

1 Vector Calculus

1.1 Vector Fields

We will focus on the mathematical description for flow.

Examples are air flow (wind), fluid flow, and electromagnetic fields.

Definition: Vector Field

Written as $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j} \rightarrow$ at each point (x, y) , there is a vector that depends on x and y .

Example

Sketch $\vec{F} = x\vec{i}$.

Basically at a point $x = 1$, then the vector is \vec{i} in the positive direction, the same applies for all x .

Exercise Sketch $\vec{F} = \langle y, x \rangle$.

Of course, vector fields can be drawn in 3-space.

Other notations are \vec{F} = vector field or if we associate x, y with a radius vector, $\vec{F}(\vec{r})$.

Example

We will look at inverse square fields.

Newton's Law of Gravitation is defined $|\vec{F}| = \frac{mMG}{r^2}$, essentially particles of mass m and M attract each other with a force $|\vec{F}|$.

Force exerted by particle of mass M to particle of mass m in the direction of a unit vector.

The direction of this is $-\frac{\vec{r}}{\|\vec{r}\|}$.

So we have $|\vec{F}(\vec{r})| = -\frac{mMG}{r^2} \cdot \frac{\vec{r}}{\|\vec{r}\|}$ and this is called the gravitational field.

As a vector this is written as $\vec{F}(\vec{r}) = \frac{c}{\|\vec{r}\|^3} \vec{r}$.

Recall the gradient is $\nabla f(x, y) = f_x(x, y)\vec{i} + f_y(x, y)\vec{j}$.

$\nabla f(x, y)$ is really a vector field in 2-space and is called the gradient vector field.

Example

Sketch the gradient field of $\phi(x, y) = x^2 + y^2$.

$\vec{\nabla}\phi = \langle 2x, 2y \rangle$.

This can be used to draw the gradient field.

At each point, the vector points in the direction of maximum increase.

Gradient vectors are long where level curves are close to each other. This is because the length of the gradient vector is equal to the value of the directional derivative.

Definition

A vector field \vec{F} is conservative in a region if it is the gradient field for some function f in that region.

\vec{F} is conservative if $\vec{F} = \vec{\nabla} f$.

f is called a potential function of \vec{F} .

Example

Consider $\vec{F} = \langle 2x, y \rangle$. Is \vec{F} conservative?

If we can write $\vec{F} = \langle 2x, y \rangle = \vec{\nabla} f$, then it is conservative.

If $f(x) = x^2 + \frac{1}{2}y^2$, then $f_x = 2x$ and $f_y = y$

Therefore, \vec{F} is conservative.

1.2 Line Integrals

What we know:

- \int is integrated over an interval
- \iint is integrated over an area in 2-space
- \iiint is integrated over an area in 3-space

Now, we are integrating along a curve in order to find mass, fluid flow, force.

Let us have a curve $C : x = x(t), y = y(t), a \leq t \leq b$.

We can integrate then as $\sum_{i=1}^n (x_i^*, y_i^*) \Delta s_i$, where Δs_i is the length of subarc.

Definition

If f is defined on a smooth curve $C : x = x(t), y = y(t)$, then the line integral of f along C is:

$$\int_C f(x, y) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

Recall, arc length is $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$.

So, $\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

Remember, $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$, so $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

Example

Evaluate $\int_C (2 + x^2 y) ds$, where C is the upper half of the unit circle.

We need a parameterization of this curve. So $C : x = \cos t, y = \sin t$.

The integral is $\int_C (2 + x^2 y) ds = \int_0^\pi (2 + \cos^2 t + \sin t) \sqrt{(-\sin t)^2 + (\cos t)^2} dt$.

This is equal to $\int_0^\pi 2 + \cos^2 t \sin t dt = 2\pi + 2/3$.

Example

Evaluate $\int_C x ds$ where C is the portions of $y = x$ and $y = x^2$ from $(0, 0)$ to $(1, 1)$.

Our first curve is $y = x$ and second curve is $y = x^2$.

Let $c_1 : y = x$, so the parameterization is $x = t, y = t, 0 \leq t \leq 1$.

So integrate $\int_0^1 t \cdot \sqrt{1^2 + 1^2} dt = \int_0^1 t\sqrt{2} dt = \frac{\sqrt{2}}{2}$.

Let $c_2 : y = x^2$. If we look at parameterization $x = t$ and $y = t^2$, we see that this will not work. So we have to parameterize as $x = 1 - t, y = (1 - t)^2, 0 \leq t \leq 1$.

So we can integrate now $\int_0^1 (1 - t) \sqrt{1 + 4(1 - t)^2} dt$.

Integrating this gives $\frac{1}{12}(5^{3/2} - 1)$.

Therefore, $\int_C x dS = \frac{\sqrt{2}}{2} + \frac{1}{12}(5^{3/2} - 1)$.

Right now, we have arc length parameterizations, so orientation does not matter. (For example, for c_2 above, we could have used parameterization $x = t, y = t^2$.)

What does this actually mean? It depends on the interpretation of the function f .

1. If $f(x, y, z)$ represents density $\int_C f(x, y, z) ds$ is the mass of a thin wire shaped like C .
2. If $f(x, y)$ is a curve, then the area underneath the curve is $A = \int_C f(x, y) ds$.

So far we have calculated line integrals with respect to arc length: $\int_C f(x, y) ds$.

