1 First-Order Differential Equations

1.1 Separable Equations

Definition

If the right-hand side of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

can be expressed as a function $g(x)$ that depends only on x times a function $p(y)$ that depends only on y, then the differential equation is called separable.

To solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)p(y)$$

multiply by $\mathrm{d}x$ and by $h(y) = 1/p(y)$ to obtain

$$h(y)\mathrm{d}y = g(x)\mathrm{d}x$$

Then integrate both sides and you end up getting $H(y) = G(x) + C$, where we have merged the two constants of integration into a single symbol C. The last equation gives an implicit solution to the differential equation.

Example

Solve the nonlinear equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-5}{y^2}$$

This can be rewritten as $y^2\mathrm{d}y = (x-5)\mathrm{d}x$. Integrating both sides results in $\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$. To get the explicit form just solve for y, which is trivial.

Example

Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-1}{x+3} \qquad y(-1) = 0$$

Doing Calc BC stuff gives us $y = 1 - \frac{1}{2}(x+3)$.

Be careful because you can be losing solutions. Ok bye!

1.2 Linear Equations

Remember a linear first-order equation is an equation that can be expressed in the form

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = b(x)$$

where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable x, not on y.

Method for solving linear equation:

- Write the equation in the standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

- Calculate the integrating factor $\mu(x)$ by the formula

$$\mu(x) = \exp\left[\int P(x)\mathrm{d}x\right]$$

- Multiply the equation in standard form by $\mu(x)$ and, recalling that the left-hand side is just $\frac{\mathrm{d}}{\mathrm{d}x}[\mu(x)y]$, obtain

$$\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\mu(x)y = \mu(x)Q(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\mu(x)y] = \mu(x)Q(x)$$

- Integrate the last equation and solve for y by dividing by $\mu(x)$ to obtain.

Example

Find the general solution to

$$\frac{1}{x}\mathrm{d}y\mathrm{d}x - \frac{2y}{x^2} = x\cos x \qquad x > 0$$

We have $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x}y = x^2\cos x$.

The integrating factor $\mu(x) = e^{\int P(x)\mathrm{d}x}$ which in this case is $e^{-2\int \frac{1}{x}\mathrm{d}x}$ and this is equivalent to $\frac{1}{x^2}$.

Using this we can multiply through in standard form then we have $\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x^3}y = \cos x$.

The left side is just $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2}\cdot\right) = \cos x$.

Integrating and solving for y we get that $y = x^2\sin x + Cx^2$.

Example

For the initial value problem

$$y' + y = \sqrt{1+\cos^2 x} \qquad y(1) = 4$$

find the value of $y(2)$.

Our $P(x)$ is 1 here, so $\mu = e^x$.

So the equation after multiplying through by it gives us that $\mu y' + \mu y = \mu\sqrt{1+\cos^2 x}$, or $e^x y' + e^x y = e^x\sqrt{1+\cos^2 x}$.

This is equivalent to basically $\frac{\mathrm{d}}{\mathrm{d}x}(e^x y) = e^x\sqrt{1+\cos^2 x}$.

This is $e^x y = \int e^x\sqrt{1+\cos^2 x}\mathrm{d}x$.

Using a calculator $y(2) = 2.127$.

Theorem 1.1: Existence and Uniqueness of Solution

Suppose $P(x)$ and $Q(x)$ are continuous on an interval (a,b) that contains the point x_0. Then for any choice of initial value y_0, there exists a unique solution $y(x)$ on (a,b) to the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Qx \qquad y(x_0) = y_0$$

In fact the solution is given for a suitable value of C.

1.3 Exact Equations

Definition: Exact Differential Form

The differential form $M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y$ is said to be exact in a rectangle R is there is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x}(x,y) = M(x,y) \qquad \text{and} \qquad \frac{\partial F}{\partial y}(x,y) = N(x,y)$$

for all (x,y) in R. That is, the total differential of $F(x,y)$ satisfies

$$\mathrm{d}F(x,y) = M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y$$

If $M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y$ is an exact differential form, then the equation

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0$$

is called an exact equation.

Theorem 1.2: Test for Exactness

Suppose the first partial derivatives of $M(x,y)$ and $N(x,y)$ are continuous in a rectangle R. Then

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0$$

is an exact equation in R if and only if the compatibility condition

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$$

holds for all (x,y) in R.

Example

Solve the differential equation

$$\frac{dy}{dx} = -\frac{2xy^2+1}{2x^2y}$$

Ok so this is not separable or linear, so we use exactness.

$$\frac{dy}{dx} + \frac{2xy^2+1}{2x^2y} = 0$$
$$dy + \frac{2xy^2+1}{2x^2y}dx = 0$$
$$\frac{2xy^2+1}{2x^2y}dx + 1dy = 0$$

This is the same form we want.

