CHAPTER 1 Introduction

1.1 Background

In the sciences and engineering, mathematical models are developed to aid in the understanding of physical phenomena. These models often yield an equation that contains some derivatives of an unknown function. Such an equation is called a **differential equation.** Two examples of models developed in calculus are the free fall of a body and the decay of a radioactive substance.

In the case of free fall, an object is released from a certain height above the ground and falls under the force of gravity.† Newton's second law, which states that an object's mass times its acceleration equals the total force acting on it, can be applied to the falling object. This leads to the equation (see Figure 1.1)

$$m\frac{d^2h}{dt^2} = -mg\,,$$

where m is the mass of the object, h is the height above the ground, d^2h/dt^2 is its acceleration, g is the (constant) gravitational acceleration, and $-mg$ is the force due to gravity. This is a differential equation containing the second derivative of the unknown height h as a function of time.

Fortunately, the above equation is easy to solve for h. All we have to do is divide by m and integrate twice with respect to t. That is,

$$\frac{d^2h}{dt^2} = -g\,,$$

so

$$\frac{dh}{dt} = -gt + c_1$$

and

$$h = h(t) = \frac{-gt^2}{2} + c_1t + c_2\,.$$

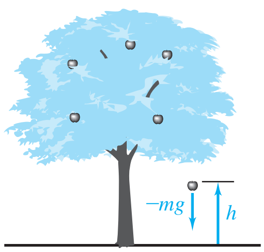


Figure 1.1 Apple in free fall

†We are assuming here that gravity is the *only* force acting on the object and that this force is constant. More general models would take into account other forces, such as air resistance.

We will see that the constants of integration, c_1 and c_2, are determined if we know the *initial* height and the *initial* velocity of the object. We then have a formula for the height of the object at time t.

In the case of radioactive decay (Figure 1.2), we begin from the premise that the rate of decay is proportional to the amount of radioactive substance present. This leads to the equation

$$\frac{dA}{dt} = -kA\,, \qquad k > 0\,,$$

where $A(>0)$ is the unknown amount of radioactive substance present at time t and k is the proportionality constant. To solve this differential equation, we rewrite it in the form

$$\frac{1}{A}dA = -k\,dt$$

and integrate to obtain

$$\int \frac{1}{A}dA = \int -k\,dt$$

$$\ln A + C_1 = -kt + C_2\,.$$

Solving for A yields

$$A = A(t) = e^{\ln A} = e^{-kt}e^{C_2 - C_1} = Ce^{-kt},$$

where C is the combination of integration constants $e^{C_2 - C_1}$. The value of C, as we will see later, is determined if the *initial* amount of radioactive substance is given. We then have a formula for the amount of radioactive substance at any future time t.

Even though the above examples were easily solved by methods learned in calculus, they do give us some insight into the study of differential equations in general. First, notice that the solution of a differential equation is a *function,* like $h(t)$ or $A(t)$, not merely a number. Second, integration† is an important tool in solving differential equations (not surprisingly!). Third, we cannot expect to get a unique solution to a differential equation, since there will be arbitrary "constants of integration." The second derivative d^2h/dt^2 in the free-fall equation gave rise to two constants, c_1 and c_2, and the first derivative in the decay equation gave rise, ultimately, to one constant, C.

Whenever a mathematical model involves the **rate of change** of one variable with respect to another, a differential equation is apt to appear. Unfortunately, in contrast to the examples for free fall and radioactive decay, the differential equation may be very complicated and difficult to analyze.

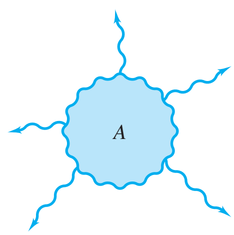


Figure 1.2 Radioactive decay

†For a review of integration techniques, see Appendix A.

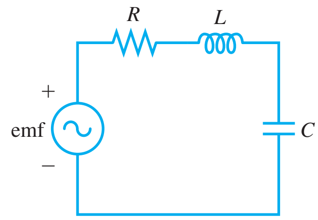


Figure 1.3 Schematic for a series RLC circuit

Differential equations arise in a variety of subject areas, including not only the physical sciences but also such diverse fields as economics, medicine, psychology, and operations research. We now list a few specific examples.

1. In banking practice, if $P(t)$ is the number of dollars in a savings bank account that pays a yearly interest rate of $r\%$ *compounded continuously*, then P satisfies the differential equation

(1) $$\frac{dP}{dt} = \frac{r}{100}P\,, \quad t \text{ in years.}$$

2. A classic application of differential equations is found in the study of an electric circuit consisting of a resistor, an inductor, and a capacitor driven by an electromotive force (see Figure 1.3). Here an application of Kirchhoff's laws† leads to the equation

(2) $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)\,,$$

where L is the inductance, R is the resistance, C is the capacitance, $E(t)$ is the electromotive force, $q(t)$ is the charge on the capacitor, and t is the time.

3. In psychology, one model of the learning of a task involves the equation

(3) $$\frac{dy/dt}{y^{3/2}(1-y)^{3/2}} = \frac{2p}{\sqrt{n}}\,.$$

Here the variable y represents the learner's skill level as a function of time t. The constants p and n depend on the individual learner and the nature of the task.

4. In the study of vibrating strings and the propagation of waves, we find the *partial* differential equation

(4) $$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0\,,^{\ddagger}$$

where t represents time, x the location along the string, c the wave speed, and u the displacement of the string, *which is a function of time and location*.

†We will discuss Kirchhoff's laws in Section 3.5.

‡*Historical Footnote:* This partial differential equation was first discovered by Jean le Rond d'Alembert (1717–1783) in 1747.

To begin our study of differential equations, we need some common terminology. If an equation involves the derivative of one variable with respect to another, then the former is called a **dependent variable** and the latter an **independent variable.** Thus, in the equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0 , \tag{5}$$

t is the independent variable and x is the dependent variable. We refer to a and k as **coefficients** in equation (5). In the equation

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x - 2y , \tag{6}$$

x and y are independent variables and u is the dependent variable.

A differential equation involving only ordinary derivatives with respect to a single independent variable is called an **ordinary differential equation.** A differential equation involving partial derivatives with respect to more than one independent variable is a **partial differential equation.** Equation (5) is an ordinary differential equation, and equation (6) is a partial differential equation.

The **order** of a differential equation is the order of the highest-order derivatives present in the equation. Equation (5) is a second-order equation because d^2x/dt^2 is the highest-order derivative present. Equation (6) is a first-order equation because only first-order partial derivatives occur.

It will be useful to classify ordinary differential equations as being either linear or nonlinear. Remember that lines (in two dimensions) and planes (in three dimensions) are especially easy to visualize, when compared to nonlinear objects such as cubic curves or quadric surfaces. For example, all the points on a line can be found if we know just two of them. Correspondingly, *linear* differential equations are more amenable to solution than nonlinear ones. Observe that the equations for lines $ax + by = c$ and planes $ax + by + cz = d$ have the feature that the variables appear in *additive combinations of their first powers only*. By analogy a **linear differential equation** is one in which the dependent variable y and its derivatives appear in additive combinations of their first powers.

More precisely, a differential equation is **linear** if it has the format

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x) , \tag{7}$$

where $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $F(x)$ depend only on the independent variable x. The additive combinations are permitted to have multipliers (coefficients) that depend on x; no restrictions are made on the nature of this x-dependence. If an ordinary differential equation is not linear, then we call it **nonlinear.** For example,

$$\frac{d^2y}{dx^2} + y^3 = 0$$

is a nonlinear second-order ordinary differential equation because of the y^3 term, whereas

$$t^3\frac{dx}{dt} = t^3 + x$$

is linear (despite the t^3 terms). The equation

$$\frac{d^2y}{dx^2} - y\frac{dy}{dx} = \cos x$$

is nonlinear because of the $y\, dy/dx$ term.

Although the majority of equations one is likely to encounter in practice fall into the *nonlinear* category, knowing how to deal with the simpler linear equations is an important first step (just as tangent lines help our understanding of complicated curves by providing local approximations).

1.1 EXERCISES

In Problems 1–12, a differential equation is given along with the field or problem area in which it arises. Classify each as an ordinary differential equation (ODE) or a partial differential equation (PDE), give the order, and indicate the independent and dependent variables. If the equation is an ordinary differential equation, indicate whether the equation is linear or nonlinear.

1. $5\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 9x = 2\cos 3t$

(mechanical vibrations, electrical circuits, seismology)

2. $\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$

(Hermite's equation, quantum-mechanical harmonic oscillator)

3. $\dfrac{dy}{dx} = \dfrac{y(2-3x)}{x(1-3y)}$

(competition between two species, ecology)

4. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(Laplace's equation, potential theory, electricity, heat, aerodynamics)

5. $y\left[1 + \left(\dfrac{dy}{dx}\right)^2\right] = C$, where C is a constant

(brachistochrone problem,† calculus of variations)

6. $\dfrac{dx}{dt} = k(4-x)(1-x)$, where k is a constant

(chemical reaction rates)

7. $\dfrac{dp}{dt} = kp(P-p)$, where k and P are constants

(logistic curve, epidemiology, economics)

8. $\sqrt{1-y}\,\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 0$

(Kidder's equation, flow of gases through a porous medium)

9. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0$

(aerodynamics, stress analysis)

10. $8\dfrac{d^4y}{dx^4} = x(1-x)$

(deflection of beams)

11. $\dfrac{\partial N}{\partial t} = \dfrac{\partial^2 N}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial N}{\partial r} + kN$, where k is a constant

(nuclear fission)

12. $\dfrac{d^2y}{dx^2} - 0.1(1-y^2)\dfrac{dy}{dx} + 9y = 0$

(van der Pol's equation, triode vacuum tube)

In Problems 13–16, write a differential equation that fits the physical description.

13. The rate of change of the population p of bacteria at time t is proportional to the population at time t.

14. The velocity at time t of a particle moving along a straight line is proportional to the fourth power of its position x.

15. The rate of change in the temperature T of coffee at time t is proportional to the difference between the temperature M of the air at time t and the temperature of the coffee at time t.

16. The rate of change of the mass A of salt at time t is proportional to the square of the mass of salt present at time t.

17. **Drag Race.** Two drivers, Alison and Kevin, are participating in a drag race. Beginning from a standing start, they each proceed with a constant acceleration. Alison covers the last $1/4$ of the distance in 3 seconds, whereas Kevin covers the last $1/3$ of the distance in 4 seconds. Who wins and by how much time?

†*Historical Footnote:* In 1630 Galileo formulated the brachistochrone problem ($\beta\rho\acute{\alpha}\chi\acute{\iota}\sigma\tau o\varsigma$ = shortest, $\chi\rho\acute{o}\nu o\varsigma$ = time), that is, to determine a path down which a particle will fall from one given point to another in the shortest time. It was reproposed by John Bernoulli in 1696 and solved by him the following year.

1.2 Solutions and Initial Value Problems

An nth-order ordinary differential equation is an equality relating the independent variable to the nth derivative (and usually lower-order derivatives as well) of the dependent variable. Examples are

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x^3 \text{ (second-order, } x \text{ independent, } y \text{ dependent)}$$

$$\sqrt{1-\left(\frac{d^2y}{dt^2}\right)}-y=0 \text{ (second-order, } t \text{ independent, } y \text{ dependent)}$$

$$\frac{d^4x}{dt^4}=xt \text{ (fourth-order, } t \text{ independent, } x \text{ dependent).}$$

Thus, a general form for an nth-order equation with x independent, y dependent, can be expressed as

$$\text{(1)} \qquad F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^ny}{dx^n}\right)=0\,,$$

where F is a function that depends on x, y, and the derivatives of y up to order n; that is, on x, y, . . . , d^ny/dx^n. We assume that the equation holds for all x in an open interval I ($a<x<b$, where a or b could be infinite). In many cases we can isolate the highest-order term d^ny/dx^n and write equation (1) as

$$\text{(2)} \qquad \frac{d^ny}{dx^n}=f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right),$$

which is often preferable to (1) for theoretical and computational purposes.

Explicit Solution

Definition 1. A function $\phi(x)$ that when substituted for y in equation (1) [or (2)] satisfies the equation for all x in the interval I is called an **explicit solution** to the equation on I.

Example 1 Show that $\phi(x)=x^2-x^{-1}$ is an explicit solution to the linear equation

$$\text{(3)} \qquad \frac{d^2y}{dx^2}-\frac{2}{x^2}y=0\,,$$

but $\psi(x)=x^3$ is not.

Solution The functions $\phi(x)=x^2-x^{-1}$, $\phi'(x)=2x+x^{-2}$, and $\phi''(x)=2-2x^{-3}$ are defined for all $x\neq 0$. Substitution of $\phi(x)$ for y in equation (3) gives

$$(2-2x^{-3})-\frac{2}{x^2}(x^2-x^{-1})=(2-2x^{-3})-(2-2x^{-3})=0\,.$$

Since this is valid for any $x \neq 0$, the function $\phi(x) = x^2 - x^{-1}$ is an explicit solution to (3) on $(-\infty, 0)$ and also on $(0, \infty)$.