We can also calculate line integrals with respect to x and y :

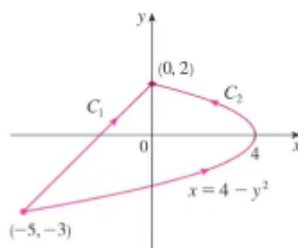
$$\int_C f(x, y) dx = \int_a^b f(x(t), y(t)) \cdot x'(t) dt$$

$$\int_C f(x, y) dy = \int_a^b f(x(t), y(t)) \cdot y'(t) dt$$

Sometimes these occur together: $\int_C P(x, y) dx + Q(x, y) dy$

Example

Evaluate $\int_C y^2 dx + x dy$ along (a) C_1 and (b) C_2 where C is as shown.



The curve c_1 is a line from $(-5, -3)$ to $(0, 2)$.

So $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$, $0 \leq t \leq 1$, $\vec{r}_0 = \langle -5, -3 \rangle$, $\vec{r}_1 = \langle 0, 2 \rangle$.

$\vec{r}(t) = \langle -5 + 5t, 5t - 3 \rangle$ $0 \leq t \leq 1$.

So, $x = 5t - 5$, $y = 5t - 3$ and $dx = 5dt$ and $dy = 5$ with $0 \leq t \leq 1$.

So we can integrate $\int_C y^2 dx + x dy$.

This is $\int_0^1 [(5t - 3)^2 \cdot 5dt + (5t - 5) \cdot 5dt] = -\frac{5}{6}$.

For c_2 , let $y = t$ and $x = 4 - t^2$ for bounds $-3 \leq t \leq 2$.

What this gives us is $dy = 1dt$ and $dx = -2tdt$.

The integral for this is $\int_{-3}^2 [(t)^2(-2tdt) + (4 - t^2) \cdot 1dt]$, which is equal to $\frac{245}{6}$.

This tells us that path matters. Line integrals in 3-space are almost the same as 2-space.

Example

Evaluate $\int_C y \sin z ds$ where $C : x = \cos t, y = \sin t, z = t$ for $0 \leq t \leq 2\pi$.

$$\int_C y \sin z ds = \int_0^{2\pi} \sin t \cdot \sin t \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} dt.$$

This is $\int_0^{2\pi} \sin^2 t \sqrt{2} dt$.

This can be rewritten as $\sqrt{2} \int_0^{2\pi} \frac{1}{2} - \frac{1}{2} \cos 2t dt$.

This is equal to $\sqrt{2}\pi$.

Example

Evaluate $\int_C y dx + z dy + x dz$ where C consists of the line segment from $(2, 0, 0)$ to $(3, 4, 5)$, then the vertical line segment from $(3, 4, 5)$ to $(3, 4, 0)$.

For the first curve we have $c_1 : \vec{r} = (1-t)\langle 2, 0, 0 \rangle + t\langle 3, 4, 5 \rangle$ and $\vec{r}_1(t) = \langle 2+t, 4t, 5t \rangle$ $0 \leq t \leq 1$.

This gives $\int_0^1 4t \cdot 1dt + 5t \cdot 4dt + (2+t) \cdot 5dt = 24.5$.

$c_2 : \vec{r} = (1-t)\langle 3, 4, 5 \rangle + t\langle 3, 4, 0 \rangle$ and get $\vec{r}_2(t) = \langle 3, 4, 5-5t \rangle$.

The integral is $\int_0^1 4(0dt) + (5-5t)(0dt) + 3(-5dt) = -15$.

The final answer is $24.5 + -15 = 9.5$.

Recall: $W = \vec{F} \cdot \vec{d}$.

Now, suppose $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a continuous force field (gravitational force field or electric force field). We may want to calculate the work done in moving the particle among a smooth curve C .

Definition

Let \vec{F} be a continuous vector field defined on a smooth curve C given by $\vec{r}(t)$.

Then the line integral of \vec{F} along C is: $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_C \vec{F} \cdot \vec{T} ds$.

This calculates work.

Example

Find the work done by $\vec{F}(x, y, z) = \langle -\frac{1}{2}x, -\frac{1}{2}y, \frac{1}{4} \rangle$ on a particle moving along the helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle, 0 \leq t \leq 3\pi$.

We have $\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$ and $\vec{F}(\vec{r}(t)) = \langle -\frac{1}{2} \cos t, -\frac{1}{2} \sin t, \frac{1}{4} \rangle$.

So $W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{3\pi} \langle -\frac{1}{2} \cos t, -\frac{1}{2} \sin t, \frac{1}{4} \rangle \cdot \langle -\sin t, \cos t, 1 \rangle dt$.

To simplify this, we have $\int_0^{3\pi} (\frac{1}{2} \cos t \sin t - \frac{1}{2} \sin t \cos t + \frac{1}{4}) dt$.

This is equal to $W = \frac{3}{4}\pi$.

Remark: $\int_C \vec{F} \cdot d\vec{r} = - \int_{-C} \vec{F} \cdot d\vec{r}$. If we reverse the path, work will be the opposite value.

Example

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ and $C : x = t, y = t^2, z = t^3, 0 \leq t \leq 1$.

$\vec{r} = \langle t, t^2, t^3 \rangle$, $\vec{r}' = \langle 1, 2t, 3t^2 \rangle$, so $\vec{F}(\vec{r}(t)) = \langle t^3, t^5, t^4 \rangle$.

So we end up getting $\int_C \vec{F} \cdot d\vec{r} = \int_0^1 t^3 + 5t^6 dt = \frac{27}{28}$.

