1 Theory of Higher-Order Linear Differential Equations

1.1 Basic Theory of Linear Differential Equations

A linear differential equation of order n is an equation that can be written in the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_0(x)y(x) = b(x)$$

where $a_0(x), a_1(x), \ldots, a_n(x)$ and $b(x)$ depend on on x, not y. When $a_0, a_1, \ldots, a_n$ are all constants, we say this equation has constant coefficients; otherwise it has variable coefficients. If $b(x) = 0$, this equation is called homogeneous; otherwwise it is nonhomogeneous.

We assume $a_0(x), a_1(x), \ldots, a_n(x)$ and $b(x)$ are all continuous on an interval I and $a_n(x) \neq 0$ on I.

We can rewrite the equation in standard form

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x)$$

where the functions $p_1(x), \ldots, p_n(x)$, and $g(x)$ are continuous on I.

Theorem 1.1

Suppose $p_1(x) \ldots p_n(x)$ and $g(x)$ are continuous on an interval (a, b) that contains the point x_0. Then, for any choice of the initial values, $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$, there exists a unique solution $y(x)$ on the whole interval (a, b) to the initial value problem

$$y^{(n)} + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x)$$

$$y(x_0) = \gamma_0, y'(x_0) = \gamma_1 \ldots y^{(n-1)}(x_0) = \gamma_{n-1}$$

Example

For the initial value problem

$$\begin{aligned} x(x-1)y''' - 3xy'' + 6x^2y' - (\cos x)y &= \sqrt{x+5} \\ y(x_0) = 1, \qquad y'(x_0) = 0, \qquad y''(x_0) &= 7 \end{aligned}$$

determine the values of x_0 and the intervals (a, b) containing x_0 for which the above theorem guarantees the existence of a unique solution on (a, b).

We know that $x \neq 0, 1$ from $x(x-1)$.

In standard form this becomes $y''' - \frac{3x}{x(x-1)}y'' + \frac{6x^2}{x(x-1)}y' - \frac{\cos x}{x(x-1)}y = \frac{\sqrt{x+5}}{x(x-1)}$.

These functions will be continuous when $x \neq 0$ and $x \neq 1$. We also know that $x \geq -5$ from the last term.

The intervals are $(-5, 0)$, $(0, 1)$ and $(1, \infty)$ in which all the x_0 can have an element from.

If we let the left-hand side of equation in the standard form define the differential operator L,

$$L[y] = \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_1 \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_n y = (D^n + p_1 D^{n-1} + \cdots + p_n)[y]$$

then the standard form equation can be expressed in the operator form

$$L[y](x) = g(x)$$

Keep in mind that L is a linear operator - that is, it satisfies

$$L[y_1 + y_2 + \cdots + y_m] = L[y_1] + L[y_2] + \cdots + L[y_m]$$

$$L[cy] = cL[y]$$

where c is any constant.

Definition: Wronksian

Let $f_1, \ldots, f_n$ be any n functions that are $(n-1)$ times differentiable.

The function

$$W[f_1, f_2, \ldots, f_n] = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronksian of $f_1, \ldots f_n$.

Theorem 1.2

Let $y_1, \ldots y_n$ be n solutions on (a, b) of

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0$$

where $p_1, \ldots, p_n$ are continuous on (a, b). If at some point x_0 in (a, b) these solutions satisfy

$$W[y_1, \ldots, y_n](x_0) \neq 0$$

then every solution of the above equation on (a, b) can be expressed in the form

$$y(x) = C_1 y_1(x) + \cdots + C_n y_n(x)$$

where $C_1, \ldots, C_n$ are constants.

Definition: Linear Dependence of Functions

The M functions $f_1, f_2, \ldots, f_m$ are said to be linearly dependent on an interval I if at least one of them can be expressed as a linear combination of the others on I; equivalently, they are linearly dependent if there exist constants $c_1, c_2, \ldots, c_m$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_m f_m(x) = 0$$

for all x in I. Otherwise, they are said to be linearly independent on I.

Example

Show that the functions $f_1(x) = e^x, f_2(x) = e^{-2x}$, and $f_3(x) = 3e^x - 2e^{-2x}$ are linearly dependent on $(-\infty, \infty)$.

We can see that $f_3 = 3f_1 - 2f_2$. We can see that f_3 is a linear combination of the other two functions.