For $\psi(x) = x^3$ we have $\psi'(x) = 3x^2$, $\psi''(x) = 6x$, and substitution into (3) gives

$$6x - \frac{2}{x^2}x^3 = 4x = 0\,,$$

which is valid only at the point $x = 0$ and not on an interval. Hence $\psi(x)$ is not a solution. ◆

Example 2 Show that for *any* choice of the constants c_1 and c_2, the function

$$\phi(x) = c_1 e^{-x} + c_2 e^{2x}$$

is an explicit solution to the linear equation

$$y'' - y' - 2y = 0\,. \tag{4}$$

Solution We compute $\phi'(x) = -c_1 e^{-x} + 2c_2 e^{2x}$ and $\phi''(x) = c_1 e^{-x} + 4c_2 e^{2x}$. Substitution of ϕ, ϕ', and ϕ'' for y, y', and y'' in equation (4) yields

$$\begin{aligned}&(c_1 e^{-x} + 4c_2 e^{2x}) - (-c_1 e^{-x} + 2c_2 e^{2x}) - 2(c_1 e^{-x} + c_2 e^{2x})\\ &\quad = (c_1 + c_1 - 2c_1)e^{-x} + (4c_2 - 2c_2 - 2c_2)e^{2x} = 0\,.\end{aligned}$$

Since equality holds for all x in $(-\infty, \infty)$, then $\phi(x) = c_1 e^{-x} + c_2 e^{2x}$ is an explicit solution to (4) on the interval $(-\infty, \infty)$ for any choice of the constants c_1 and c_2. ◆

As we will see in Chapter 2, the methods for solving differential equations do not always yield an explicit solution for the equation. We may have to settle for a solution that is defined implicitly. Consider the following example.

Example 3 Show that the relation

$$y^2 - x^3 + 8 = 0 \tag{5}$$

implicitly defines a solution to the nonlinear equation

$$\frac{dy}{dx} = \frac{3x^2}{2y} \tag{6}$$

on the interval $(2, \infty)$.

Solution When we solve (5) for y, we obtain $y = \pm\sqrt{x^3 - 8}$. Let's try $\phi(x) = \sqrt{x^3 - 8}$ to see if it is an explicit solution. Since $d\phi/dx = 3x^2/\left(2\sqrt{x^3 - 8}\right)$, both ϕ and $d\phi/dx$ are defined on $(2, \infty)$. Substituting them into (6) yields

$$\frac{3x^2}{2\sqrt{x^3 - 8}} = \frac{3x^2}{2\left(\sqrt{x^3 - 8}\right)}\,,$$

which is indeed valid for all x in $(2, \infty)$. [You can check that $\psi(x) = -\sqrt{x^3 - 8}$ is also an explicit solution to (6).] ◆

Implicit Solution

Definition 2. A relation $G(x, y) = 0$ is said to be an **implicit solution** to equation (1) on the interval I if it defines one or more explicit solutions on I.

Example 4 Show that

$$x + y + e^{xy} = 0 \tag{7}$$

is an implicit solution to the nonlinear equation

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0\,. \tag{8}$$

Solution First, we observe that we are unable to solve (7) directly for y in terms of x alone. However, for (7) to hold, we realize that any change in x requires a change in y, so we expect the relation (7) to define implicitly at least one function $y(x)$. This is difficult to show directly but can be rigorously verified using the **implicit function theorem**† of advanced calculus, which guarantees that such a function $y(x)$ exists that is also differentiable (see Problem 30).

Once we know that y is a differentiable function of x, we can use the technique of implicit differentiation. Indeed, from (7) we obtain on differentiating with respect to x and applying the product and chain rules,

$$\frac{d}{dx}(x + y + e^{xy}) = 1 + \frac{dy}{dx} + e^{xy}\left(y + x\frac{dy}{dx}\right) = 0$$

or

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0\,,$$

which is identical to the differential equation (8). Thus, relation (7) is an implicit solution on some interval guaranteed by the implicit function theorem. ◆

Example 5 Verify that for every constant C the relation $4x^2 - y^2 = C$ is an implicit solution to

$$y\frac{dy}{dx} - 4x = 0\,. \tag{9}$$

Graph the solution curves for $C = 0, \pm 1, \pm 4$. (We call the collection of all such solutions a *one-parameter family of solutions*.)

Solution When we implicitly differentiate the equation $4x^2 - y^2 = C$ with respect to x, we find

$$8x - 2y\frac{dy}{dx} = 0\,,$$

†See *Vector Calculus*, 6th ed, by J. E. Marsden and A. J. Tromba (Freeman, San Francisco, 2013).

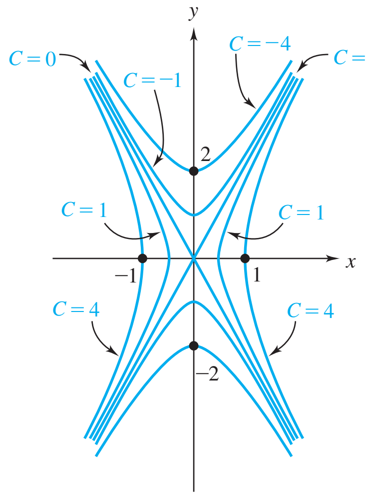


Figure 1.4 Implicit solutions $4x^2 - y^2 = C$

which is equivalent to (9). In Figure 1.4 we have sketched the implicit solutions for $C = 0, \pm 1, \pm 4$. The curves are hyperbolas with common asymptotes $y = \pm 2x$. Notice that the implicit solution curves (with C arbitrary) fill the entire plane and are nonintersecting for $C \neq 0$. For $C = 0$, the implicit solution gives rise to the two explicit solutions $y = 2x$ and $y = -2x$, both of which pass through the origin. ◆

For brevity we hereafter use the term *solution* to mean either an explicit or an implicit solution.

In the beginning of Section 1.1, we saw that the solution of the *second*-order free-fall equation invoked two arbitrary constants of integration c_1, c_2:

$$h(t) = \frac{-gt^2}{2} + c_1 t + c_2 ,$$

whereas the solution of the *first*-order radioactive decay equation contained a single constant C:

$$A(t) = Ce^{-kt} .$$

It is clear that integration of the simple *fourth*-order equation

$$\frac{d^4y}{dx^4} = 0$$

brings in four undetermined constants:

$$y(x) = c_1 x^3 + c_2 x^2 + c_3 x + c_4 .$$

It will be shown later in the text that in general the methods for solving nth-order differential equations evoke n arbitrary constants. In most cases, we will be able to evaluate these constants if we know n initial values $y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)$.

Initial Value Problem

Definition 3. By an **initial value problem** for an nth-order differential equation

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0\,,$$

we mean: Find a solution to the differential equation on an interval I that satisfies at x_0 the n initial conditions

$$\begin{aligned} y(x_0) &= y_0\,, \\ \frac{dy}{dx}(x_0) &= y_1\,, \\ &\vdots \\ \frac{d^{n-1}y}{dx^{n-1}}(x_0) &= y_{n-1}\,, \end{aligned}$$

where $x_0 \in I$ and $y_0, y_1, \ldots, y_{n-1}$ are given constants.

In the case of a first-order equation, the initial conditions reduce to the single requirement

$$y(x_0) = y_0\,,$$

and in the case of a second-order equation, the initial conditions have the form

$$y(x_0) = y_0\,, \qquad \frac{dy}{dx}(x_0) = y_1\,.$$

The terminology *initial conditions* comes from mechanics, where the independent variable x represents *time* and is customarily symbolized as t. Then if t_0 is the starting time, $y(t_0) = y_0$ represents the initial location of an object and $y'(t_0)$ gives its initial velocity.

Example 6 Show that $\phi(x) = \sin x - \cos x$ is a solution to the initial value problem

$$\textbf{(10)} \qquad \frac{d^2y}{dx^2} + y = 0\,; \qquad y(0) = -1\,, \qquad \frac{dy}{dx}(0) = 1\,.$$

Solution Observe that $\phi(x) = \sin x - \cos x$, $d\phi/dx = \cos x + \sin x$, and $d^2\phi/dx^2 = -\sin x + \cos x$ are all defined on $(-\infty, \infty)$. Substituting into the differential equation gives

$$(-\sin x + \cos x) + (\sin x - \cos x) = 0\,,$$

which holds for all $x \in (-\infty, \infty)$. Hence, $\phi(x)$ is a solution to the differential equation in (10) on $(-\infty, \infty)$. When we check the initial conditions, we find

$$\begin{aligned} \phi(0) &= \sin 0 - \cos 0 = -1\,, \\ \frac{d\phi}{dx}(0) &= \cos 0 + \sin 0 = 1\,, \end{aligned}$$

which meets the requirements of (10). Therefore, $\phi(x)$ is a solution to the given initial value problem. ◆

Example 7 As shown in Example 2, the function $\phi(x) = c_1e^{-x} + c_2e^{2x}$ is a solution to

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

for any choice of the constants c_1 and c_2. Determine c_1 and c_2 so that the initial conditions

$$y(0) = 2 \quad \text{and} \quad \frac{dy}{dx}(0) = -3$$

are satisfied.

Solution To determine the constants c_1 and c_2, we first compute $d\phi/dx$ to get $d\phi/dx = -c_1e^{-x} + 2c_2e^{2x}$. Substituting in our initial conditions gives the following system of equations:

$$\begin{cases} \phi(0) & = c_1e^0 + c_2e^0 = 2 \ , \\ \dfrac{d\phi}{dx}(0) & = -c_1e^0 + 2c_2e^0 = -3 \ , \end{cases} \qquad \text{or} \qquad \begin{cases} c_1 + c_2 & = 2 \ , \\ -c_1 + 2c_2 & = -3 \ . \end{cases}$$

Adding the last two equations yields $3c_2 = -1$, so $c_2 = -1/3$. Since $c_1 + c_2 = 2$, we find $c_1 = 7/3$. Hence, the solution to the initial value problem is $\phi(x) = (7/3)e^{-x} - (1/3)e^{2x}$. ◆

We now state an existence and uniqueness theorem for first-order initial value problems. We presume the differential equation has been cast into the format

$$\frac{dy}{dx} = f(x, y) \ .$$

Of course, the right-hand side, $f(x, y)$, must be well defined at the starting value x_0 for x and at the stipulated initial value $y_0 = y(x_0)$ for y. The hypotheses of the theorem, moreover, require *continuity* of both f and $\partial f/\partial y$ for x in some interval $a < x < b$ containing x_0, and for y in some interval $c < y < d$ containing y_0. Notice that the set of points in the xy-plane that satisfy $a < x < b$ and $c < y < d$ constitutes a *rectangle*. Figure 1.5 on page 12 depicts this "rectangle of continuity" with the initial point (x_0, y_0) in its interior and a sketch of a portion of the solution curve contained therein.

Existence and Uniqueness of Solution

Theorem 1. Consider the initial value problem

$$\frac{dy}{dx} = f(x, y) \ , \qquad y(x_0) = y_0 \ .$$

If f and $\partial f/\partial y$ are continuous functions in some rectangle

$$R = \{(x, y): a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.†

†We remark that the continuity of f alone in such a rectangle is enough to guarantee the existence of a solution to the initial value problem in some open interval containing x_0, but uniqueness may not hold (see Example 9).

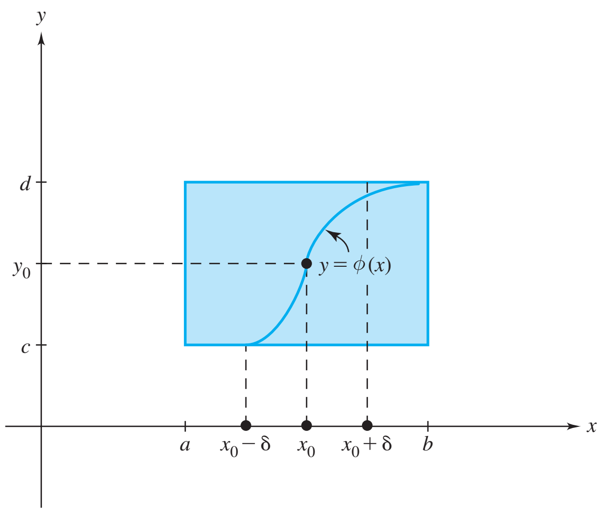


Figure 1.5 Layout for the existence–uniqueness theorem

The preceding theorem tells us two things. First, when an equation satisfies the hypotheses of Theorem 1, we are assured that a solution to the initial value problem exists. Naturally, it is desirable to know whether the equation we are trying to solve actually has a solution before we spend too much time trying to solve it. Second, when the hypotheses are satisfied, there is a **unique** solution to the initial value problem. This uniqueness tells us that if we can find a solution, then it is the *only* solution for the initial value problem. Graphically, the theorem says that there is only one solution curve that passes through the point (x_0, y_0). In other words, for this first-order equation, two solutions cannot cross anywhere in the rectangle. Notice that the existence and uniqueness of the solution holds only in *some* neighborhood $(x_0 - \delta, x_0 + \delta)$. Unfortunately, the theorem does not tell us the span (2δ) of this neighborhood (merely that it is not zero). Problem 18 elaborates on this feature.

Problem 19 gives an example of an equation with no solution. Problem 29 displays an initial value problem for which the solution is not unique. Of course, the hypotheses of Theorem 1 are not met for these cases.

