)$ and $y = h(u,v)$, then the Jacobian of x and y with respect to u and v is:

$$\frac{\partial(x,y)}{\partial(u,v)} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

We use this to give us the extra factor when converting integrals.

It is similar to converting in polar (or cylindrical), where you added r, or simliar to spherical when you added $\rho^2 \sin\phi$.

So

$$\iint\limits_R f(x,y)dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

Example

Consider $x = r\cos\theta$ and $y = r\sin\theta$. Find the Jacobian.

The determinant of this will be $\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r^2\cos^2\theta + r\sin^2\theta = r$

Example

Evaluate $\iint\limits_R 3xydA$ where R is bounded by $x - 2y = 0$, $x - 2y = -4$, $x + y = 4$, and $x + y = 1$.

We let $u = x - 2y$, $v = x + y$, so $u = 0, -4$ and $v = 4, 1$. The region of this is rectangular, much easier to solve than if you were to do it based on y.

For change of variables, first you need to find the Jacobian.

We need $x(u, v)$ and $y(u, v)$.

So we have $y = \frac{1}{3}(v - u)$ and $x = \frac{1}{3}(u + 2v)$ from the values of u and v.

The jacobian is then $\begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}$.

Then we have to do change of variables in the integral.

So the integral is $\int_1^4 \int_{-4}^0 3\left(\frac{1}{3}(u + 2v)\right)\left(\frac{1}{3}(v - u)\right) \cdot \left|\frac{1}{3}\right| dudv$.

This integral results in $\frac{164}{9}$.

Now for 3 variables.

Example

Find the Jacobian of $x = \rho \sin\phi \cos\theta$, $y = x = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$.

The Jacobian becomes $\begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$

From this, we get and doing some simplifications gives you the determinant of this, which is $\rho^2 \sin\phi$.