Consider $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}, \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Then $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \langle P, Q, R \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle dt = \int_a^b (P \cdot x'(t) + Qy'(t) + Rz'(t)) dt$.

So $\int_C \vec{F} \cdot d\vec{r} = \int_C Pdx + Qdy + Rdz$.

Example

Calculate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle \cos x, \sin x \rangle$ and $C : \vec{r}(t) = t\vec{i} + t^2\vec{j}, -1 \leq t \leq 2$.

$\vec{F} = \langle \cos t, \sin t \rangle$

$\int_C \vec{F} \cdot d\vec{r} = \int_{-1}^2 \langle \cos t, \sin t \rangle \cdot \langle 1, 2t \rangle dt = \int_{-1}^2 \cos t + 2t \sin t dt$.

When we integrate all of this we get $\sin t - 2t \cos t + 2 \sin t$ from these bounds and the answer is $3 \sin 2 + 3 \sin 1 - 4 \cos 2 - 2 \cos 1$.

1.3 The Fundamental Theorem for Line Integrals

Recall the Fundamental Theorem of Calculus: $\int_a^b F'(x)dx = F(b) - F(a)$

Now, we have the Fundamental Theorem of Line Integrals.

Theorem 1.1: Fundamental Theorem of Line Integrals

Let C be a smooth curve given by $\vec{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function whose gradient ∇f is continuous on C . Then $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$, where f is a potential function.

Example

Confirm that $f(x, y) = x^2y - \frac{1}{2}y^2$ is a potential function for $\vec{F}(x, y) = 2xy\vec{i} + (x^2 - y)\vec{j}$. Then evaluate $\int_C \vec{F} \cdot d\vec{r}$ for C where C is a curve from $(-1, 4)$ to $(1, 2)$.

The potential function is: $\vec{\nabla} f = \vec{F}$ and $\langle 2xy, x^2 - y \rangle = \langle 2xy, x^2 - y \rangle$.

Then we can just see can integrate $\int_C \vec{\nabla} f \cdot d\vec{r}$ and get $f(1, 2) - f(-1, 4) = 4$.

This value represents the work done in moving a particle from $(-1, 4)$ to $(1, 2)$. Previously we would have to use $\vec{r}(t)$ for a curve and parameterize and integrate.

Recall that $\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}$ (value of line integral depends on the path)

But: $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$ as long as C_1 and C_2 have the same endpoints (from the theorem above).

This is true for line integrals of a conservative vector field ($\vec{F} = \vec{\nabla} f$). In general, $\int_C \vec{F} \cdot d\vec{r}$ is independent of path if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any 2 paths C_1, C_2 , that have the same initial points and endpoints.

Line Integrals along Closed Paths: The following are equivalent

1. $\vec{F}(x, y) = \langle f(x, y), g(x, y) \rangle$ is conservative.
2. $\int_C \vec{F} \cdot d\vec{r} = 0$ (same initial/endpoint)
3. $\int_C \vec{F} \cdot d\vec{r}$ is independent of path from any point P to any point Q .

Any single one of these statements implies the other two statements for closed paths.

Simple Path: a path that goes from one point to another point.

Not Simple: A path that may curve back on itself.

Simply-Connected region: closed, non-intersecting

Multiply-Connected region: region that has holes in the domain

So the theorem only applies when \vec{F} is conservative. Then $\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$ where $\nabla f = \vec{F}$.

Conservative Field Test:

Let $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$ be a vector field on an open, simply-connected region D . If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D , then \vec{F} is conservative.

Example

Show that $\vec{F}(x, y) = \langle 2xy^3, 1 + 3x^2y^2 \rangle$ is conservative. Then find the potential function.

Previously, we would have found the potential function.

We have $\frac{\partial P}{\partial y} = 6xy^2$ and $\frac{\partial Q}{\partial x} = 6xy^2$, so it is conservative.

So integrating $\int \frac{\partial F}{\partial x} dx = \int 2xy^3 dx$ and get $f = x^2y^3 + k(y)$.

Now we can use $\frac{\partial f}{\partial y} = 3x^2y^2 + k'(y)$.

So we can see that this has to be equal to $3x^2y^2 + k'(y) = 1 + 3x^2y^2$.

So we have $\int k'(y) = \int 1 dy$, so $k(y) = y + C$.

Therefore, $f(x, y) = x^2y^3 + y + C$.

Example

Find the work done by the vector field $\vec{F}(x, y) = (y^3 + 1)\vec{i} + (3xy^2 + 1)\vec{j}$ in moving a particle from $(0, 0)$ to $(2, 0)$.

Work is $\int_C \vec{F} \cdot d\vec{r}$.

Before, we would have parameterized.

Now we can ask if \vec{F} is conservative.

$$\frac{\partial P}{\partial y} = 3y^2 = \frac{\partial Q}{\partial x} = 3y^2.$$

Because the vector fields are conservative, then $\int_C \vec{F} \cdot d\vec{r} = f(2, 0) - f(0, 0)$, where $\nabla f = \vec{F}$.

Start $\int \frac{\partial f}{\partial y} dy = \int 3xy^2 + 1 dy$ and we get $f = xy^3 + y + k(x)$.

Now $\frac{\partial f}{\partial x} = y^3 + k'(x) = y^3 + 1$.

So $\int k'(x) = \int 1 dx$ and $k(x) = x + C$.

Therefore $f(x, y) = xy^3 + y + x + C$.

Work is $f(2, 0) - f(0, 0) = 2 - 0 = 2$.