When initial value problems are used to model physical phenomena, many practitioners tacitly presume the conclusions of Theorem 1 to be valid. Indeed, for the initial value problem to be a reasonable model, we certainly expect it to have a solution, since physically "something does happen." Moreover, the solution should be unique in those cases when repetition of the experiment under identical conditions yields the same results.[†]

The proof of Theorem 1 involves converting the initial value problem into an integral equation and then using Picard's method to generate a sequence of successive approximations that converge to the solution. The conversion to an integral equation and Picard's method are discussed in Project A at the end of this chapter. A detailed discussion and proof of the theorem are given in Chapter 13.[‡]

[†]At least this is the case when we are considering a deterministic model, as opposed to a probabilistic model.

[‡]All references to Chapters 11–13 refer to the expanded text, *Fundamentals of Differential Equations and Boundary Value Problems,* 7th ed.

Example 8 For the initial value problem

$$3\frac{dy}{dx} = x^2 - xy^3 \;, \qquad y(1) = 6 \;, \tag{11}$$

does Theorem 1 imply the existence of a unique solution?

Solution Dividing by 3 to conform to the statement of the theorem, we identify $f(x, y)$ as $(x^2 - xy^3)/3$ and $\partial f/\partial y$ as $-xy^2$. Both of these functions are continuous in any rectangle containing the point (1, 6), so the hypotheses of Theorem 1 are satisfied. It then follows from the theorem that the initial value problem (11) has a unique solution in an interval about $x = 1$ of the form $(1 - \delta, 1 + \delta)$, where δ is some positive number. ◆

Example 9 For the initial value problem

$$\frac{dy}{dx} = 3y^{2/3} \;, \qquad y(2) = 0 \;, \tag{12}$$

does Theorem 1 imply the existence of a unique solution?

Solution Here $f(x, y) = 3y^{2/3}$ and $\partial f/\partial y = 2y^{-1/3}$. Unfortunately $\partial f/\partial y$ is not continuous or even defined when $y = 0$. Consequently, there is no rectangle containing $(2, 0)$ in which both f and $\partial f/\partial y$ are continuous. Because the hypotheses of Theorem 1 do not hold, we cannot use Theorem 1 to determine whether the initial value problem does or does not have a unique solution. It turns out that this initial value problem has more than one solution. We refer you to Problem 29 and Project G of Chapter 2 for the details. ◆

In Example 9 suppose the initial condition is changed to $y(2) = 1$. Then, since f and $\partial f/\partial y$ are continuous in any rectangle that contains the point $(2, 1)$ but does not intersect the x-axis—say, $R = \{(x, y): 0 < x < 10, 0 < y < 5\}$—it follows from Theorem 1 that this *new* initial value problem has a unique solution in some interval about $x = 2$.

1.2 EXERCISES

1. (a) Show that $\phi(x) = x^2$ is an explicit solution to
$$x\frac{dy}{dx} = 2y$$
on the interval $(-\infty, \infty)$.

(b) Show that $\phi(x) = e^x - x$ is an explicit solution to
$$\frac{dy}{dx} + y^2 = e^{2x} + (1 - 2x)e^x + x^2 - 1$$
on the interval $(-\infty, \infty)$.

(c) Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to $x^2d^2y/dx^2 = 2y$ on the interval $(0, \infty)$.

2. (a) Show that $y^2 + x - 3 = 0$ is an implicit solution to $dy/dx = -1/(2y)$ on the interval $(-\infty, 3)$.

(b) Show that $xy^3 - xy^3 \sin x = 1$ is an implicit solution to
$$\frac{dy}{dx} = \frac{(x\cos x + \sin x - 1)y}{3(x - x\sin x)}$$
on the interval $(0, \pi/2)$.

In Problems 3–8, determine whether the given function is a solution to the given differential equation.

3. $y = \sin x + x^2 \,,\qquad \dfrac{d^2y}{dx^2} + y = x^2 + 2$

4. $x = 2\cos t - 3\sin t \,,\qquad x'' + x = 0$

5. $\theta = 2e^{3t} - e^{2t} \,,\qquad \dfrac{d^2\theta}{dt^2} - \theta\dfrac{d\theta}{dt} + 3\theta = -2e^{2t}$

6. $x = \cos 2t\,, \qquad \dfrac{dx}{dt} + tx = \sin 2t$

7. $y = e^{2x} - 3e^{-x}\,, \qquad \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$

8. $y = 3\sin 2x + e^{-x}\,, \qquad y'' + 4y = 5e^{-x}$

In Problems 9–13, determine whether the given relation is an implicit solution to the given differential equation. Assume that the relationship does define y implicitly as a function of x and use implicit differentiation.

9. $x^2 + y^2 = 4\,, \qquad \dfrac{dy}{dx} = \dfrac{x}{y}$

10. $y - \ln y = x^2 + 1\,, \qquad \dfrac{dy}{dx} = \dfrac{2xy}{y-1}$

11. $e^{xy} + y = x - 1\,, \qquad \dfrac{dy}{dx} = \dfrac{e^{-xy} - y}{e^{-xy} + x}$

12. $x^2 - \sin(x+y) = 1\,, \qquad \dfrac{dy}{dx} = 2x\sec(x+y) - 1$

13. $\sin y + xy - x^3 = 2\,,$

$$y'' = \frac{6xy' + (y')^3\sin y - 2(y')^2}{3x^2 - y}$$

14. Show that $\phi(x) = c_1 \sin x + c_2 \cos x$ is a solution to $d^2y/dx^2 + y = 0$ for any choice of the constants c_1 and c_2. Thus, $c_1 \sin x + c_2 \cos x$ is a two-parameter family of solutions to the differential equation.

15. Verify that $\phi(x) = 2/(1 - ce^x)$, where c is an arbitrary constant, is a one-parameter family of solutions to

$$\frac{dy}{dx} = \frac{y(y-2)}{2}\,.$$

Graph the solution curves corresponding to $c = 0$, ± 1, ± 2 using the same coordinate axes.

16. Verify that $x^2 + cy^2 = 1$, where c is an arbitrary nonzero constant, is a one-parameter family of implicit solutions to

$$\frac{dy}{dx} = \frac{xy}{x^2 - 1}$$

and graph several of the solution curves using the same coordinate axes.

17. Show that $\phi(x) = Ce^{3x} + 1$ is a solution to $dy/dx - 3y = -3$ for any choice of the constant C. Thus, $Ce^{3x} + 1$ is a one-parameter family of solutions to the differential equation. Graph several of the solution curves using the same coordinate axes.

18. Let $c > 0$. Show that the function $\phi(x) = (c^2 - x^2)^{-1}$ is a solution to the initial value problem $dy/dx = 2xy^2$, $y(0) = 1/c^2$, on the interval $-c < x < c$. Note that this solution becomes unbounded as x approaches $\pm c$. Thus, the solution exists on the interval $(-\delta, \delta)$ with $\delta = c$, but not for larger δ. This illustrates that in Theorem 1 the existence interval can be quite small (if c is small) or quite large (if c is large). Notice also that there is no clue from the equation $dy/dx = 2xy^2$ itself, or from the initial value, that the solution will "blow up" at $x = \pm c$.

19. Show that the equation $(dy/dx)^2 + y^2 + 4 = 0$ has no (real-valued) solution.

20. Determine for which values of m the function $\phi(x) = e^{mx}$ is a solution to the given equation.

(a) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 5y = 0$

(b) $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 0$

21. Determine for which values of m the function $\phi(x) = x^m$ is a solution to the given equation.

(a) $3x^2\dfrac{d^2y}{dx^2} + 11x\dfrac{dy}{dx} - 3y = 0$

(b) $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 5y = 0$

22. Verify that the function $\phi(x) = c_1e^x + c_2e^{-2x}$ is a solution to the linear equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

for any choice of the constants c_1 and c_2. Determine c_1 and c_2 so that each of the following initial conditions is satisfied.

(a) $y(0) = 2\,, \quad y'(0) = 1$
(b) $y(1) = 1\,, \quad y'(1) = 0$

In Problems 23–28, determine whether Theorem 1 implies that the given initial value problem has a unique solution.

23. $\dfrac{dy}{dx} = y^4 - x^4\,, \qquad y(0) = 7$

24. $\dfrac{dy}{dt} - ty = \sin^2 t\,, \qquad y(\pi) = 5$

25. $3x\dfrac{dx}{dt} + 4t = 0\,, \qquad x(2) = -\pi$

26. $\dfrac{dx}{dt} + \cos x = \sin t\,, \qquad x(\pi) = 0$

27. $y\dfrac{dy}{dx} = x\,, \qquad y(1) = 0$

28. $\dfrac{dy}{dx} = 3x - \sqrt[3]{y-1}\,, \qquad y(2) = 1$

29. **(a)** For the initial value problem (12) of Example 9, show that $\phi_1(x) \equiv 0$ and $\phi_2(x) = (x-2)^3$ are solutions. Hence, this initial value problem has multiple solutions. (See also Project G in Chapter 2.)

(b) Does the initial value problem $y' = 3y^{2/3}$, $y(0) = 10^{-7}$, have a unique solution in a neighborhood of $x = 0$?

30. Implicit Function Theorem. *Let $G(x, y)$ have continuous first partial derivatives in the rectangle $R = \{(x, y): a < x < b, c < y < d\}$ containing the point (x_0, y_0). If $G(x_0, y_0) = 0$ and the partial derivative $G_y(x_0, y_0) \neq 0$, then there exists a differentiable function $y = \phi(x)$, defined in some interval $I = (x_0 - \delta, x_0 + \delta)$, that satisfies $G(x, \phi(x)) = 0$ for all $x \in I$.*

The implicit function theorem gives conditions under which the relationship $G(x, y) = 0$ defines y implicitly as a function of x. Use the implicit function theorem to show that the relationship $x + y + e^{xy} = 0$, given in Example 4, defines y implicitly as a function of x near the point $(0, -1)$.

31. Consider the equation of Example 5,

$$y\frac{dy}{dx} - 4x = 0 \, . \tag{13}$$

(a) Does Theorem 1 imply the existence of a unique solution to (13) that satisfies $y(x_0) = 0$?

(b) Show that when $x_0 \neq 0$, equation (13) can't possibly have a solution in a neighborhood of $x = x_0$ that satisfies $y(x_0) = 0$.

(c) Show that there are two distinct solutions to (13) satisfying $y(0) = 0$ (see Figure 1.4 on page 9).

1.3 Direction Fields

The existence and uniqueness theorem discussed in Section 1.2 certainly has great value, but it stops short of telling us anything about the *nature* of the solution to a differential equation. For practical reasons we may need to know the value of the solution at a certain point, or the intervals where the solution is increasing, or the points where the solution attains a maximum value. Certainly, knowing an explicit representation (a formula) for the solution would be a considerable help in answering these questions. However, for many of the differential equations that we are likely to encounter in real-world applications, it will be impossible to find such a formula. Moreover, even if we are lucky enough to obtain an implicit solution, using this relationship to determine an explicit form may be difficult. Thus, we must rely on other methods to analyze or approximate the solution.

One technique that is useful in visualizing (graphing) the solutions to a first-order differential equation is to sketch the direction field for the equation. To describe this method, we need to make a general observation. Namely, a first-order equation

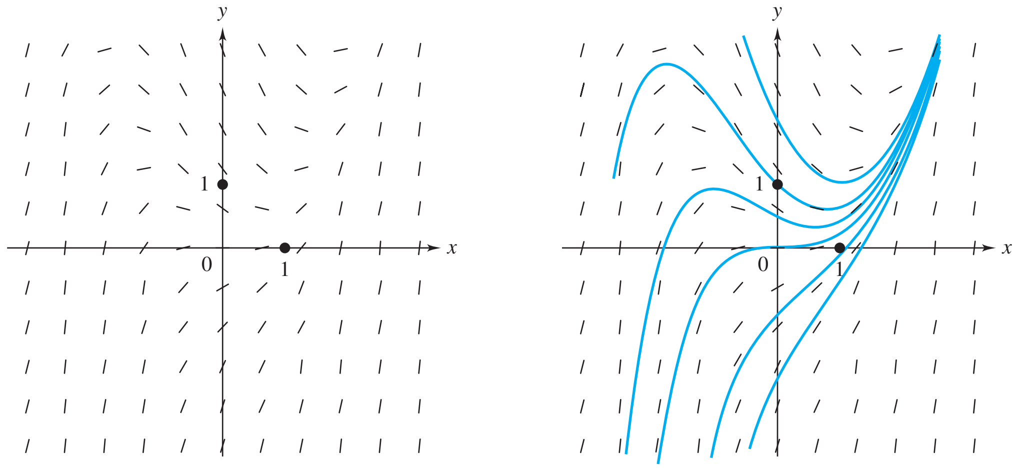
$$\frac{dy}{dx} = f(x, y)$$

specifies a slope at each point in the xy-plane where f is defined. In other words, it gives the direction that a graph of a solution to the equation must have at each point. Consider, for example, the equation

$$\frac{dy}{dx} = x^2 - y \, . \tag{1}$$

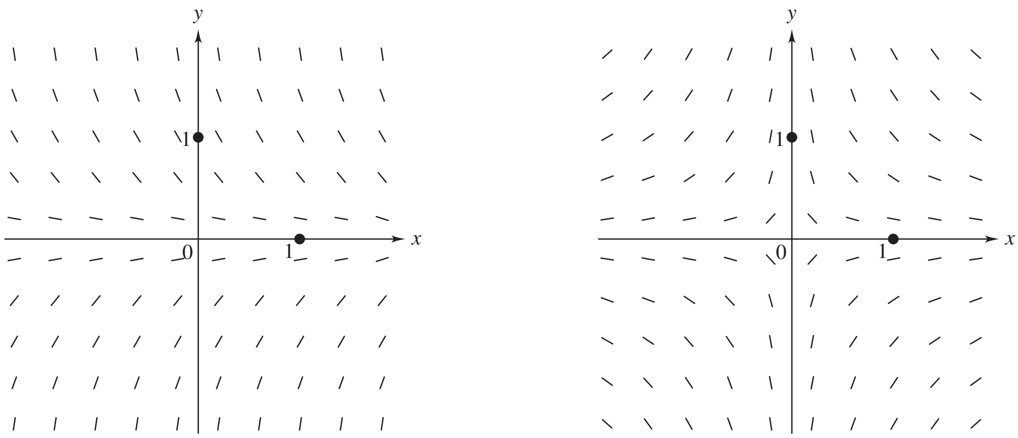
The graph of a solution to (1) that passes through the point $(-2, 1)$ must have slope $(-2)^2 - 1 = 3$ at that point, and a solution through $(-1, 1)$ has zero slope at that point.