We have a set of constants, not all zero that $c_1f_1 + c_2f_2 + c_3f_3 = 0$ from $3f_1 - 2f_2 - 1f_3$, so the set of $\{f_1, f_2, f_3\}$ is linearly dependent.

If you do the Wronksian of the functions: $W[f_1, f_2, f_3]$, we get 0 which eans that it is linearly dependent. The process of writing the Wronksian takes a lot of paper, so it is easier likely to do the $c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$ method.

To prove that functions $f_1, f_2, \ldots, f_m$ are linearly independent, a convenient approach is to assume the equation defined in the linear dependence definition holds and show that this forces $c_1 = c_2 = \cdots = c_m = 0$.

Example

Show that the functions $f_1(x) = x, f_2(x) = x^2$, and $f_3(x) = 1 - 2x^2$ are linearly independent on $(-\infty, \infty)$.

Assume $c_1f_1 + c_2f_2 + c_3f_3 = 0$. If we can show this, then we can show its independence.

From this we will get $c_1x + c_2x^2 + c_3(1 - 2x^2) = 0$.

If we let $x = 0$, we get $c_3 = 0$.

If we let $x = 1$, we get $c_1 + c_2 - c_3 = 0$ and if we let $x = -1$, we get $-c_1 + c_2 - c_3 = 0$.

From this we see that $c_1 + c_2 = 0$ and $-c_1 + c_2 = 0$.

The functions are linearly independent when c_1, c_2 and c_3 are equal to 0, so x, x^2, and $1 - 2x^2$ are linearly independent.

There are other ways to do this as well.

Theorem 1.3

If $y_1, y_2, \ldots, y_n$ are n solutions to $y^{(n)} + p_1y^{(n-1)} + \cdots + p_ny = 0$ on the interval (a, b), with $p_1, p_2, \ldots, p_n$ continuous on (a, b), then the following statements are equivalent

1. $y_1, y_2, \ldots, y_n$ are linearly dependent on (a, b).
2. The Wronksian $W[y_1, y_2, \ldots, y_n](x_0)$ is zero at some point x_0 in (a, b).
3. The Wronksian $W[y_1, y_2, \ldots, y_n](x)$ is identically zero on (a, b).

The contrapositives of these statements are also equivalent:

1. $y_1, y_2, \ldots, y_n$ are linearly independent on (a, b).
2. The Wronksian $W[y_1, y_2, \ldots, y_n](x_0)$ is nonzero at some point x_0 in (a, b).
3. The Wronksian $W[y_1, y_2, \ldots, y_n](x)$ is never zero on (a, b)

Whenever the last 3 are met, $\{y_1, y_2, \ldots, y_n\}$ is called a fundamental solution set for linear independence theorem on (a, b).

It is useful to keep in mind the following sets consist of functions that are linearly independencet on every open interval (a, b):

$$\{1, x, x^2, \ldots, x^n\}$$
$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx\}$$
$$\{e^{\alpha_1 x}, e^{\alpha_2 x}, \ldots, e^{\alpha_n x}\}$$

where α_i are distinct constants.

Theorem 1.4

Let $y_p(x)$ be a particular solution to the nonhomogeneous equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x)$$

on the interval (a, b) with $p_1, p_2, \ldots, p_n$ continuous on (a, b), and let $\{y_1, \ldots, y_n\}$ be a fundamental solution set for the corresponding homogeneous equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0$$

Then every solution of the original nonhomogeneous equation on the interval (a, b) can be expressed in the form

$$y(x) = y_p(x) + C_1y_1(x) + \cdots + C_ny_n(x)$$

Example

Find a general solution on the interval $(-\infty, \infty)$ to

$$L[y] = y''' - 2y'' - y' + 2y = 2x^2 - 2x - 4 - 24e^{-2x}$$

given that $y_{p_1}(x) = x^2$ is a particular solution to $L[y] = 2x^2 - 2x - 4, y_{p_2}(x) = e^{-2x}$ is a particular solution to $L[y] = -12e^{-12x}$, and that $y_1(x) = e^{-x}, y_2(x) = e^x$, and $y_3(x) = e^{2x}$ are solutions to the corresponding homogeneous equation.

We know that $\{e^{-x}, e^x, e^{2x}\}$ is a fundamental solution set for homogeneous equations so we have $C_1e^{-x} + C_2e^x + C_3e^{2x}$.