If \vec{F} is conservative, then \vec{F} is independent of path. If you cannot remember how to find f , choose any path from $(0, 0)$ to $(2, 0)$.

For example, we could have done $\vec{r}(t) = \langle 2t, 0 \rangle, 0 \leq t \leq 1$, so work is $\int_0^1 \langle 1, 1 \rangle \cdot \langle 2, 0 \rangle dt = \int_0^1 2 dt = 2$.

Options for $\int_C \vec{F} \cdot d\vec{r}$ (work)

1. Parameterize Curve

If \vec{F} is conservative/independent of path

2. Find f (potential function)

3. Use any path from P to Q

Example

$\vec{F} = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$. Find f such that $\nabla f = \vec{F}$.

Let f_x, f_y, f_z be the terms in \vec{F} in order.

So $\int f_x dx = \int y^2 dx$, or $f = xy^2 + k(y, z)$.

Now $f_y = 2xy + k_y(y, z) = 2xy + e^{3z}$.

Integrating gives $\int k_y(y, z) dy = \int e^{3z} dy$ so $k(y, z) = ye^{3z} + k(z)$.

So right now we have $f = xy^2 + ye^{3z} + k(z)$.

So $f_z = 3ye^{3z} + k'(z) = 3ye^{3z}$ so $k'(z) = 0$.

Therefore $f(x, y, z) = xy^2 + ye^{3z} + C$

We can check by finding f_x, f_y , and f_z from this.

1.4 Green's Theorem

George Green (1793-1841) was a self-taught british mathematician and physicist.

Theorem 1.2: Green's Theorem

Let R be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve oriented counterclockwise.

If $f(x, y)$ and $g(x, y)$ are continuous and have continuous first partials on some open set containing R , then

$$\int_C f(x, y)dx + g(x, y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

Proof. We need to show $\int_C f(x, y)dx = \iint_R -\frac{\partial f}{\partial y}dA$

Let c_1 and c_2 be boundary curves. Then $\int_C f(X, y)dx = \int_{c_1} f(x, y)dx + \int_{c_2} f(x, y)dx$.

This is equivalent to $\int_C f(X, y)dx = \int_{c_1} f(x, y)dx - \int_{-c_2} f(x, y)dx$.

When we parameterize this, we get $c_1 : x = t, y = g_1(t)$ and $c_2 : x = t, y = g_2(t)$.

So the integral $\int_C f(x, y)dx = \int_a^b f(t, g_1(t))dt - \int_a^b f(t, g_2(t))dt$.

Then we have $-\int_a^b [f(t, g_2(t))dt - f(t, g_1(t))]dt$. This is equal to $-\int_a^b f(t, y)$ between the values $y = g_1(t)$ and $y = g_2(t)$ wrt t .

We can write this now as a double integral $-\int_a^b \int_{g_1(t)}^{g_2(t)} \frac{\partial f}{\partial y} dy dx = -\iint_R \frac{\partial f}{\partial y} dA$. \square

Example

$\int_C y^3 dx + (x^3 + 3xy^2) dy$ where C is below. Also find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle y^3, x^3 + 3xy^2 \rangle$.

C goes from $(0,0)$ to $(1,1)$ and goes from the line $y = x$ then the line $y = x^3$.

This is closed and counterclockwise, so we can use Green's Theorem.

This integral becomes $\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$.

So we have $\iint_R (3x^2 + 3y^2) - 3y^2 dA = \int_0^1 \int_{x^3}^x 3x^2 dy dx = \frac{1}{4}$.

For the other part, we need to do the line integral of $\int_C \vec{F} \cdot d\vec{r}$.

$\int_C \vec{F} \cdot d\vec{r} = \frac{1}{4}$, it is basically the same as above.

If we have a closed path, then $\oint_C f(x,y)dx + g(x,y)dy = \iint_R \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} dA$.

Sometimes we get the orientation, such as \oint_C or \oint_C .

If \vec{F} is conservative, then $\oint_C \vec{F} \cdot d\vec{r} = 0$.

Another Application of Green's Theorem:

Area = $\iint_R dA = \oint_C x dy = \oint_C -y dx = \frac{1}{2} \oint_C -y dx + x dy$.

Example

Use a line integral to find the area of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

This is an ellipse.

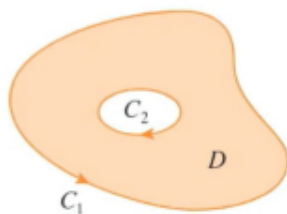
If we parameterize this, we get $x = a \cos t$ and $y = b \sin t$ for $0 \leq t \leq 2\pi$.

We have $A = \oint_C x dy = \int_0^{2\pi} a \cos t \cdot b \cos t dt$.

This is equal to $ab \int_0^{2\pi} \cos^2 t dt = ab \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2} \cos 2t \right) dt$.

This is equal to $A = \pi ab$.

For multiply-connected regions:



We want to calculate the line integral for this.

For the outside curve, we are finding $\oint_{C_1} f(x,y)dx + g(x,y)dy$.

We have to go in and exit, so the inner region contributes nothing.

For the second curve, it is oriented clockwise so we have to reverse the orientation.

So we add $\oint_{c_2} f(x, y)dx + g(x, y)dy$.

Using Green's Theorem, this becomes $\iint_R \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} dA + \iint_R \frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} dA$.

Now we will see if the curve has negative orientation and if we can use Green's Theorem.