A plot of short line segments drawn at various points in the xy-plane showing the slope of the solution curve there is called a **direction field** for the differential equation. Because the direction field gives the "flow of solutions," it facilitates the drawing of any particular solution (such as the solution to an initial value problem). In Figure 1.6(a) on page 16 we have sketched the direction field for equation (1) and in Figure 1.6(b) we have drawn several solution curves in color.



(a) (b)

Figure 1.6 (a) Direction field for $dy/dx = x^2 - y$ (b) Solutions to $dy/dx = x^2 - y$



(a) $\dfrac{dy}{dx} = -2y$ (b) $\dfrac{dy}{dx} = -\dfrac{y}{x}$

Figure 1.7 (a) Direction field for $dy/dx = -2y$ (b) Direction field for $dy/dx = -y/x$

Some other interesting direction field patterns are displayed in Figure 1.7. Depicted in Figure 1.7(a) is the pattern for the radioactive decay equation $dy/dx = -2y$ (recall that in Section 1.1 we analyzed this equation in the form $dA/dt = -kA$). From the flow patterns, we can see that all solutions tend asymptotically to the positive x-axis as x gets larger. In other words, any material decaying according to this law eventually dwindles to practically nothing. This is consistent with the solution formula we derived earlier,

$$A = Ce^{-kt}, \quad \text{or} \quad y = Ce^{-2x}.$$

From the direction field in Figure 1.7(b), we can anticipate that all solutions to $dy/dx = -y/x$ also approach the x-axis as x approaches infinity (plus or minus infinity, in

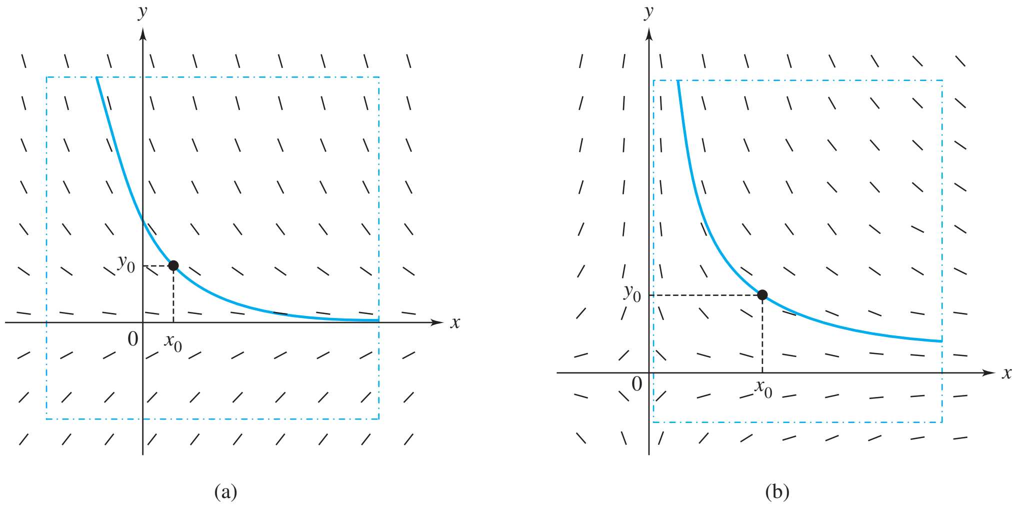


Figure 1.8 (a) A solution for $dy/dx = -2y$ (b) A solution for $dy/dx = -y/x$

fact). But more interesting is the observation that no solution can make it across the y-axis; $|y(x)|$ "blows up" as x goes to zero from either direction. Exception: On close examination, it appears the function $y(x) \equiv 0$ might just make it through this barrier. As a matter of fact, in Problem 19 you are invited to show that the solutions to this differential equation are given by $y = C/x$, with C an arbitrary constant. So they do diverge at $x = 0$, unless $C = 0$.

Let's interpret the existence–uniqueness theorem of Section 1.2 for these direction fields. For Figure 1.7(a), where $dy/dx = f(x, y) = -2y$, we select a starting point x_0 and an initial value $y(x_0) = y_0$, as in Figure 1.8(a). Because the right-hand side $f(x, y) = -2y$ is continuously differentiable for all x and y, we can enclose any initial point (x_0, y_0) in a "rectangle of continuity." We conclude that the equation has one and only one solution curve passing through (x_0, y_0), as depicted in the figure.

For the equation

$$\frac{dy}{dx} = f(x, y) = -\frac{y}{x},$$

the right-hand side $f(x, y) = -y/x$ does not meet the continuity conditions when $x = 0$ (i.e., for points on the y-axis). However, for any *nonzero* starting value x_0 and *any* initial value $y(x_0) = y_0$, we *can* enclose (x_0, y_0) in a rectangle of continuity that excludes the y-axis, as in Figure 1.8(b). Thus, we can be assured of one and only one solution curve passing through such a point.

The direction field for the equation

$$\frac{dy}{dx} = 3y^{2/3}$$

is intriguing because Example 9 of Section 1.2 showed that the hypotheses of Theorem 1 do *not* hold in any rectangle enclosing the initial point $x_0 = 2$, $y_0 = 0$. Indeed, Problem 29 of that section demonstrated the violation of uniqueness by exhibiting *two* solutions, $y(x) \equiv 0$ and $y(x) = (x - 2)^3$, passing through $(2, 0)$. Figure 1.9(a) on page 18 displays this direction field, and Figure 1.9(b) demonstrates how both solution curves can successfully "negotiate" this flow pattern.

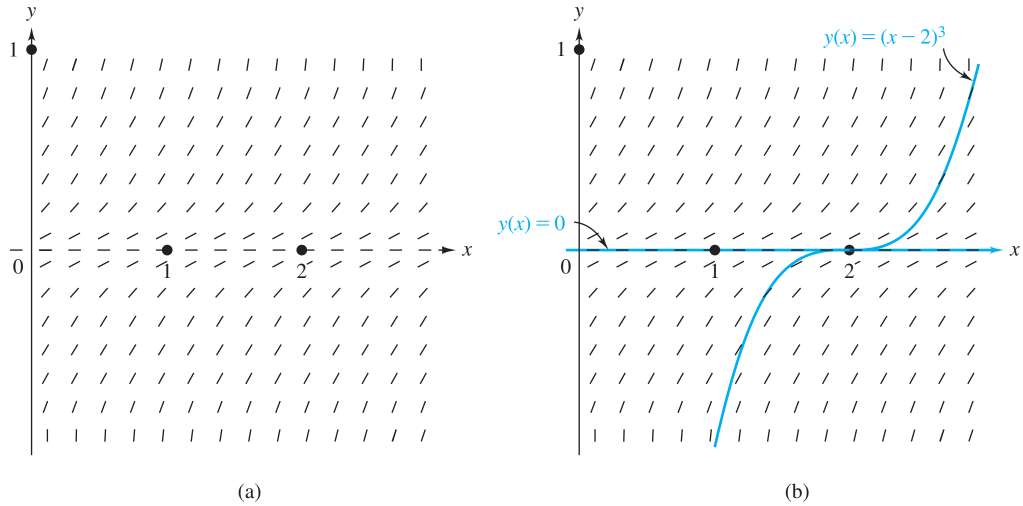


Figure 1.9 (a) Direction field for $dy/dx = 3y^{2/3}$ (b) Solutions for $dy/dx = 3y^{2/3}$, $y(2) = 0$

Clearly, a sketch of the direction field of a first-order differential equation can be helpful in visualizing the solutions. However, such a sketch is not sufficient to enable us to trace, unambiguously, the solution curve passing through a given initial point (x_0, y_0). If we tried to trace one of the solution curves in Figure 1.6(b) on page 16, for example, we could easily "slip" over to an adjacent curve. For nonunique situations like that in Figure 1.9(b), as one negotiates the flow along the curve $y = (x-2)^3$ and reaches the inflection point, one cannot decide whether to turn or to (literally) go off on the tangent $(y = 0)$.

Example 1 The *logistic equation* for the population p (in thousands) at time t of a certain species is given by

$$\frac{dp}{dt} = p(2-p)\ . \tag{2}$$

(Of course, p is nonnegative. The interpretation of the terms in the logistic equation is discussed in Section 3.2.) From the direction field sketched in Figure 1.10 on page 19, answer the following:

(a) If the initial population is 3000 $[$that is, $p(0) = 3]$, what can you say about the limiting population $\lim_{t\to+\infty} p(t)$?

(b) Can a population of 1000 ever decline to 500?

(c) Can a population of 1000 ever increase to 3000?

Solution

(a) The direction field indicates that all solution curves $[$other than $p(t) \equiv 0]$ will approach the horizontal line $p = 2$ as $t \to +\infty$; that is, this line is an asymptote for all positive solutions. Thus, $\lim_{t\to+\infty} p(t) = 2$.

(b) The direction field further shows that populations greater than 2000 will steadily decrease, whereas those less than 2000 will increase. In particular, a population of 1000 can never decline to 500.

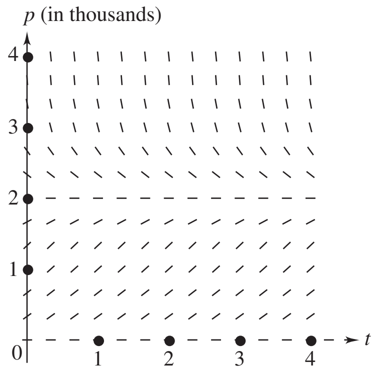


Figure 1.10 Direction field for logistic equation

(c) As mentioned in part (b), a population of 1000 will increase with time. But the direction field indicates it can never reach 2000 or any larger value; i.e., the solution curve cannot cross the line $p = 2$. Indeed, the constant function $p(t) \equiv 2$ is a solution to equation (2), and the uniqueness part of Theorem 1, page 11, precludes intersections of solution curves. ◆

Notice that the direction field in Figure 1.10 has the nice feature that the slopes do not depend on t; that is, the slopes are the same along each horizontal line. The same is true for Figures 1.8(a) and 1.9. This is the key property of so-called **autonomous equations** $y' = f(y)$, where the right-hand side is a function of the dependent variable only. Project B, page 33, investigates such equations in more detail.

Hand sketching the direction field for a differential equation is often tedious. Fortunately, several software programs have been developed to obviate this task†. When hand sketching is necessary, however, the **method of isoclines** can be helpful in reducing the work.

The Method of Isoclines

An **isocline** for the differential equation

$$y' = f(x, y)$$

is a set of points in the xy-plane where all the solutions have the same slope dy/dx; thus, it is a level curve for the function $f(x, y)$. For example, if

$$y' = f(x, y) = x + y\,, \tag{3}$$

the isoclines are simply the curves (straight lines) $x + y = c$ or $y = -x + c$. Here c is an arbitrary constant. But c can be interpreted as the numerical value of the slope dy/dx of every solution curve as it crosses the isocline. (Note that c is not the slope of the isocline itself; the latter is, obviously, -1.) Figure 1.11(a) on page 20 depicts the isoclines for equation (3).

†Appendix G describes various web sites and commercial software that sketch direction fields and automate most of the differential equation algorithms discussed in this book.