Another form we can get is $(2xy^2+1)dx + 2x^2y dy = 0$.

Another form we can get is $1dx + \frac{2x^2y}{2xy^2+1}dy = 0$.

We are now looking for a $F(x, y) = c$ and we know this is true when $\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$.

So the second one of these is probably the best, so we now have $m = 2xy^2 + 1$ and $n = 2x^2y$.

Doing the partial of m with respect to y we get $4xy$ and the partial of n with respect to x is $4xy$ and these are the same.

Let $F(x, y) = x^2y^2 + x = C$. The partial of this function with respect to x is $2xy^2 + 1$ and the partial of this function with respect to y is $2x^2y$ and this is the same as previous.

Method for Solving Exact Equations:

- If $Mdx + Ndy = 0$ is exact, then $\partial F/\partial x = M$. Integrate this last equation with respect to x to get

$$F(x, y) = \int M(x, y)dx + g(y)$$

- To determine $g(y)$, take the partial derivative with respect to y of both sides of the aboev equation and substitute N for $\partial F/\partial y$. We can now solve for $g'(y)$.
- Integrate $g'(y)$ to obtain $g(y)$ up to a numerical constant. Substituting $g(y)$ into the equation from step 1 gives $F(x, y)$
- The solution to $Mdx + Ndy = 0$ is given implicitly by

$$F(x, y) = C$$

(Alternatively, starting with $\partial F/\partial y = N$, the implicit solution can be foudn by first integrating with respect to y.)

Example

Solve

$$(2xy - \sec^2 x)\mathrm{d}x + (x^2 + 2y)\mathrm{d}y = 0$$

Let m be the first term and n be the second term, and the partial derivatives of these are the same, so they are exact.

Let $F(x, y) = \int 2xy - \sec^2 x\mathrm{d}x$. When we integrate this, we get $yx^2 - \tan x$. In this case, the constant is anything with y, so the integral is equivalent to $yx^2 - \tan x + g(y)$.

Now we take the $\frac{\partial F}{\partial y} = x^2 - 0 + g'(y)$. These two are n so $x^2 + 2y = x^2 + g'(y)$, so solving for $g(y)$ we get that this is equal to $y^2 + C$.

So $F(x, y) = xy^2 - \tan x + y^2 = C$.

Exercise Solve $(1 + e^x y + xe^x y)\mathrm{d}x + (xe^x + 2)\mathrm{d}y = 0$.

Solution: $x + xye^x + 2y = C$.

Example

Solve

$$(x + 3x^3 \sin y)\mathrm{d}x + (x^4 \cos y)\mathrm{d}y = 0$$

Doing the partials originally makes them not equal to each other.

We can get this to exact form by multiplying through by x^{-1}. When we do this we get $(1+3x^2 \sin y)\mathrm{d}x + x^3 \cos y\mathrm{d}y = 0$ and the partials of these are the same.

x^{-1} is called an integrating factor.

Integrating m with respect to x, we get that $F(x, y) = \int 1 + 3x^2 \sin y\mathrm{d}x = x + \sin y \cdot x^3 + g(y)$.

Doing the partial of F with respect to y, we get $\frac{\partial F}{\partial y} = x^3 \cos y = 0 + x^3 \cos y + g'(y)$, and this gets that $g(y) = C$.

So the answer is $x + x^3 \sin y = C$.

1.4 Special Integrating Factors

Definition

If the equation

$$M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0$$

is not exact, but the equation

$$\mu(x, y)M(x, y)\mathrm{d}x + \mu(x, y)N(x, y)\mathrm{d}y = 0$$

which results from multiplying the first equation by the function $\mu(x, y)$, is exact, then $\mu(x, y)$ is called an integrating factor of the first equation.

Theorem 1.3: Special Integrating Factors

If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu(x) = \exp\left[\int \left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right) \mathrm{d}x\right]$$

is an integrating factor for an equation. If $(\partial N/\partial x - \partial M/\partial y)/M$ is continuous and depends only on y, then

$$\mu(y) = \exp\left[\int \left(\frac{\partial M/\partial x - \partial N/\partial y}{M}\right) \mathrm{d}y\right]$$

is an integrating factor for the same equation.

Method for Finding Special Integrating Factors:

If $M\mathrm{d}x + N\mathrm{d}y = 0$ is neither separable nor linear, compute $\partial M/\partial y$ and $\partial N/\partial x$. If $\partial M/\partial y = \partial N/\partial x$, then the equation is exact. If it is not exact, consider

$$\frac{\partial M/\partial y - \partial N/\partial x}{N}$$

If this is a function of just x, then an integrating factor is given by the formula above of $\mu(x)$. If not consider

$$\frac{\partial N/\partial x - \partial M/\partial y}{M}$$

If this is a function of just y¡ then an integrating factor is given by above of $\mu(y)$.