Example

Evaluate $\int_C x^4 dx + xy dx$ where C is the triangle curve consisting of line segments from $(0, 0)$ to $(0, 1)$, to $(1, 0)$, and then back to $(0, 0)$.

We need the curve to have the reverse orientation to use Green's Theorem.

So we can just do $-\iint_{-C} (y - 0) dA = -\iint_R y dA$.

1.5 Curl and Divergence

Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$. Then $\text{curl } \vec{F} = \vec{\nabla} \times \vec{F}$.

Curl relates to the rotational properties of a fluid at a point.

Example

Find $\text{curl } \vec{F}$ for $\vec{F} = \langle xz, xyz, -y^2 \rangle$.

The cross product $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$.

This is equal to $\langle \frac{\partial}{\partial y} - \frac{\partial}{\partial z}(xyz), -[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)], \frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz) \rangle$.

This is equal to $\text{curl } \vec{F} = \langle -y(2+x), x, yz \rangle$.

Theorem 1.3

$\text{curl}(\vec{\nabla} f) = \vec{0}$ (Note f must have continuous 2nd order partials)

Proof. $\text{curl}(\vec{\nabla} f) = \vec{\nabla} \times \vec{\nabla} f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$

This is equal to $\langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right), \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \rangle$.

Everything subtracts from each other, so $0\vec{i} + 0\vec{j} + 0\vec{k}$ is the result, or $\vec{0}$. \square

We care about this because if \vec{F} is a conservative vector field, then $\vec{\nabla} f = \vec{F}$.

So the theorem also says if \vec{F} is conservative, then $\text{curl } \vec{F} = \vec{0}$.

Theorem 1.4

If \vec{F} is a vector field defined on all of \mathbb{R}^3 whose component functions have continuous partial derivatives and $\text{curl } \vec{F} = \vec{0}$, then \vec{F} is a conservative vector field.

Example

Show that $\vec{F}(x, y, z) = y^2 z^3 \vec{i} + 2xyz^3 \vec{j} + 3xy^2 z^2 \vec{k}$ is conservative and find the potential function.

Before, we would use $\vec{F}(x, y) = P\vec{i} + Q\vec{j}$ and find the partials.

Now we can show in 3-space that $\text{curl } \vec{F} = \vec{0}$.

Remember that $\text{curl } \vec{F} = \vec{\nabla} \times \vec{F}$.

$$\text{This is equal to } \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix}$$

This gives $\langle 6xyz^2 - 6xyz^2, -(3y^2 z^2 - 3y^2 z^2), 2yz^3 - 2yz^3 \rangle = \vec{0}$. \vec{F} is a conservative vector field.

Start with $\int f_x dx = \int y^2 z^3 dx$, so $f = xy^2 z^3 + k(y, z)$.

Now we have $f_y = 2xyz^3 + k'(y, z) = 2xyz^3$, so then $\int k'(y, z) = \int 0$ so $k(y, z) = k(z)$.

Now we have $f = xy^2 z^3 + k(z)$.

$f_z = 3xy^2 z^2 + k'(z)$ and from this we can see that $k'(z) = \text{a constant}$.

So our potential function is $f(x, y, z) = xy^2 z^3 + k$.

Divergence: $\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F}$.

Divergence relates to the way in which fluid flows away or towards a point.

Example

Find $\text{div } \vec{F}$ for $\vec{F} = \langle xz, xyz, -y^2 \rangle$.

So $\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle xz, xyz, -y^2 \rangle$.

This gives us $\text{div } \vec{F} = z + xz + 0$, so $\text{div } \vec{F} = z + xz$. This is called a scalar field.

Laplacian: ∇^2

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

If applied to $f(x, y, z) \rightarrow \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

So, $\nabla^2 f = \text{div}(\vec{\nabla} f)$.

Vector Forms of Green's Theorem:

Consider $\vec{F} = P\vec{i} + Q\vec{j}$. Suppose all conditions for Green's Theorem are satisfied. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy = \iint_D (Q_x - P_y) dA$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}.$$

So we can rewrite Green's theorem as:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl } \vec{F}) \cdot \vec{k} dA$$

This is equal to $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k} \cdot \vec{k} dA$ or $\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$.

1.6 Surface Integrals

Recall: $\int, \iint, \iiint, \int_C ds$, or $\left(\int_C d\vec{r}\right)$ or \oint_C - Green's theorem)

The idea now is integrating over a smooth parametric surface.

Take $\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$.

Then form Riemann sum: $\sum_{i=1}^m \sum_{j=1}^n f(P_{ij}^*) \Delta S_{ij}$.

If we take the limit $\lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(P_{ij}^*) \Delta S_{ij}$ this becomes equal to $\iint_S f(x, y, z) dS$.

The above expression represents the surface integral of f over S .

To evaluate:

1. If parameterize: $\iint_S f(x, y, z) dS = \iint_D f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| dA$. (Note $|\vec{r}_u \times \vec{r}_v|$ denotes the surface area of a surface S)

2. If rectangular: $\iint_S f(x, y, z) dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA$

Interpretation:

1. Mass
2. $\iint_S dS$ is surface area
3. Mass/center of mass: $m = \iint_S \rho(x, y, z) dS$ and $\bar{x} = \frac{1}{m} \iint_S x \cdot \rho(x, y, z) dS$

Example

Evaluate $\iint_S x^2 dS$ over the sphere $S : x^2 + y^2 + z^2 = 1$.

Let $x = \sin \phi \cos \theta$, $y = \sin \phi \sin \theta$ and $z = \cos \phi$.