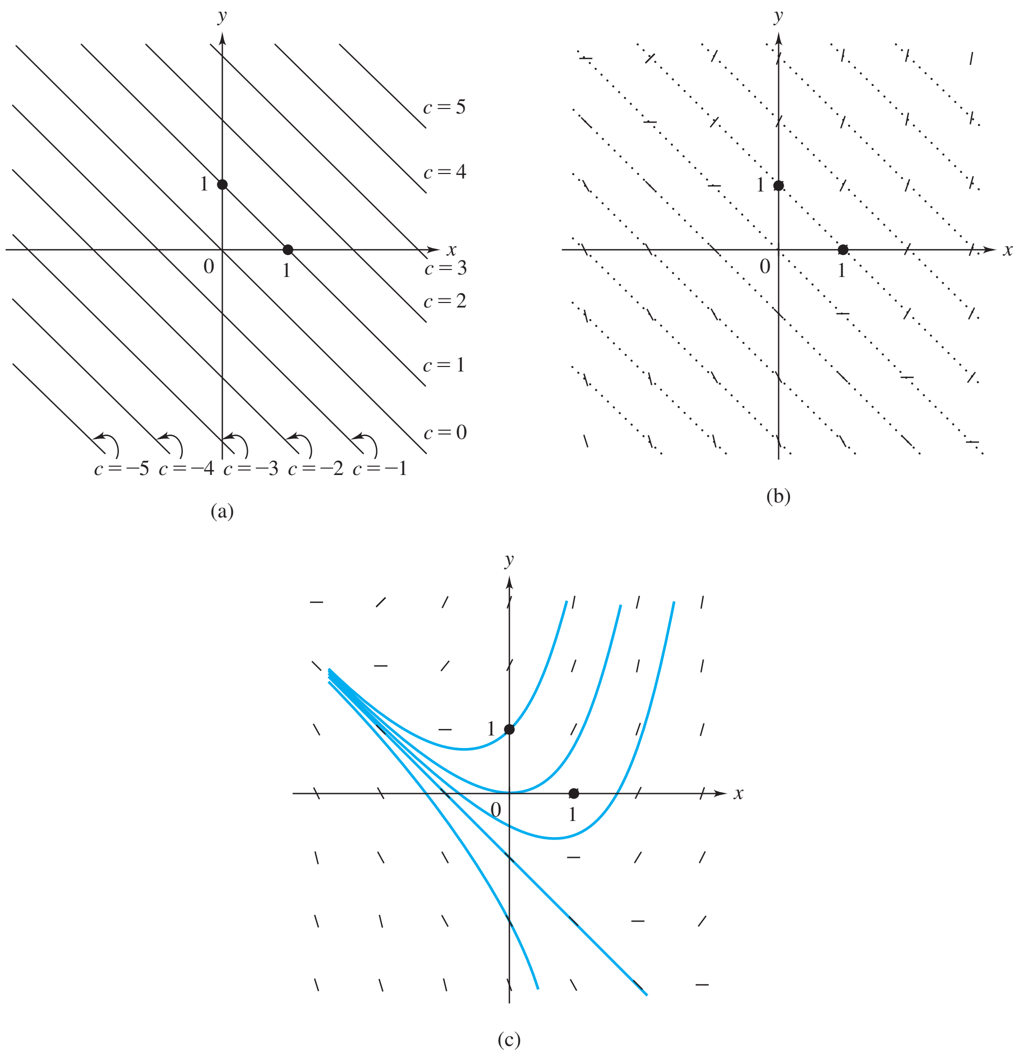


Figure 1.11 (a) Isoclines for $y' = x + y$ (b) Direction field for $y' = x + y$ (c) Solutions to $y' = x + y$

To implement the method of isoclines for sketching direction fields, we draw hash marks with slope c along the isocline $f(x, y) = c$ for a few selected values of c. If we then erase the underlying isocline curves, the hash marks constitute a part of the direction field for the differential equation. Figure 1.11(b) depicts this process for the isoclines shown in Figure 1.11(a), and Figure 1.11(c) displays some solution curves.

Remark. The isoclines themselves are not always straight lines. For equation (1) at the beginning of this section (page 15), they are parabolas $x^2 - y = c$. When the isocline curves are complicated, this method is not practical.

1.3 EXERCISES

1. The direction field for $dy/dx = 4x/y$ is shown in Figure 1.12.
   (a) Verify that the straight lines $y = \pm 2x$ are solution curves, provided $x \neq 0$.
   (b) Sketch the solution curve with initial condition $y(0) = 2$.
   (c) Sketch the solution curve with initial condition $y(2) = 1$.
   (d) What can you say about the behavior of the above solutions as $x \to +\infty$? How about $x \to -\infty$?

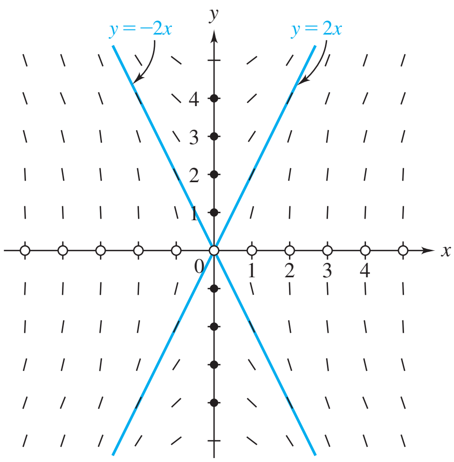


Figure 1.12 Direction field for $dy/dx = 4x/y$

2. The direction field for $dy/dx = 2x + y$ is shown in Figure 1.13.
   (a) Sketch the solution curve that passes through $(0, -2)$. From this sketch, write the equation for the solution.

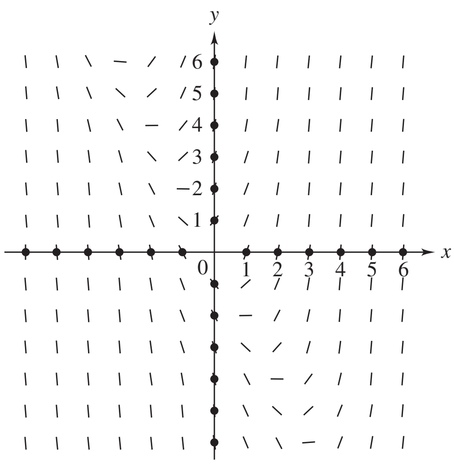


Figure 1.13 Direction field for $dy/dx = 2x + y$

   (b) Sketch the solution curve that passes through $(-1, 3)$.
   (c) What can you say about the solution in part (b) as $x \to +\infty$? How about $x \to -\infty$?

3. A model for the velocity v at time t of a certain object falling under the influence of gravity in a viscous medium is given by the equation

$$\frac{dv}{dt} = 1 - \frac{v}{8}.$$

From the direction field shown in Figure 1.14, sketch the solutions with the initial conditions $v(0) = 5$, 8, and 15. Why is the value $v = 8$ called the "terminal velocity"?

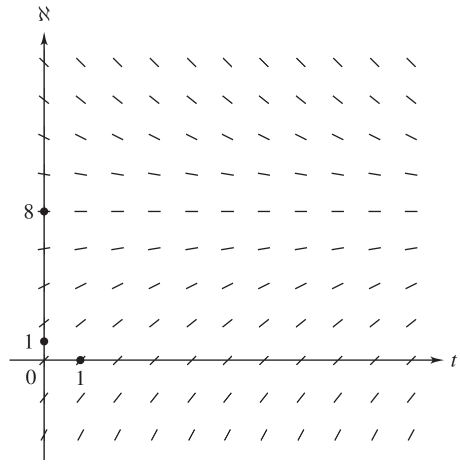


Figure 1.14 Direction field for $\frac{dv}{dt} = 1 - \frac{v}{8}$

4. If the viscous force in Problem 3 is nonlinear, a possible model would be provided by the differential equation

$$\frac{dv}{dt} = 1 - \frac{v^3}{8}.$$

Redraw the direction field in Figure 1.14 to incorporate this v^3 dependence. Sketch the solutions with initial conditions $v(0) = 0$, 1, 2, 3. What is the terminal velocity in this case?

5. The logistic equation for the population (in thousands) of a certain species is given by

$$\frac{dp}{dt} = 3p - 2p^2.$$

   (a) Sketch the direction field by using either a computer software package or the method of isoclines.

(b) If the initial population is 3000 [that is, $p(0) = 3$], what can you say about the limiting population $\lim_{t\to+\infty} p(t)$?
(c) If $p(0) = 0.8$, what is $\lim_{t\to+\infty} p(t)$?
(d) Can a population of 2000 ever decline to 800?

6. Consider the differential equation

$$\frac{dy}{dx} = x + \sin y .$$

(a) A solution curve passes through the point $(1, \pi/2)$. What is its slope at this point?
(b) Argue that every solution curve is increasing for $x > 1$.
(c) Show that the second derivative of every solution satisfies

$$\frac{d^2y}{dx^2} = 1 + x\cos y + \frac{1}{2}\sin 2y .$$

(d) A solution curve passes through $(0, 0)$. Prove that this curve has a relative minimum at $(0, 0)$.

7. Consider the differential equation

$$\frac{dp}{dt} = p(p-1)(2-p)$$

for the population p (in thousands) of a certain species at time t.
(a) Sketch the direction field by using either a computer software package or the method of isoclines.
(b) If the initial population is 4000 [that is, $p(0) = 4$], what can you say about the limiting population $\lim_{t\to+\infty} p(t)$?
(c) If $p(0) = 1.7$, what is $\lim_{t\to+\infty} p(t)$?
(d) If $p(0) = 0.8$, what is $\lim_{t\to+\infty} p(t)$?
(e) Can a population of 900 ever increase to 1100?

8. The motion of a set of particles moving along the x-axis is governed by the differential equation

$$\frac{dx}{dt} = t^3 - x^3 ,$$

where $x(t)$ denotes the position at time t of the particle.
(a) If a particle is located at $x = 1$ when $t = 2$, what is its velocity at this time?
(b) Show that the acceleration of a particle is given by

$$\frac{d^2x}{dt^2} = 3t^2 - 3t^3x^2 + 3x^5 .$$

(c) If a particle is located at $x = 2$ when $t = 2.5$, can it reach the location $x = 1$ at any later time? [*Hint*: $t^3 - x^3 = (t-x)(t^2 + xt + x^2)$.]

9. Let $\phi(x)$ denote the solution to the initial value problem

$$\frac{dy}{dx} = x - y, \qquad y(0) = 1 .$$

(a) Show that $\phi''(x) = 1 - \phi'(x) = 1 - x + \phi(x)$.
(b) Argue that the graph of ϕ is decreasing for x near zero and that as x increases from zero, $\phi(x)$ decreases until it crosses the line $y = x$, where its derivative is zero.
(c) Let x^* be the abscissa of the point where the solution curve $y = \phi(x)$ crosses the line $y = x$. Consider the sign of $\phi''(x^*)$ and argue that ϕ has a relative minimum at x^*.
(d) What can you say about the graph of $y = \phi(x)$ for $x > x^*$?
(e) Verify that $y = x - 1$ is a solution to $dy/dx = x - y$ and explain why the graph of $\phi(x)$ always stays above the line $y = x - 1$.
(f) Sketch the direction field for $dy/dx = x - y$ by using the method of isoclines or a computer software package.
(g) Sketch the solution $y = \phi(x)$ using the direction field in part (f).

10. Use a computer software package to sketch the direction field for the following differential equations. Sketch some of the solution curves.
(a) $dy/dx = \sin x$
(b) $dy/dx = \sin y$
(c) $dy/dx = \sin x \sin y$
(d) $dy/dx = x^2 + 2y^2$
(e) $dy/dx = x^2 - 2y^2$

In Problems 11–16, draw the isoclines with their direction markers and sketch several solution curves, including the curve satisfying the given initial conditions.

11. $dy/dx = -x/y$, $\quad y(0) = 4$
12. $dy/dx = y$, $\quad y(0) = 1$
13. $dy/dx = 2x$, $\quad y(0) = -1$
14. $dy/dx = x/y$, $\quad y(0) = -1$
15. $dy/dx = 2x^2 - y$, $\quad y(0) = 0$
16. $dy/dx = x + 2y$, $\quad y(0) = 1$

17. From a sketch of the direction field, what can one say about the behavior as x approaches $+\infty$ of a solution to the following?

$$\frac{dy}{dx} = 3 - y + \frac{1}{x}$$

18. From a sketch of the direction field, what can one say about the behavior as x approaches $+\infty$ of a solution to the following?

$$\frac{dy}{dx} = -y$$

19. By rewriting the differential equation $dy/dx = -y/x$ in the form

$$\frac{1}{y}dy = \frac{-1}{x}dx$$

integrate both sides to obtain the solution $y = C/x$ for an arbitrary constant C.

20. A bar magnet is often modeled as a magnetic dipole with one end labeled the north pole N and the opposite end labeled the south pole S. The magnetic field for the magnetic dipole is symmetric with respect to rotation about the axis passing lengthwise through the center of the bar. Hence we can study the magnetic field by restricting ourselves to a plane with the bar magnet centered on the x-axis.

For a point P that is located a distance r from the origin, where r is much greater than the length of the magnet, the **magnetic field lines** satisfy the differential equation

$$\frac{dy}{dx} = \frac{3xy}{2x^2 - y^2} \tag{4}$$

and the **equipotential lines** satisfy the equation

$$\frac{dy}{dx} = \frac{y^2 - 2x^2}{3xy} \,. \tag{5}$$

(a) Show that the two families of curves are perpendicular where they intersect. [*Hint:* Consider the slopes of the tangent lines of the two curves at a point of intersection.]

(b) Sketch the direction field for equation (4) for $-5 \le x \le 5$, $-5 \le y \le 5$. You can use a software package to generate the direction field or use the method of isoclines. The direction field should remind you of the experiment where iron filings are sprinkled on a sheet of paper that is held above a bar magnet. The iron filings correspond to the hash marks.

(c) Use the direction field found in part (b) to help sketch the magnetic field lines that are solutions to (4).