We know that $L[x^2] = 2x^2 - 2x - 4$ and $L[e^{-2x}] = -12e^{-2x}$. From the former, we have $L[2e^{-2x}] = -24e^{-2x}$.

We know that $Ly_p = 2x^2 - 2x - 4 - 24e^{-2x}$. We also know that $L[x^2 - 2e^{-2x}] = L[x^2] - 2L[e^{-2x}] = 2x^2 - 2x - 4 - 24e^{-2x}$.

The solution of the nonhomogeneous equation is $x^2 - 2e^{-2x}$.

The general solution is therefore $y(x) = x^2 - 2x^{-2x} + C_1e^{-2x} + C_2e^x + C_3e^{2x}$.

1.2 Homogeneous Linear Equations with Constant Coefficients

Consider the homogeneous linear nth-order differential equation with constant coefficients

$$a_ny^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_1y'(x) + a_0y(x) = 0$$

e^{rx} is a solution to the equation, provided r is a root of the auxiliary (or characteristic equation)

$$P(r) = a_nr^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0$$

Distinct real roots: If the roots $r_1, r_2, \ldots, r_n$ of the auxiliary equation are real and distinct, then the n solutions to the first equation defined are

$$y_1(x) = e^{r_1x}, \quad y_2(x) = e^{r_2x} \quad , \ldots, \quad , y_n(x) = e^{r_nx}$$

Example

Find a general solution to

$$y''' - 2y'' - 5y' + 6y = 0$$

Using the auxiliary equation we get $r^3 - 2r^2 - 5r + 6 = 0$ from this.

From algebra, we know that the possible roots are $\pm 1, \pm 2, \pm 3, \pm 6$.

Let's assume $r = 1$ is a solution. From synthetic division, we see that $r = 1$ is a root. Now we can see that $(r-1)(r^2 - r - 6)$ is a solution.

Factoring this gives $(r-1)(r-3)(r+2)$.

The general solution is $y = C_1 e^x + C_2 e^{3x} + C_3 e^{-2x}$.

Looking at complex roots: If $\alpha + i\beta(\alpha, \beta$ real) is a complex root of the auxiliary equation, then so is its complex conjugate $\alpha - i\beta$. If we accept complex-valued functions as solutions, then both $e^{(\alpha + i\beta)x}$ and $e^{(\alpha - i\beta)x}$ are solutions to the original homogeneous linear equation. The real-valued functions (which are linearly independent) corresponding to the complex roots $\alpha \pm i\beta$ are

$$e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)$$

Example

Find a general solution to

$$y''' + y'' + 3y' - 5y = 0$$

The auxiliary equation is $r^3 + r^2 + 3r - 5$.

The possible roots are $\pm 1, \pm 5$.

We know that 1 works from synthetic division and we get $(r-1)(r^2 + 2r + 5)$.

From $r^2 + 2r + 5$, we get that $-1 \pm 2i$ are the roots of this.

We get $c_1 e^x + c_2 e^{-x}\cos 2x + c_3 e^{-x}\sin 2x$.

If r_1 is a root of multiplicity m, then the m linearly independent solutions are

$$e^{r_1 x}, \quad xe^{r_1 x}, \quad x^2 e^{r_1 x}, \ldots, x^{m-1} e^{r_1} x$$

If $\alpha + i\beta$ is a repeated complex root of multiplicity m, then the $2m$ linearly independent real-valued solutions are

$$e^{\alpha x}\cos(\beta x), xe^{\alpha x}\cos(\beta x), \ldots, x^{m-1}e^{\alpha x}\cos(\beta x)$$
$$e^{\alpha x}\sin(\beta x), xe^{\alpha x}\sin(\beta x), \ldots, x^{m-1}e^{\alpha x}\sin(\beta x)$$

Example

Find a general solution to
$$y^{(4)} - y^{(3)} - 3y'' + 5y' - 2y = 0$$

The auxiliary equation is $r^4 - r^3 - 3r^2 + 5r - 2 = 0$.

The possible roots are $\pm 1, \pm 2$.

We know that $r = 1$ works from plugging in. Using synthetic division, we get $(r-1)(r^3 - 3r + 2)$.

From the $r^3 - 3r + 2$ term, we can factor this to $(r-1)(r^2 + r - 2)$.

The auxiliary equation ends up being $(r-1)^3(r+2)$.

The general solution ends up being $c_1 e^x + C_2 x e^x + C_3 x^2 e^x + C_4 e^{-2x}$.