So $\vec{r}(\phi, \theta)$.

So cross product $\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\ -\sin \phi \sin \theta & \sin \phi \cos \theta & 0 \end{vmatrix} =$

$\langle \sin^2 \phi \cos \theta, \sin^2 \phi \sin \theta, \sin \phi \cos \phi \cos^2 \theta + \sin \phi \cos \phi \sin^2 \theta \rangle$.

So we get $\langle \sin^2 \phi \cos \theta, \sin^2 \phi \sin \theta, \sin \phi \cos \phi \rangle$.

So the magnitude of this is $\sqrt{\sin^4 \phi \cos^2 \theta + \sin^4 \phi \sin^2 \theta + \sin^2 \phi \cos^2 \phi} = \sqrt{\sin^4 \phi + \sin^2 \phi \cos^2 \phi} = \sqrt{\sin^2 \phi} = \sin \phi$.

Now we have $\iint_S x^2 dS = \int_0^{2\pi} \int_0^\pi (\sin \phi \cos \theta)^2 \cdot \sin \phi d\phi d\theta$.

This is equal to $\int_0^{2\pi} \cos^2 \theta d\theta \cdot \int_0^\pi \sin^3 \phi d\phi$.

The first integral gives $\int_0^{2\pi} \left[\frac{1}{2} + \frac{1}{2} \cos 2\theta \right] d\theta = \pi$.

The second integral becomes $\int_0^\pi \sin \phi (1 - \cos^2 \phi) d\phi = \int_0^\pi (\sin \phi - \sin \phi \cos^2 \phi) d\phi = \frac{4}{3}$.

These are being multiplied, so $\pi \cdot \frac{4}{3} = \frac{4}{3}\pi$.

Example

Evaluate $\iint_S (y^2 + 2ys) dS$ where S : first octant of $2x + y + 2z = 6$.

We will use the $\iint_D f(x, y, g(x, y))$ to solve this.

We have $z = \frac{1}{2}(6 - 2x - y)$ and $z_x = -1, z_y = -\frac{1}{2}$. Then we have $\sqrt{1^2 + \frac{1}{2}^2 + 1} = \frac{3}{2}$.

The surface integral becomes $\iint_D [y^2 + 2y(\frac{1}{2}(6 - 2x - y))] \cdot \frac{3}{2} dA$.

We end up getting $\frac{3}{2} \int_0^3 \int_0^{6-2x} (y^2 + 6y - 2xy - y^2) dy dx$.

This evaluates to $\frac{243}{2}$.

Surface Integrals of Vector Fields:

Main Applications:

1. Line integrals calculate work.
2. Surface integrals calculate flux.

Flow Fields - think about $\vec{F}(x, y, z)$ representing velocity of a fluid particle at the point (x, y, z) .

Oriented Surface:

"orientable": surface has 2 sides (outside and inside)

"non-orientable": one side only

\vec{n} is a unit vector at each point. $-\vec{n}$ is a unit normal vector in opposite direction. Unit normals will never have an abrupt change in direction.

Orientation of a smooth parametric surface is defined by \vec{n} .

$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ gives a unit vector with a positive orientation.

Example

Find the natural orientation for $\vec{r}(u, v) = \langle \cos u, v, -\sin u \rangle, 0 \leq u \leq 2\pi, 0 \leq v \leq 3$.

The cross product is $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin u & 0 & -\cos u \\ 0 & 1 & 0 \end{vmatrix} = \langle \cos u, 0, -\sin u \rangle$.

The magnitude $|\vec{r}_u \times \vec{r}_v| = 1$ and $\vec{n} = \langle \cos u, 0, -\sin u \rangle$.

When $u = 0, v = 0$, then $\vec{n} = \langle 1, 0, 0 \rangle$.

Outward is positive orientation at this point, and negative is inward.

Flux conditions:

- We will only deal with incompressible fluids (uniform density)
- We will assume that the velocity of a fluid at a fixed point does not vary with time (steady-state)

Definition: Flux

The volume of a fluid that passes through a surface in one unit of time.

This depends on

1. speed of fluid (greater speed gives greater volume)
2. how surface is positioned relative to flow. Maximum flux is when fluid is perpendicular to surface and no flux happens when nothing is passing through.
3. Area of surface (larger area gives more flux)

How do we calculate flux?

$$\text{Flux} = \phi = \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS = \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| dA$$

$$\text{Flux: } \phi = \iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA$$

Example

Find the flux of the vector field $\vec{F}(x, y, z) = z\vec{i} + y\vec{j} + x\vec{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Writing in terms of ϕ and θ , we have $r(\phi, \theta) = \langle \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \rangle$.

$$\text{Crossing gives } \vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\ -\sin \phi \sin \theta & \sin \phi \cos \theta & 0 \end{vmatrix} = \langle \sin^2 \phi \cos \theta, \sin^2 \phi \sin \theta, \sin \phi \cos \phi \rangle.$$

$$\text{So } \phi = \iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA$$

$$\text{We have that } \vec{F}(\vec{r}(\phi, \theta)) \cdot (\vec{r}_\phi \times \vec{r}_\theta) = \cos \phi \sin^2 \phi \cos \theta + \sin^3 \phi \sin^2 \theta + \sin^2 \phi \cos \phi \cos \theta.$$

$$\text{Flux is } \int_0^{2\pi} \int_0^\pi (\cos \phi \sin^2 \phi \cos \theta + \sin^3 \phi \sin^2 \theta + \sin^2 \phi \cos \phi \cos \theta) d\phi d\theta.$$

$$\text{We can pull out some things and get } 2 \int_0^{2\pi} \sin^2 \phi \cos \phi d\phi \cdot \int_0^{2\pi} \cos \theta d\theta + \int_0^\pi \sin^3 \phi d\phi \cdot \int_0^{2\pi} \sin^2 \theta d\theta.$$

$$\text{Simplifying this gives } \int_0^\pi \sin^3 \phi d\phi \cdot \int_0^{2\pi} \sin^2 \theta d\theta = \frac{4\pi}{3}.$$

Now for Non-Parametric Surfaces:

Suppose $z = g(x, y)$ is a surface oriented upward. Rewrite this function as $z - g(x, y) = 0$.