(d) Apply the statement of part (a) to the curves in part (c) to sketch the equipotential lines that are solutions to (5). The magnetic field lines and the equipotential lines are examples of *orthogonal trajectories*. (See Problem 32 in Exercises 2.4, page 65.)[†]

1.4 The Approximation Method of Euler

Euler's method (or the tangent-line method) is a procedure for constructing approximate solutions to an initial value problem for a first-order differential equation

$$y' = f(x, y)\,, \qquad y(x_0) = y_0\,. \tag{1}$$

It could be described as a "mechanical" or "computerized" implementation of the informal procedure for hand sketching the solution curve from a picture of the direction field. As such, we will see that it remains subject to the failing that it may skip across solution curves. However, under fairly general conditions, iterations of the procedure do converge to true solutions.

The method is illustrated in Figure 1.15 on page 24. Starting at the initial point (x_0, y_0), we follow the straight line with slope $f(x_0, y_0)$, the tangent line, for some distance to the point (x_1, y_1). Then we reset the slope to the value $f(x_1, y_1)$ and follow this line to (x_2, y_2). In this way we construct polygonal (broken line) approximations to the solution. As we take smaller spacings between points (and thus employ more points), we may expect to converge to the true solution.

To be more precise, assume that the initial value problem (1) has a unique solution $\phi(x)$ in some interval centered at x_0. Let h be a fixed positive number (called the *step size*) and consider the equally spaced points[‡]

$$x_n := x_0 + nh\,, \qquad n = 0, 1, 2, \ldots\,.$$

[†] Equations (4) and (5) can be solved using the method for homogeneous equations in Section 2.6 (see Exercises 2.6, Problem 46).

[‡] The symbol $:=$ means "is defined to be."

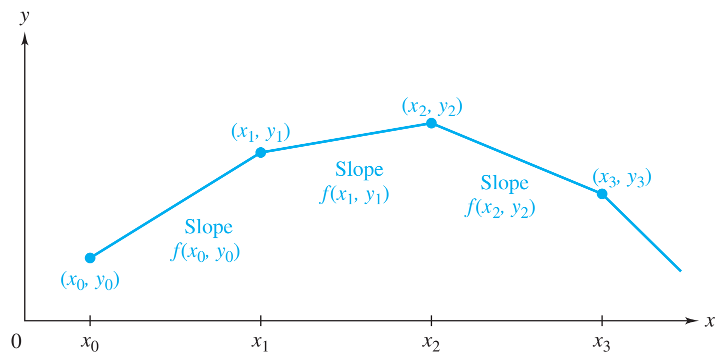


Figure 1.15 Polygonal-line approximation given by Euler's method

The construction of values y_n that approximate the solution values $\phi(x_n)$ proceeds as follows. At the point (x_0, y_0), the slope of the solution to (1) is given by $dy/dx = f(x_0, y_0)$. Hence, the tangent line to the solution curve at the initial point (x_0, y_0) is

$$y = y_0 + (x - x_0)f(x_0, y_0) .$$

Using this tangent line to approximate $\phi(x)$, we find that for the point $x_1 = x_0 + h$

$$\phi(x_1) \approx y_1 \coloneqq y_0 + hf(x_0, y_0) .$$

Next, starting at the point (x_1, y_1), we construct the line with slope given by the direction field at the point (x_1, y_1)—that is, with slope equal to $f(x_1, y_1)$. If we follow this line[†] [namely, $y = y_1 + (x - x_1)f(x_1, y_1)$] in stepping from x_1 to $x_2 = x_1 + h$, we arrive at the approximation

$$\phi(x_2) \approx y_2 \coloneqq y_1 + hf(x_1, y_1) .$$

Repeating the process (as illustrated in Figure 1.15), we get

$$\phi(x_3) \approx y_3 \coloneqq y_2 + hf(x_2, y_2) ,$$
$$\phi(x_4) \approx y_4 \coloneqq y_3 + hf(x_3, y_3) , \qquad \text{etc.}$$

This simple procedure is **Euler's method** and can be summarized by the recursive formulas

(2) $$x_{n+1} = x_n + h ,$$

(3) $$y_{n+1} = y_n + hf(x_n, y_n) , \qquad n = 0, 1, 2, \ldots .$$

[†]Because y_1 is an *approximation* to $\phi(x_1)$, we cannot assert that this line is tangent to the solution curve $y = \phi(x)$.

Example 1 Use Euler's method with step size $h = 0.1$ to approximate the solution to the initial value problem

$$y' = x\sqrt{y}, \qquad y(1) = 4 \tag{4}$$

at the points $x = 1.1, 1.2, 1.3, 1.4$, and 1.5.

Solution Here $x_0 = 1$, $y_0 = 4$, $h = 0.1$, and $f(x, y) = x\sqrt{y}$. Thus, the recursive formula (3) for y_n is

$$y_{n+1} = y_n + hf(x_n, y_n) = y_n + (0.1)x_n\sqrt{y_n}\,.$$

Substituting $n = 0$, we get

$$x_1 = x_0 + 0.1 = 1 + 0.1 = 1.1\,,$$
$$y_1 = y_0 + (0.1)x_0\sqrt{y_0} = 4 + (0.1)(1)\sqrt{4} = 4.2\,.$$

Putting $n = 1$ yields

$$x_2 = x_1 + 0.1 = 1.1 + 0.1 = 1.2\,,$$
$$y_2 = y_1 + (0.1)x_1\sqrt{y_1} = 4.2 + (0.1)(1.1)\sqrt{4.2} \approx 4.42543\,.$$

Continuing in this manner, we obtain the results listed in Table 1.1. For comparison we have included the exact value (to five decimal places) of the solution $\phi(x) = (x^2 + 7)^2/16$ to (4), which can be obtained using separation of variables (see Section 2.2). As one might expect, the approximation deteriorates as x moves farther away from 1. ◆

TABLE 1.1 Computations for $y' = x\sqrt{y}$, $y(1) = 4$

n	x_n	Euler's Method	Exact Value
0	1	4	4
1	1.1	4.2	4.21276
2	1.2	4.42543	4.45210
3	1.3	4.67787	4.71976
4	1.4	4.95904	5.01760
5	1.5	5.27081	5.34766

Given the initial value problem (1) and a *specific* point x, how can Euler's method be used to approximate $\phi(x)$? Starting at x_0, we can take one giant step that lands on x, or we can take several smaller steps to arrive at x. If we wish to take N steps, then we set $h = (x - x_0)/N$ so that the step size h and the number of steps N are related in a specific way. For example, if $x_0 = 1.5$ and we wish to approximate $\phi(2)$ using 10 steps, then we would take $h = (2 - 1.5)/10 = 0.05$. It is expected that the more steps we take, the better will be the approximation. (But keep in mind that more steps mean more computations and hence greater accumulated roundoff error.)

Example 2 Use Euler's method to find approximations to the solution of the initial value problem

$$y' = y\,, \qquad y(0) = 1 \tag{5}$$

at $x = 1$, taking 1, 2, 4, 8, and 16 steps.

Remark. Observe that the solution to (5) is just $\phi(x) = e^x$, so Euler's method will generate algebraic approximations to the transcendental number $e = 2.71828\ldots$.

Solution Here $f(x, y) = y$, $x_0 = 0$, and $y_0 = 1$. The recursive formula for Euler's method is

$$y_{n+1} = y_n + hy_n = (1 + h)y_n\,.$$

To obtain approximations at $x = 1$ with N steps, we take the step size $h = 1/N$. For $N = 1$, we have

$$\phi(1) \approx y_1 = (1 + 1)(1) = 2\,.$$

For $N = 2$, $\phi(x_2) = \phi(1) \approx y_2$. In this case we get

$$\begin{aligned} y_1 &= (1 + 0.5)(1) = 1.5\,, \\ \phi(1) \approx y_2 &= (1 + 0.5)(1.5) = 2.25\,. \end{aligned}$$

For $N = 4$, $\phi(x_4) = \phi(1) \approx y_4$, where

$$\begin{aligned} y_1 &= (1 + 0.25)(1) = 1.25\,, \\ y_2 &= (1 + 0.25)(1.25) = 1.5625\,, \\ y_3 &= (1 + 0.25)(1.5625) = 1.95313\,, \\ \phi(1) \approx y_4 &= (1 + 0.25)(1.95313) = 2.44141\,. \end{aligned}$$

(In the above computations, we have rounded to five decimal places.) Similarly, taking $N = 8$ and 16, we obtain even better estimates for $\phi(1)$. These approximations are shown in Table 1.2. For comparison, Figure 1.16 on page 27 displays the polygonal-line approximations to e^x using Euler's method with $h = 1/4$ $(N = 4)$ and $h = 1/8$ $(N = 8)$. Notice that the smaller step size yields the better approximation. ◆

TABLE 1.2 Euler's Method for $y' = y$, $y(0) = 1$

N	h	Approximation for $\phi(1) = e$
1	1.0	2.0
2	0.5	2.25
4	0.25	2.44141
8	0.125	2.56578
16	0.0625	2.63793

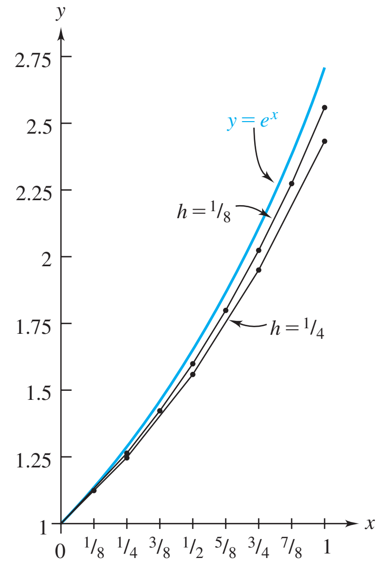


Figure 1.16 Approximations of e^x using Euler's method with $h = 1/4$ and $1/8$

How good (or bad) is Euler's method? In judging a numerical scheme, we must begin with two fundamental questions. Does the method converge? And, if so, what is the rate of convergence? These important issues are discussed in Section 3.6, where improvements in Euler's method are introduced (see also Problems 12 and 13 of this section).

Example 3 Suppose $v(t)$ satisfies the initial value problem

$$\frac{dv}{dt} = -3 - 2v^2\,, \quad v(0) = 2\,.$$

By experimenting with Euler's method, determine to within one decimal place (± 0.1) the value of $v(0.2)$ and the time it will take $v(t)$ to reach zero.

Solution Determining rigorous estimates of the accuracy of the answers obtained by Euler's method can be quite a challenging problem. The common practice is to repeatedly approximate $v(0.2)$ and the zero crossing, using smaller and smaller values of h, until the digits of the computed values stabilize at the required accuracy level. For this example, Euler's algorithm yields the following values:

$h = 0.1$	$v(0.2) \approx 0.4380$	$v(0.3) \approx 0.0996$	$v(0.4) \approx -0.2024$
$h = 0.05$	$v(0.2) \approx 0.6036$	$v(0.35) \approx 0.0935$	$v(0.4) \approx -0.0574$
$h = 0.025$	$v(0.2) \approx 0.6659$	$v(0.375) \approx 0.0750$	$v(0.4) \approx -0.0003$
$h = 0.0125$	$v(0.2) \approx 0.6938$		
$h = 0.00625$	$v(0.2) \approx 0.7071$		

Acknowledging the remote possibility that finer values of h might reveal aberrations, we state with reasonable confidence that $v(0.2) = 0.7 \pm 0.1$. The Intermediate Value Theorem would imply that $v(t_0) = 0$ at some time t_0 satisfying $0.375 < t_0 < 0.4$ if the computations were perfect; they clearly provide evidence that $t_0 = 0.4 \pm 0.1$. ◆

1.4 EXERCISES

In many of the problems below, it will be helpful to have a calculator or computer available.[†] *You may also find it convenient to write a program for solving initial value problems using Euler's method. (Remember, all trigonometric calculations are done in radians.)*

In Problems 1–4, use Euler's method to approximate the solution to the given initial value problem at the points $x = 0.1, 0.2, 0.3, 0.4,$ *and* 0.5, *using steps of size* 0.1 $(h = 0.1)$.

1. $dy/dx = -x/y\,, \qquad y(0) = 4$
2. $dy/dx = y(2 - y)\,, \qquad y(0) = 3$
3. $dy/dx = x + y\,, \qquad y(0) = 1$
4. $dy/dx = x/y\,, \qquad y(0) = -1$

5. Use Euler's method with step size $h = 0.1$ to approximate the solution to the initial value problem
$$y' = x - y^2\,, \qquad y(1) = 0$$
at the points $x = 1.1, 1.2, 1.3, 1.4$, and 1.5.

6. Use Euler's method with step size $h = 0.2$ to approximate the solution to the initial value problem
$$y' = \frac{1}{x}(y^2 + y)\,, \qquad y(1) = 1$$
at the points $x = 1.2, 1.4, 1.6$, and 1.8.

7. Use Euler's method to find approximations to the solution of the initial value problem
$$y' = 1 - \sin y\,, \qquad y(0) = 0$$
at $x = \pi$, taking 1, 2, 4, and 8 steps.

8. Use Euler's method to find approximations to the solution of the initial value problem
$$\frac{dx}{dt} = 1 + t\sin(tx)\,, \qquad x(0) = 0$$
at $t = 1$, taking 1, 2, 4, and 8 steps.