Example

Solve $(2x^2 + y)\mathrm{d}x + (x^2 y - x)\mathrm{d}y = 0$

When we do the partials, we get that $1 \neq 2xy - 1$.

So lets look at $\frac{\partial m/\partial y - \partial n/\partial x}{N}$, which is $\frac{1-(2xy-1)}{x^2 y - x} = \frac{-2}{x}$ which is just a function of x. So we have that $\mu = e^{\int -\frac{2}{x}\mathrm{d}x}$, so we don't have to look at the one in terms of y.

Doing the integral of all this gives us that $e^{-2\ln x} = x^{-2}$. So when we multiply through by x^{-2}, we get that $(2 + x^{-2}y)\mathrm{d}x + (y - x^{-1})\mathrm{d}y = 0$.

The partials are equal to each other, so this equation is now exact.

Now we find $F(x, y)$ by integrating m, so $\int(2 + x^{-2}y)\mathrm{d}x = 2x + -x^{-1}y + g(y) = F(x)$

Now we differentiation with respect to y so $\frac{\partial F}{\partial y} = y - x^{-1} = -x^{-1} + g'(y)$, so $g(y) = \frac{y^2}{2}$

The solution is therefore $2x - x^{-1}y + \frac{y^2}{2} = C$.

1.5 Substitutions and Transformations

Substitution Procedure:

- Identify the type of equation and determine the appropriate substitution or transformation
- Rewrite the original equation in terms of new variables
- Solve the transformed equation
- Express the solution in terms of the original variables

Definition: Homogeneous Equation

If the right-hand side of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

can be expressed as a function of the ratio y/x alone, then we say the equation is homogeneous.

To solve a homogeneous equation, use the substitution $v = \frac{y}{x}$; $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ to transform the equation into a separable equation.

Example

Solve $(xy + y^2 + x^2)\mathrm{d}x - x^2\mathrm{d}y = 0$.

Solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$ we get that this is equal to $\frac{-x^2-y^2-xy}{-x^2}$ and this simplifies to $1 + \left(\frac{y}{x}\right)^2 + \frac{y}{x}$.

This is equivalent to $v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v^2 + v$. We end up getting that $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2+1}{x}$ and this can be done by separation. The solution is $y = x\tan(\ln|x| + C)$ after solving.

To solve an equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = G(ax+by)$, use the substitution $z = ax+by$ to transform the equation into a separable equation.

Example

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = y - x - 1 + (x - y + 2)^{-1}$

First we have $\frac{\mathrm{d}y}{\mathrm{d}x} = -(x - y) - 1 + (x - y + 2)^{-1}$

So substituting with $z = x - y$, we have that $\frac{\mathrm{d}z}{=}1 - \frac{\mathrm{d}z}{\mathrm{d}x}$. Knowing this, the equation is equal to $1 - \frac{\mathrm{d}z}{\mathrm{d}x} = -z - 1 + (z + 2)^{-1}$. From this this simplifies to $\frac{\mathrm{d}z}{\mathrm{d}x} = z + 2 - (z + 2)^{-1}$.

So now we write this into a separable equation with $\frac{(z+2)\mathrm{d}z}{(z+2)^2-1} = \mathrm{d}x$.

Separating by parts and substituting gives $(x - y + 2)^2 = ce^{2x} + 1$

Definition: Bernoulli Equation

A first-order equation that can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

where $P(x)$ and $Q(x)$ are continuous on the interval (a, b) and n is a real number, is called a Bernoulli equation.

To solve a Bernoulli equation use the substitution $v = y^{1-n}$ to transform the equation into a linear equation.

Example

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = -\frac{5}{2}xy^3$.

From above, we have $v = y^{-2}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = -2y^{-3}\frac{\mathrm{d}u}{\mathrm{d}x}$ and $-\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}x} = y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}$.

So multiplying through by y^{-3} and substituting, we get that $-\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}x} - 5v = -\frac{5}{2}x$.

This is equal to $\frac{\mathrm{d}v}{\mathrm{d}x} + 10v = 5x$. The integrating factor here is $\mu = e^{\int P(x)\mathrm{d}x}$, which is e^{10x} in this case.

Multiplying through by μ, we get that $e^{10x}\frac{\mathrm{d}v}{\mathrm{d}x} + 10ve^{10x} = 5xe^{10}x$ and the LHS should be equal to $\frac{\mathrm{d}}{\mathrm{d}x}(e^{10x}v) = 5xe^{10x}$.

Using elementary integration techniques the answer is $y^{-2} = \frac{x}{2} - \frac{1}{20} + Ce^{-10x}$.