For \vec{n} , use $\vec{n} = \frac{\langle -z_x, -z_y, 1 \rangle}{|\sqrt{z_x^2 + z_y^2 + 1}|}$. If we call $z - g(x, y) = 0$ as $G(x, y, z)$, then $\vec{n} = \frac{\vec{\nabla} G}{|\vec{\nabla} G|}$.

So, if oriented upwards, $\iint_S \vec{F} \cdot \langle -z_x, -z_y, 1 \rangle dA$. Downwards, we get $\iint_S \vec{F} \cdot \langle z_x, z_y, -1 \rangle dA$.

Example

Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle y, x, z \rangle$ and S is the boundary of the solid region E enclosed by $z = 1 - x^2 - y^2$ and $z = 0$. Assume outward orientation.

When we draw this, we can call the upwards orientation s_1 and the downwards orientation s_2 .

For s_1 : $\iint_{s_1} \vec{F} \cdot d\vec{S}$.

This is $\iint_D \langle y, x, 1 - x^2 - y^2 \rangle \cdot \langle 2x, 2y, 1 \rangle dA = \iint_D (2xy + 2xy + 1 - x^2 - y^2) dA$.

This is equal to $\iint_D (1 + 4xy - x^2 - y^2) dA$. Polar will make this integral easier.

The integral becomes $\int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos \theta \sin \theta - r^2) \cdot dr d\theta = \frac{\pi}{2}$.

For s_2 , we have $z = 0$.

So we can rewrite $\iint_{s_2} \vec{F} \cdot d\vec{S} = \iint_D \langle y, x, 0 \rangle \cdot \langle 0, 0, -1 \rangle dA$.

This is $\iint_D 0 dA = 0$.

So, the flux is $\frac{\pi}{2}$.

1.7 Stokes' Theorem

Recall: $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$.

Green's theorem applies for closed curves C on the xy -plane.

We previously talked about writing Green's Theorem as $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl} \vec{F} \cdot \vec{K} dA$.

We could do this because $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle \cdot \langle 0, 0, 1 \rangle$.

So, Green's theorem can be tied to curl.

Recall the right-hand rule, the surface on the left is the positive orientation.

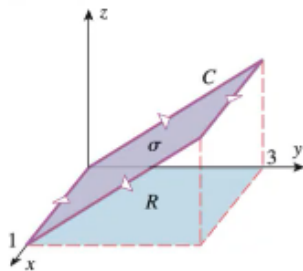
Theorem 1.5: Stokes' Theorem

Let S be an oriented piecewise smooth surface bounded by a curve C with positive orientation.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot d\vec{s}$$

Example

$\vec{F}(x, y, z) = \langle x^2, 4xy^3, xy^2 \rangle$. Find the work done by \vec{F} on a partial traversing $z = y$ as shown.



Previously, we could have used Green's theorem if the curve was on the xy -plane.

We could have also parameterized the 4 lines and set up 4 line integrals.

Using Stokes' Theorem, we must ask if the surface is smooth (yes), do we have a curve with positive orientation (yes if \vec{n} is down).

$$\text{So we have } W = \int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot d\vec{s}.$$

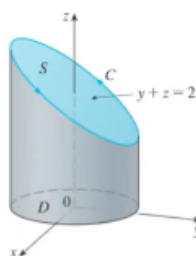
$$\text{Finding } \text{curl} \vec{F} = \langle 2xy, -y^2, 4y^3 \rangle.$$

$$\text{To find } d\vec{s}, \text{ we find } \langle z_x, z_y, -1 \rangle = \langle 0, 1, -1 \rangle.$$

$$\text{So the integral becomes } W = \iint_R \langle 2xy, -y^2, 4y^3 \rangle \cdot \langle 0, 1, -1 \rangle dA = \int_0^1 \int_0^3 (-y^2 - 4y^3) dy dx = -90.$$

Example

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = -y^2\vec{i} + x\vec{j} + z^2\vec{k}$ and C is the curve of intersection of $y + z = 2$ and $x^2 + y^2 = 1$. Orient C to be counterclockwise when viewed from above.



The surface is smooth, and the curve c has positive orientation so Stokes' theorem can be used.

$$\text{The } \text{curl} \vec{F} \text{ is } \langle 0, 0, 1 + 2y \rangle.$$

$$y + z = 2 \text{ gives } z = 2 - y, \text{ and since } \vec{n} \text{ is up, we have } \langle -z_x, -z_y, 1 \rangle = \langle 0, 1, 1 \rangle.$$

$$\text{So the line integral is } \int_C \vec{F} \cdot d\vec{r} = \iint_D \langle 0, 0, 1 + 2y \rangle \cdot \langle 0, 1, 1 \rangle dA.$$

This is $\iint_D (1 + 2y) dA$. Since the region is a circle, polar is a good option.

$$\text{So we have } \int_0^{2\pi} \int_0^1 (1 + 2r \sin \theta) \cdot r dr d\theta = \pi$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \text{ is a special case of Stokes' Theorem.}$$

Example

Verify Stokes' Theorem for $\vec{F}(x, y, z) = \langle 2z, 3x, 5y \rangle$ where $S : 4 - x^2 - y^2, z \geq 0$, upward orientation and $C : x^2 + y^2 = 4$ on xy -plane (positively oriented).