9. Use Euler's method with $h = 0.1$ to approximate the solution to the initial value problem
$$y' = \frac{1}{x^2} - \frac{y}{x} - y^2\,, \qquad y(1) = -1$$
on the interval $1 \le x \le 2$. Compare these approximations with the actual solution $y = -1/x$ (verify!) by graphing the polygonal-line approximation and the actual solution on the same coordinate system.

10. Use the strategy of Example 3 to find a value of h for Euler's method such that $y(1)$ is approximated to within ± 0.01, if $y(x)$ satisfies the initial value problem
$$y' = x - y\,, \qquad y(0) = 0\,.$$
Also find, to within ± 0.05, the value of x_0 such that $y(x_0) = 0.2$. Compare your answers with those given by the actual solution $y = e^{-x} + x - 1$ (verify!). Graph the polygonal-line approximation and the actual solution on the same coordinate system.

11. Use the strategy of Example 3 to find a value of h for Euler's method such that $x(1)$ is approximated to within ± 0.01, if $x(t)$ satisfies the initial value problem
$$\frac{dx}{dt} = 1 + x^2\,, \qquad x(0) = 0\,.$$
Also find, to within ± 0.02, the value of t_0 such that $x(t_0) = 1$. Compare your answers with those given by the actual solution $x = \tan t$ (verify!).

12. In Example 2 we approximated the transcendental number e by using Euler's method to solve the initial value problem
$$y' = y\,, \qquad y(0) = 1\,.$$
Show that the Euler approximation y_n obtained by using the step size $1/n$ is given by the formula
$$y_n = \left(1 + \frac{1}{n}\right)^n, \qquad n = 1, 2, \ldots$$
Recall from calculus that
$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e\,,$$
and hence Euler's method converges (theoretically) to the correct value.

13. Prove that the "rate of convergence" for Euler's method in Problem 12 is comparable to $1/n$ by showing that
$$\lim_{n\to\infty}\frac{e - y_n}{1/n} = \frac{e}{2}\,.$$
[*Hint:* Use L'Hôpital's rule and the Maclaurin expansion for $\ln(1 + t)$.]

14. Use Euler's method with the spacings $h = 0.5, 0.1, 0.05, 0.01$ to approximate the solution to the initial value problem
$$y' = 2xy^2\,, \qquad y(0) = 1$$
on the interval $0 \le x \le 2$. (The explanation for the erratic results lies in Problem 18 of Exercises 1.2.)

Heat Exchange. *There are basically two mechanisms by which a physical body exchanges heat with its environment. The* **contact** *heat transfer across the body's surface is driven by the difference in the body's temperature and*

[†] Appendix G describes various web sites and commercial software that sketch direction fields and automate most of the differential equation algorithms discussed in this book.

that of the environment; this is known as Newton's law of cooling. However, heat transfer also occurs due to **thermal radiation,** *which according to Stefan's law of radiation is governed by the difference of the fourth powers of these temperatures. In most cases one of these modes dominates the other. Problems 15 and 16 invite you to simulate each mode numerically for a given set of initial conditions.*

15. Newton's Law of Cooling. Newton's law of cooling states that the rate of change in the temperature $T(t)$ of a body is proportional to the difference between the temperature of the medium $M(t)$ and the temperature of the body. That is,

$$\frac{dT}{dt} = K[M(t) - T(t)] ,$$

where K is a constant. Let $K = 0.04\ (\text{min})^{-1}$ and the temperature of the medium be constant, $M(t) \equiv 293$ kelvins. If the body is initially at 360 kelvins, use Euler's method with $h = 3.0$ min to approximate the temperature of the body after

(a) 30 minutes.
(b) 60 minutes.

16. Stefan's Law of Radiation. Stefan's law of radiation states that the rate of change in temperature of a body at $T(t)$ kelvins in a medium at $M(t)$ kelvins is proportional to $M^4 - T^4$. That is,

$$\frac{dT}{dt} = K(M(t)^4 - T(t)^4) ,$$

where K is a constant. Let $K = 2.9 \times 10^{-10}\ (\text{min})^{-1}$ and assume that the medium temperature is constant, $M(t) \equiv 293$ kelvins. If $T(0) = 360$ kelvins, use Euler's method with $h = 3.0$ min to approximate the temperature of the body after

(a) 30 minutes.
(b) 60 minutes.

Chapter 1 Summary

In this chapter we introduced some basic terminology for differential equations. The **order** of a differential equation is the order of the highest derivative present. The subject of this text is **ordinary** differential equations, which involve derivatives with respect to a single independent variable. Such equations are classified as **linear** or **nonlinear.**

An **explicit solution** of a differential equation is a function of the independent variable that satisfies the equation on some interval. An **implicit solution** is a relation between the dependent and independent variables that implicitly defines a function that is an explicit solution. A differential equation typically has infinitely many solutions. In contrast, some theorems ensure that a unique solution exists for certain **initial value problems** in which one must find a solution to the differential equation that also satisfies given initial conditions. For an nth-order equation, these conditions refer to the values of the solution and its first $n - 1$ derivatives at some point.

Even if one is not successful in finding explicit solutions to a differential equation, several techniques can be used to help analyze the solutions. One such method for first-order equations views the differential equation $dy/dx = f(x, y)$ as specifying directions (slopes) at points on the plane. The conglomerate of such slopes is the **direction field** for the equation. Knowing the "flow of solutions" is helpful in sketching the solution to an initial value problem. Furthermore, carrying out this method algebraically leads to numerical approximations to the desired solution. This numerical process is called **Euler's method.**

REVIEW PROBLEMS FOR CHAPTER 1

In Problems 1–6, identify the independent variable, dependent variable, and determine whether the equation is linear or nonlinear.

1. $5\dfrac{dx}{dt} + 5x^2 + 3 = 0$

2. $3r - \cos\theta \dfrac{dr}{d\theta} = \sin\theta$

3. $y^3 \dfrac{d^2x}{dy^2} + 3x - \dfrac{8}{y-1} = 0$

2.2 Separable Equations

A simple class of first-order differential equations that can be solved using integration is the class of **separable equations.** These are equations

(1) $$\frac{dy}{dx} = f(x, y) ,$$

that can be rewritten to isolate the variables x and y (together with their differentials dx and dy) on opposite sides of the equation, as in

$$h(y)\,dy = g(x)\,dx .$$

So the original right-hand side $f(x, y)$ must have the factored form

$$f(x, y) = g(x) \cdot \frac{1}{h(y)} .$$

More formally, we write $p(y) = 1/h(y)$ and present the following definition.

Separable Equation

Definition 1. If the right-hand side of the equation

$$\frac{dy}{dx} = f(x, y)$$

can be expressed as a function $g(x)$ that depends only on x times a function $p(y)$ that depends only on y, then the differential equation is called **separable.**†

In other words, a first-order equation is separable if it can be written in the form

$$\frac{dy}{dx} = g(x)p(y) .$$

For example, the equation

$$\frac{dy}{dx} = \frac{2x + xy}{y^2 + 1}$$

is separable, since (if one is sufficiently alert to detect the factorization)

$$\frac{2x + xy}{y^2 + 1} = x\frac{2 + y}{y^2 + 1} = g(x)p(y) .$$

However, the equation

$$\frac{dy}{dx} = 1 + xy$$

admits no such factorization of the right-hand side and so is not separable.

Informally speaking, one solves separable equations by performing the separation and then integrating each side.

---

†*Historical Footnote:* A procedure for solving separable equations was discovered implicitly by Gottfried Leibniz in 1691. The explicit technique called separation of variables was formalized by John Bernoulli in 1694.

Method for Solving Separable Equations

To solve the equation

(2) $$\frac{dy}{dx} = g(x)p(y)$$

multiply by dx and by $h(y) := 1/p(y)$ to obtain

$$h(y)\,dy = g(x)\,dx\,.$$

Then integrate both sides:

$$\int h(y)\,dy = \int g(x)\,dx\,,$$

(3) $$H(y) = G(x) + C\,,$$

where we have merged the two constants of integration into a single symbol C. The last equation gives an implicit solution to the differential equation.

Caution: Constant functions $y \equiv c$ such that $p(c) = 0$ are also solutions to (2), but will not be included in (3) (see remarks on page 45).

We will look at the mathematical justification of this "streamlined" procedure shortly, but first we study some examples.

Example 1 Solve the nonlinear equation

$$\frac{dy}{dx} = \frac{x-5}{y^2}\,.$$

Solution Following the streamlined approach, we separate the variables and rewrite the equation in the form

$$y^2\,dy = (x-5)\,dx\,.$$

Integrating, we have

$$\int y^2\,dy = \int (x-5)\,dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} - 5x + C\,,$$

and solving this last equation for y gives

$$y = \left(\frac{3x^2}{2} - 15x + 3C\right)^{1/3}.$$

Since C is a constant of integration that can be any real number, $3C$ can also be any real number. Replacing $3C$ by the single symbol K, we then have

$$y = \left(\frac{3x^2}{2} - 15x + K\right)^{1/3}.$$

If we wish to abide by the custom of letting C represent an arbitrary constant, we can go one step further and use C instead of K in the final answer. This solution family is graphed in Figure 2.3 on page 43. ◆

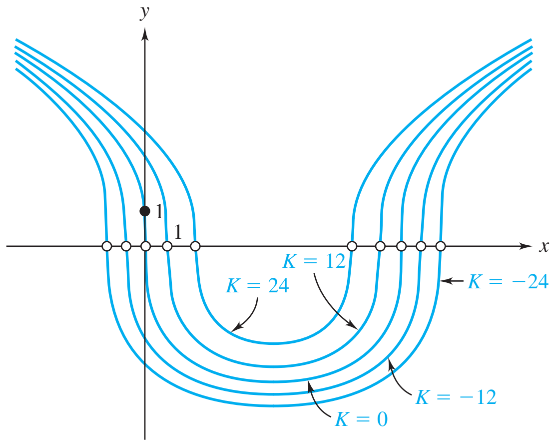


Figure 2.3 Family of solutions for Example 1[†]

As Example 1 attests, separable equations are among the easiest to solve. However, the procedure does require a facility for computing integrals. Many of the procedures to be discussed in the text also require a familiarity with the techniques of integration. For this reason we have provided a review of integration methods in Appendix A and a brief table of integrals at the back of the book.

Example 2 Solve the initial value problem

(4) $$\frac{dy}{dx} = \frac{y-1}{x+3}, \qquad y(-1) = 0 .$$

Solution Separating the variables and integrating gives

$$\frac{dy}{y-1} = \frac{dx}{x+3},$$

$$\int \frac{dy}{y-1} = \int \frac{dx}{x+3},$$

(5) $$\ln|y-1| = \ln|x+3| + C .$$

At this point, we can either solve for y explicitly (retaining the constant C) or use the initial condition to determine C and then solve explicitly for y. Let's try the first approach.

Exponentiating equation (5), we have

$$e^{\ln|y-1|} = e^{\ln|x+3|+C} = e^C e^{\ln|x+3|},$$

(6) $$|y-1| = e^C|x+3| = C_1|x+3| ,$$

where $C_1 := e^C$.[‡] Now, depending on the values of y, we have $|y-1| = \pm(y-1)$; and similarly, $|x+3| = \pm(x+3)$. Thus, (6) can be written as

$$y - 1 = \pm C_1(x+3) \quad \text{or} \quad y = 1 \pm C_1(x+3) ,$$

[†]The gaps in the curves reflect the fact that in the original differential equation, y appears in the denominator, so that $y = 0$ must be excluded.

[‡]Recall that the symbol $:=$ means "is defined to be."

where the choice of sign depends (as we said) on the values of x and y. Because C_1 is a *positive* constant (recall that $C_1 = e^C > 0$), we can replace $\pm C_1$ by K, where K now represents an *arbitrary* nonzero constant. We then obtain

$$y = 1 + K(x + 3)\ . \tag{7}$$

Finally, we determine K such that the initial condition $y(-1) = 0$ is satisfied. Putting $x = -1$ and $y = 0$ in equation (7) gives

$$0 = 1 + K(-1 + 3) = 1 + 2K\ ,$$

and so $K = -1/2$. Thus, the solution to the initial value problem is

$$y = 1 - \frac{1}{2}(x + 3) = -\frac{1}{2}(x + 1)\ . \tag{8}$$

Alternative Approach. The second approach is to first set $x = -1$ and $y = 0$ in equation (5) and solve for C. In this case, we obtain

$$\begin{aligned} \ln|0 - 1| &= \ln|-1 + 3| + C\ , \\ 0 = \ln 1 &= \ln 2 + C\ , \end{aligned}$$

and so $C = -\ln 2$. Thus, from (5), the solution y is given implicitly by

$$\ln(1 - y) = \ln(x + 3) - \ln 2\ .$$

Here we have replaced $|y - 1|$ by $1 - y$ and $|x + 3|$ by $x + 3$, since we are interested in x and y near the initial values $x = -1$, $y = 0$ (for such values, $y - 1 < 0$ and $x + 3 > 0$). Solving for y, we find

$$\begin{aligned} \ln(1 - y) &= \ln(x + 3) - \ln 2 = \ln\left(\frac{x + 3}{2}\right), \\ 1 - y &= \frac{x + 3}{2}\ , \\ y &= 1 - \frac{1}{2}(x + 3) = -\frac{1}{2}(x + 1)\ , \end{aligned}$$

which agrees with the solution (8) found by the first method. ◆

Example 3 Solve the nonlinear equation

$$\frac{dy}{dx} = \frac{6x^5 - 2x + 1}{\cos y + e^y}\ . \tag{9}$$

Solution Separating variables and integrating, we find

$$\begin{aligned} (\cos y + e^y)\, dy &= (6x^5 - 2x + 1)\, dx\ , \\ \int (\cos y + e^y)\, dy &= \int (6x^5 - 2x + 1)\, dx\ , \\ \sin y + e^y &= x^6 - x^2 + x + C\ . \end{aligned}$$

At this point, we reach an impasse. We would like to solve for y explicitly, but we cannot. This is often the case in solving nonlinear first-order equations. Consequently, when we say "solve the equation," we must on occasion be content if only an implicit form of the solution has been found. ◆

The separation of variables technique, as well as several other techniques discussed in this book, entails rewriting a differential equation by performing certain algebraic operations on it. "Rewriting $dy/dx = g(x)p(y)$ as $h(y)\,dy = g(x)\,dx$" amounts to dividing both sides by $p(y)$. You may recall from your algebra days that doing this can be treacherous. For example, the equation $x(x-2) = 4(x-2)$ has two solutions: $x = 2$ and $x = 4$. But if we "rewrite" the equation as $x = 4$ by dividing both sides by $(x-2)$, we lose track of the root $x = 2$. Thus, we should record the zeros of $(x-2)$ itself before dividing by this factor.