Example

Find a general solution to
$$y^{(4)} - 8y^{(3)} + 26y'' - 40y' + 25y = 9$$

The auxiliary equation is $r^4 - 8r^3 + 26r^2 - 40r + 25 = 0$.

Let's assume we are told that $r_1 = 2 + i$ and $r_2 = 2 - i$.

This means that $(r - (2+i))(r - (2-i)) = r^2 - 4r + 5$ is a factor.

Dividing $r^4 - 8r^3 + 26r^2 - 40r + 25$ from this gives us $r^2 - 4r + 5$.

We know the roots are $2+i, 2-i, 2+i, 2-i$.

Since $2+i$ and $2-i$ have multiplicity of two, then the solution is $y = C_1 e^{2x}\cos x + C_2 e^{2x}\sin x + C_3 x e^{2x}\cos x + C_4 x e^{2x}\sin x$.

1.3 Undetermined Coefficients and the Annihilator Method

Previously we used the Method of Undetermined Coefficients to find a particular solution to a nonhomogeneous linear second-order constant coefficient equation

$$L[y] = (aD^2 + bD + c)[y] = f(x)$$

when $f(x)$ had a particular form (a product of a polynomial, an exponential, and a sinusoid) by observing a solution form y_p must resemble f. We also had to make accomodations when y_p was a solution to the homogeneous equation $L[y] = 0$.

The annihilator method uses the observation that suitable types of nonhomogeneities $f(x)$ are themselves solutions to homogeneous differential equations with constant coefficients.

1. Any nonhomogeneous term of the form $f(x) = e^{rx}$ satisfies $(D - r)[f] = 0$
2. Any nonhomogeneous term of the form $f(x) = x^k e^{rx}$ satisfies $(D-r)^m[f] = 0$ for $k = 0, 1, \ldots, m-1$.
3. Any nonhomogeneous term of the form $f(x) = \cos\beta x$ or $\sin\beta x$ satisfies $(D^2 + \beta^2)[f] = 0$
4. Any nonhomogeneous term of the form $f(x) = x^k e^{\alpha x}\cos\beta x$ or $x^k e^{\alpha x}\sin\beta x$ satisfies $[(D-\alpha)^2 + \beta^2]^m[f] = 0$ for $k = 0, 1, \ldots, m-1$.

We have that D^n annihilates polynomial of degree $n - r$.

We have that $D - r$ annihilates e^{rx}.

We have that $(D-r)^k$ annihilates $x^{k-1}e^{rx}$

We have that $D^2 - 2\alpha D + (\alpha^2 + \beta^2)$ annihilates $e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x$.

If have a power of x^{k-1} to the above, then raise the above to the power of k to annihilate this. If we just have $\cos \beta x$ or $\sin \beta x$, then the operator becomes $D^2 + \beta^2$.

Definition

A linear differential operator A is said to annihilate a function f if

$$A[f](x) = 0$$

for all x. That is, A annihilates f if f is a solution to the homogeneous linear differential equation above on $(-\infty, \infty)$.

Example

Find a differential operator that annihilates

$$6xe^{-4x} + 5e^x \sin 2x$$

We know that $(D + 4)^2$ will annihilate $6xe^{-4x}$.

We saw that the form that annihilates the other part of the equation is $D^2 - 2\alpha D + (\alpha^2 + \beta^2)$.

We know that $\alpha = 1$ and $\beta = 2$.

The operator that will annihilate that term is $D^2 - 2D + 5$, so this term annihilates $5e^x \sin 2x$.

The sum will be annihilated by multiplying $(D + 4)^2$ and $(D^2 - 2D + 5)$.

Example

Find a general solution to

$$y'' - y = xe^x + \sin x$$

Method 1: Undetermined Coefficients

The homogeneous equation is $m^2 - 1$, so the solution to the homogeneous equation is $y_c = c_1e^x + c_2e^{-x}$

The form of the particular solution looks like $y_p = (Ax + B)e^x + C\sin x + D\cos x$.

Let's find the form of xe^x first.

We have that $y_p = (Ax + B)e^x$, then the derivative is $Ae^x + (Ax + B)e^x = Axe^x + (A + B)e^x$. The second derivative is $Ae^x + Axe^x + (A + B)e^x = Axe^x + (2A + B)e^x$.

Plugging this in gives $Axe^x + (2A + B)e^x - (Ax + B)e^x = xe^x$. We end up getting $2Ae^x = xe^x$.