We start with $\vec{r} = \langle 2 \cos t, 2 \sin t, 0 \rangle$.

We use $\int_C \vec{F} \cdot d\vec{r} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$.

So the integral is $\int_0^{2\pi} 12 \cos^2 t dt$ after the dot product and this is equal to 12π .

Using $\iint_S \text{curl} \vec{F} \cdot d\vec{S}$ should give us similar answer.

The $\text{curl} \vec{F}$ is $\langle 5, 2, 3 \rangle$, and S has upward orientation, so $\langle -z_x, -z_y, 1 \rangle = \langle 2x, 2y, 1 \rangle$.

So the integral is $\iint_R \langle 5, 2, 3 \rangle \cdot \langle 2x, 2y, 1 \rangle = \iint_R (10x + 4y + 3) dA$.

The integral is $\int_0^{2\pi} \int_0^2 (10r \cos \theta + 4r \sin \theta + 3) r dr d\theta = 12\pi$.

Example

$\iint_S \text{curl} \vec{F} \cdot d\vec{S}$ for $\vec{F} = \langle xz, yz, xy \rangle$ where S is the part of sphere $x^2 + y^2 + z^2 = 4$ that lies inside $x^2 + y^2 = 1$ above the xy -plane.

We can use a line integral for this.

We need to find c first:

$x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$, so $1 + z^2 = 4$, so we can get $z = \sqrt{3}$ from this.

So $C : x^2 + y^2 = 1, z = \sqrt{3}$.

We have $\vec{r} = \langle \cos t, \sin t, \sqrt{3} \rangle$, so $\vec{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$.

Therefore $\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle \sqrt{3} \cos t, \sqrt{3} \sin t, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt$.

This is equal to $\int_0^{2\pi} (-\sqrt{3} \cos t \sin t + \sqrt{3} \sin t \cos t) dt = 0$.

1.8 The Divergence Theorem

Recall that given $\vec{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle$, that $\text{div} \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$ (fluid flow)

Theorem 1.6: Divergence Theorem

Let E be a solid (closed) and let S be the boundary surface of E , given with positive (outward) orientation. Then

$$\phi = \iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div} \vec{F} dV$$

This is like using the potential function to calculate work when you have a conservative vector field.

Example

Find the flux of the vector field $\vec{F}(x, y, z) = z\vec{i} + y\vec{j} + x\vec{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$. We will assume outward orientation.

It is closed and $\text{div}\vec{F} = 0 + 1 + 0 = 1$.

The flux is $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E 1 \cdot dV$.

E is unit sphere $x^2 + y^2 + z^2 \leq 1$ and S is boundary $x^2 + y^2 + z^2 = 1$.

The options are set up bounds (spherical coordinates) or to find the volume with a triple integral noting the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

The flux is $\frac{4}{3}\pi$.

Example

Find the (outward) flux of the vector field $\vec{F} = \langle x, y^2, z \rangle$ across S : tetrahedron bounded by $2x + 2y + z = 6$ in the 1st octant.

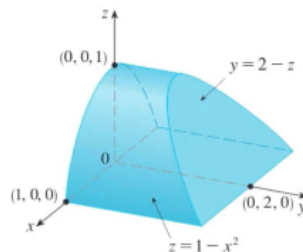
Divergence theorem can be used because the solid is closed. $\text{div}\vec{F} = 2 + 2y$.

Flux is $\int_0^3 \int_0^{3-x} \int_0^{6-2x-2y} (2 + 2y) dz dy dx$.

The answer of the integral is 31.5.

Example

Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle xy, y^2 + e^{xz^2}, \sin(xy) \rangle$ and S is the surface of the region bounded by $z = 1 - x^2$, $z = 0$, $y = 0$, and $y + z = 2$.



We get $\text{div}\vec{F} = 3y$.

The flux is $\phi = \iint_S \vec{F} \cdot d\vec{S} = \int_{-1}^1 \int_0^{1-x^2} \int_0^{2-z} 3y dy dz dx$.

The answer is $\frac{184}{35}$.

Sources and Sinks

A source is a point where fluid enters flow. ($\text{div}\vec{F}(p_0) > 0$)

A sink is a point where fluid leaves flow. ($\text{div}\vec{F}(p_0) < 0$)

Example

Determine if $\vec{F} = (y + z)\vec{i} - xz^3\vec{j} + (x^2 \sin y)\vec{k}$ is free of sources and sinks. If not, locate them.

We have $\text{div}\vec{F} = 0$, so there are no sources and no sinks.

Example

Find all sources and sinks for $\vec{F}(x, y, z) = \langle xy, -xy, y^2 \rangle$.

The divergence of \vec{F} is $\operatorname{div} \vec{F} = y - x$.

Sources will happen when $y - x > 0$ and sinks will happen when $y - x < 0$.

So sources at all points where $x < y$ and there are sinks at all points where $x > y$.