Because of the overlap with the homogeneous equation, the particular solution is actually $y_p = x(Ax + B)e^x = (Ax^2 + Bx)e^x$.

The first derivative of this is $(2Ax + B)e^x + (Ax^2 + Bx)e^x = [Ax^2 + (2A + B)x + B]e^x$. The second derivative is $(2Ax + 2A + B)e^x + [Ax^2 + (2A + B)x + B]e^x = [Ax^2 + (4A + B)x + (2A + 2B)]e^x$.

Plugging this in gives $[Ax^2 + (4A + B)x + (2A + 2B)]e^x - (Ax^2 + Bx)e^x = xe^x$.

Simplifying this gives $4A = 1$ and $2A + 2B = 0$. From this we get $A = 1/4$ and $B = -1/4$.

The solution for $y_p = (1/4x^2 - 1/4x)e^x = x(\frac{1}{4}x - \frac{1}{4})e^x$.

Now we need to solve the other part of y_p.

Doing derivatives and plugging in stuff we get $C = -1/2$ and $D = 0$, so $y_p = x(\frac{1}{4}x - \frac{1}{4})e^x - \frac{1}{2}\sin x$.

Therefore $y = c_1e^x + c_2e^{-x} + x(-\frac{1}{4}x - \frac{1}{4})e^x - \frac{1}{2}\sin x$

Method 2: Annihilator Method We know that $(D - 1)^2$ annihilates xe^x.

We know that for $\sin x$ the form is $D^2 - 2\alpha D + (\alpha^2 + \beta^2)$.

So $D^2 + 1$ annihilates $\sin x$.

$(D - 1)^2(D^2 + 1)$ annihilates $xe^x + \sin x$.

Rewrite the equation using differential operator notation. We end up getting $(D^2 - 1)y = xe^x + \sin x$.

This gives $(D + 1)(D - 1)y = xe^x + \sin x$. Applying $(D - 1)^2(D^2 + 1)$ to both sides, we get $(D + 1)(D - 1)^3(D^2 + 1)y = (D - 1)^2(D + 1)[xe^x + \sin x]$.

We get that $(D + 1)(D - 1)^3(D^2 + 1)y = 0$.

We would have $y = c_1e^{-x} + c_2e^x + c_3xe^x + c_4x^2e^x + c_5\sin x + c_6\cos x$ as the general solution to the homogeneous equation.

The particular solution is exactly what we got in the same form using the annihilator method.

Belpw for me later *Exercise* Find a general solution to $y''' - 3y'' + 4y = xe^2x$ pls later anastasia come back

1.4 Method Of Variation of Parameters

The method of undetermined coefficients and the annihilator method work only for linear equations with constant coefficients and when the nonhomogeneous term is a solution to some homogeneous linear equation with constant coefficients. The method of variation of parameters discussed in chapter 4 generalizes to higher-order linear equations with variable coefficients.

Our goal is to find a solution to the standard form equation

$$L[y](x) = g(x)$$

where $L[y] = y^{(n)} + p_1y^{(n-1)} + \cdots + p_ny$ and the coefficient functions $p_1, p_2, \ldots, p_n$ as well as g are continuous on (a, b).

A general solution to $L[y](x) = 0$ is $y_h(x) = C_1 y_1(x) + \cdots + C_n y_n(x)$.

In the method of variation of parameters, there exists a particular solution to the standard form equation of the form

$$y_p(x) = v_1(x)y_1(x) + \cdots + v_n(x)y_n(x)$$

The functions $v_1', v_2', \ldots, v_n'$ must satisfy the system

$$\begin{aligned} y_1 v_1' + \cdots + y_n v_n' &= 0 \\ \vdots \quad\ \ \vdots\ \vdots\ &\vdots\ \vdots \\ y_1^{(n-2)} v_1' + \cdots + y_n^{(n-2)} v_n' &= 0 \\ y_1^{(n-1)} v_1' + \cdots + y_n^{(n-1)} v_n' &= g \end{aligned}$$

Solving the system using Cramer's Rule, we find that $v_k'(x) = \frac{g(x)W_k(x)}{W[y_1,\ldots,y_n](x)}$ where $k = 1, \ldots, n$. and where $W_k(x)$ is the determinant of a matrix obtained from the Wronksian $W[y_1, \ldots, y_n](x)$ by replacing the kth column by Col$[0, \ldots, 0, 1]$.

Example

Find a general solution to the Cauchy-Euler equation

$$x^3y''' + x^2y'' - 2xy' + 2y = x^3 \sin x, \qquad x > 0$$

From Cauchy Euler, we see that $y = x^r, y' = rx^{r-1}, y'' = r(r-1)x^{r-2}, y''' = r(r-1)(r-2)x^{r-3}$.

Plugging this in the homogeneous equation gives $x^3 \cdot r(r-1)(r-2)x^{r-3} + x^2r(r-1)x^{r-2} - 2xrx^{r-1} + 2x^r = 0$.

This gives $x^r[r^3 - 3r^2 + 2r] + x^r[r^2 - r] - 2x^r[r] + 2x^r = 0$. Factoring this gives $x^r[r^3 - 3r^2 + 2r + r^2 - r - 2r + 2] = 0$.

Assuming $x \neq 0$, we get $r^3 - 2r^2 - r + 2 = 0$. Factoring this gives $(r-2)(r-1)(r+1) = 0$.

The general solution to the homogeneous equation is $y = c_1x^2 + c_2x^{-1} + c_3x$.

From above we see that $y_1 = x^2, y_2x^{-1}, y_3 = x$.

The particular solution will be of the form $y_p = v_1x^2 + v_2x^{-1} + v_3x$.

Starting with the Wronksian of x^2, x^{-1}, x.

Before, we get $g(x) = \frac{x^3 \sin x}{x^3} = \sin x$. This comes from dividing the $x^3 \sin x$ by the leading coefficient.

The next determinant for v_1 is the same as the original Wronksian above, but the first column has $0, 0, \sin x$ instead of $x^2, 2x, 2$.

The determinant for v_2 is the same as the original, but the second column is replaced by $0, 0, \sin x$ instead of $x^{-1}, -x^{-2}, 2x^{-3}$.

The determinant for v_3 is the same as the original, but the third column is replaced by $0, 0 \sin x$ instea dof $x, 1, 0$.

For the original Wronksian, we get $x(4x^{-2} + 2x^{-2}) - 1(2x^{-1} - 2x^{-1}) = 6x^{-1} = W$.

For the Wronksian of v_1, we get $\sin x(x^{-1} + x^{-1}) = 2x^{-1} \sin x$.

For the Wronksian of v_2, we get $-\sin x(x^2 - 2x^2) = x^2 \sin x$.

For the Wronksian of v_3, we get $\sin x(-1-2) = -3 \sin x$.

So we get $v_1' = \frac{W_1}{W} = \frac{2x^{-1} \sin x}{6x^{-1}} = \frac{1}{3} \sin x$

We get $v_2' = \frac{W^2}{W} = \frac{x^2 \sin x}{6x^{-1}} = \frac{1}{6}x^3 \sin x$

We get $v_3' = \frac{W^3}{W} = \frac{-3 \sin x}{6x^{-1}} = -\frac{1}{2}x \sin x$.

Integrating, we get $v_1 = -\frac{1}{3} \cos x$.

For v_3, we have $-\frac{1}{2} \int x \sin x \mathrm{d}x$. Let $u = x$ and $\mathrm{d}v = \sin x$. From this, $v = -\cos x$ and $\mathrm{d}u = 1$.

Integrating by parts should give $v_3 = \frac{1}{2}x \cos x - \frac{1}{2} \sin x$.

For v_2, we get $u = x^3$ then $3x^2, 6x, 6, 0$. and for $\mathrm{d}v$ we get $\sin x, -\cos x, \sin x, -\cos x, \sin x$.

This is tabular integration by parts. We get $v_2 = \frac{1}{6}[-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x]$.

Simplifying this gives $v_2 = -\frac{1}{6}x^3 \cos x + \frac{1}{2}x^2 \sin x + x \cos x - \sin x$.

We can now get y_p.

Yea, so y_p is simply $y_p = [-\frac{1}{3} \cos x]x^2 + [-\frac{1}{6}x^3 \cos x + \frac{1}{2}x^2 \sin x + x \cos x - \sin x]x^{-1} + [\frac{1}{2}x \cos x - \frac{1}{2} \sin x]x$.

Simplifying this gives $\cos x - x^{-1} \sin x$.

So the general solution is $y = \cos x - x^{-1} \sin x + c_1x^2 + c_2x^{-1} + c_3x$.