By the same token we must take note of the zeros of $p(y)$ in the separable equation $dy/dx = g(x)p(y)$ prior to dividing. After all, if (say) $g(x)p(y) = (x-2)^2(y-13)$, then observe that the constant function $y(x) \equiv 13$ solves the differential equation $dy/dx = g(x)p(y)$:

$$\frac{dy}{dx} = \frac{d(13)}{dx} = 0\,,$$

$$g(x)p(y) = (x-2)^2(13-13) = 0\,.$$

Indeed, in solving the equation of Example 2,

$$\frac{dy}{dx} = \frac{y-1}{x+3}\,,$$

we obtained $y = 1 + K(x+3)$ as the set of solutions, where K was a *nonzero* constant (since K replaced $\pm e^C$). But notice that the constant function $y \equiv 1$ (which in this case corresponds to $K = 0$) is also a solution to the differential equation. The reason we lost this solution can be traced back to a division by $y - 1$ in the separation process. (See Problem 30 for an example of where a solution is lost and cannot be retrieved by setting the constant $K = 0$.)

Formal Justification of Method

We close this section by reviewing the separation of variables procedure in a more rigorous framework. The original differential equation (2) is rewritten in the form

(10) $$h(y)\frac{dy}{dx} = g(x)\,,$$

where $h(y) := 1/p(y)$. Letting $H(y)$ and $G(x)$ denote antiderivatives (indefinite integrals) of $h(y)$ and $g(x)$, respectively—that is,

$$H'(y) = h(y)\,, \qquad G'(x) = g(x)\,,$$

we recast equation (10) as

$$H'(y)\frac{dy}{dx} = G'(x)\,.$$

By the chain rule for differentiation, the left-hand side is the derivative of the composite function $H\big(y(x)\big)$:

$$\frac{d}{dx}H\big(y(x)\big) = H'\big(y(x)\big)\frac{dy}{dx}\,.$$

Thus, if $y(x)$ is a solution to equation (2), then $H\big(y(x)\big)$ and $G(x)$ are two functions of x that have the same derivative. Therefore, they differ by a constant:

(11) $$H\big(y(x)\big) = G(x) + C\,.$$

Equation (11) agrees with equation (3), which was derived informally, and we have thus verified that the latter can be used to construct implicit solutions.

2.2 EXERCISES

In Problems 1–6, determine whether the given differential equation is separable.

1. $\dfrac{dy}{dx} - \sin(x+y) = 0$

2. $\dfrac{dy}{dx} = 4y^2 - 3y + 1$

3. $\dfrac{ds}{dt} = t\ln(s^{2t}) + 8t^2$

4. $\dfrac{dy}{dx} = \dfrac{ye^{x+y}}{x^2+2}$

5. $(xy^2 + 3y^2)\,dy - 2x\,dx = 0$

6. $s^2 + \dfrac{ds}{dt} = \dfrac{s+1}{st}$

In Problems 7–16, solve the equation.

7. $x\dfrac{dy}{dx} = \dfrac{1}{y^3}$

8. $\dfrac{dx}{dt} = 3xt^2$

9. $\dfrac{dx}{dt} = \dfrac{t}{xe^{t+2x}}$

10. $\dfrac{dy}{dx} = \dfrac{x}{y^2\sqrt{1+x}}$

11. $x\dfrac{dv}{dx} = \dfrac{1-4v^2}{3v}$

12. $\dfrac{dy}{dx} = \dfrac{\sec^2 y}{1+x^2}$

13. $\dfrac{dy}{dx} = 3x^2(1+y^2)^{3/2}$

14. $\dfrac{dx}{dt} - x^3 = x$

15. $(x + xy^2)\,dx + e^{x^2}y\,dy = 0$

16. $y^{-1}\,dy + ye^{\cos x}\sin x\,dx = 0$

In Problems 17–26, solve the initial value problem.

17. $\dfrac{dy}{dx} = (1+y^2)\tan x\,, \qquad y(0) = \sqrt{3}$

18. $y' = x^3(1-y)\,, \qquad y(0) = 3$

19. $\dfrac{1}{2}\dfrac{dy}{dx} = \sqrt{y+1}\cos x\,, \qquad y(\pi) = 0$

20. $x^2\dfrac{dy}{dx} = \dfrac{4x^2 - x - 2}{(x+1)(y+1)}\,, \qquad y(1) = 1$

21. $\dfrac{1}{\theta}\dfrac{dy}{d\theta} = \dfrac{y\sin\theta}{y^2+1}\,, \qquad y(\pi) = 1$

22. $x^2\,dx + 2y\,dy = 0\,, \qquad y(0) = 2$

23. $\dfrac{dy}{dt} = 2t\cos^2 y\,, \qquad y(0) = \pi/4$

24. $\dfrac{dy}{dx} = 8x^3e^{-2y}\,, \qquad y(1) = 0$

25. $\dfrac{dy}{dx} = x^2(1+y)\,, \qquad y(0) = 3$

26. $\sqrt{y}\,dx + (1+x)\,dy = 0\,, \qquad y(0) = 1$

27. Solutions Not Expressible in Terms of Elementary Functions. As discussed in calculus, certain indefinite integrals (antiderivatives) such as $\int e^{x^2}dx$ cannot be expressed in finite terms using elementary functions. When such an integral is encountered while solving a differential equation, it is often helpful to use definite integration (integrals with variable upper limit). For example, consider the initial value problem

$$\frac{dy}{dx} = e^{x^2}y^2\,, \qquad y(2) = 1\,.$$

The differential equation separates if we divide by y^2 and multiply by dx. We integrate the separated equation from $x = 2$ to $x = x_1$ and find

$$\begin{aligned}\int_{x=2}^{x=x_1} e^{x^2}\,dx &= \int_{x=2}^{x=x_1}\frac{dy}{y^2}\\ &= -\frac{1}{y}\bigg|_{x=2}^{x=x_1}\\ &= -\frac{1}{y(x_1)} + \frac{1}{y(2)}\,.\end{aligned}$$

If we let t be the variable of integration and replace x_1 by x and $y(2)$ by 1, then we can express the solution to the initial value problem by

$$y(x) = \left(1 - \int_2^x e^{t^2}dt\right)^{-1}.$$

Use definite integration to find an explicit solution to the initial value problems in parts (a)–(c).

(a) $dy/dx = e^{x^2}\,, \qquad y(0) = 0$

(b) $dy/dx = e^{x^2}y^{-2}\,, \qquad y(0) = 1$

(c) $dy/dx = \sqrt{1+\sin x}\,(1+y^2)\,, \qquad y(0) = 1$

(d) Use a numerical integration algorithm (such as Simpson's rule, described in Appendix C) to approximate the solution to part (b) at $x = 0.5$ to three decimal places.

28. Sketch the solution to the initial value problem

$$\frac{dy}{dt} = 2y - 2yt\,, \qquad y(0) = 3$$

and determine its maximum value.

29. Uniqueness Questions. In Chapter 1 we indicated that in applications most *initial value problems* will have a unique solution. In fact, the existence of unique solutions was so important that we stated an existence and uniqueness theorem, Theorem 1, page 11. The method for separable equations can give us a solution, but it may not give us all the solutions (also see Problem 30). To illustrate this, consider the equation $dy/dx = y^{1/3}$.

(a) Use the method of separation of variables to show that

$$y = \left(\frac{2x}{3} + C\right)^{3/2}$$

is a solution.

(b) Show that the initial value problem $dy/dx = y^{1/3}$ with $y(0) = 0$ is satisfied for $C = 0$ by $y = (2x/3)^{3/2}$ for $x \ge 0$.

(c) Now show that the constant function $y \equiv 0$ also satisfies the initial value problem given in part (b). Hence, this initial value problem does not have a unique solution.

(d) Finally, show that the conditions of Theorem 1 on page 11 are not satisfied.

(The solution $y \equiv 0$ was lost because of the division by zero in the separation process.)

30. As stated in this section, the separation of equation (2) on page 42 requires division by $p(y)$, and this may disguise the fact that the roots of the equation $p(y) = 0$ are actually constant solutions to the differential equation.

(a) To explore this further, separate the equation

$$\frac{dy}{dx} = (x-3)(y+1)^{2/3}$$

to derive the solution,

$$y = -1 + (x^2/6 - x + C)^3 .$$

(b) Show that $y \equiv -1$ satisfies the original equation $dy/dx = (x-3)(y+1)^{2/3}$.

(c) Show that there is no choice of the constant C that will make the solution in part (a) yield the solution $y \equiv -1$. Thus, we lost the solution $y \equiv -1$ when we divided by $(y+1)^{2/3}$.

31. Interval of Definition. By looking at an initial value problem $dy/dx = f(x, y)$ with $y(x_0) = y_0$, it is not always possible to determine the domain of the solution $y(x)$ or the interval over which the function $y(x)$ satisfies the differential equation.

(a) Solve the equation $dy/dx = xy^3$.

(b) Give explicitly the solutions to the initial value problem with $y(0) = 1$; $y(0) = 1/2$; $y(0) = 2$.

(c) Determine the domains of the solutions in part (b).

(d) As found in part (c), the domains of the solutions depend on the initial conditions. For the initial value problem $dy/dx = xy^3$ with $y(0) = a$, $a > 0$, show that as a approaches zero from the right the domain approaches the whole real line $(-\infty, \infty)$ and as a approaches $+\infty$ the domain shrinks to a single point.

(e) Sketch the solutions to the initial value problem $dy/dx = xy^3$ with $y(0) = a$ for $a = \pm 1/2, \pm 1$, and ± 2.

32. Analyze the solution $y = \phi(x)$ to the initial value problem

$$\frac{dy}{dx} = y^2 - 3y + 2 , \qquad y(0) = 1.5$$

using approximation methods and then compare with its exact form as follows.

(a) Sketch the direction field of the differential equation and use it to guess the value of $\lim_{x\to\infty} \phi(x)$.

(b) Use Euler's method with a step size of 0.1 to find an approximation of $\phi(1)$.

(c) Find a formula for $\phi(x)$ and graph $\phi(x)$ on the direction field from part (a).

(d) What is the exact value of $\phi(1)$? Compare with your approximation in part (b).

(e) Using the exact solution obtained in part (c), determine $\lim_{x\to\infty} \phi(x)$ and compare with your guess in part (a).

33. Mixing. Suppose a brine containing 0.3 kilogram (kg) of salt per liter (L) runs into a tank initially filled with 400 L of water containing 2 kg of salt. If the brine enters at 10 L/min, the mixture is kept uniform by stirring, and the mixture flows out at the same rate. Find the mass of salt in the tank after 10 min (see Figure 2.4). [*Hint:* Let A denote the number of kilograms of salt in the tank at t min after the process begins and use the fact that

rate of increase in A = rate of input − rate of exit.

A further discussion of mixing problems is given in Section 3.2.]

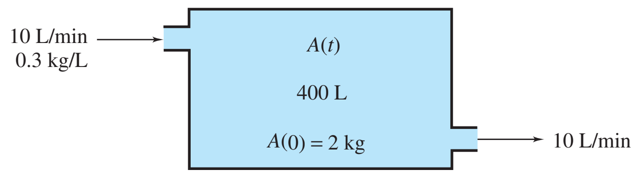


Figure 2.4 Schematic representation of a mixing problem

34. Newton's Law of Cooling. According to Newton's law of cooling, if an object at temperature T is immersed in a medium having the constant temperature M, then the rate of change of T is proportional to the difference of temperature $M - T$. This gives the differential equation

$$dT/dt = k(M - T) .$$

(a) Solve the differential equation for T.

(b) A thermometer reading 100°F is placed in a medium having a constant temperature of 70°F. After 6 min, the thermometer reads 80°F. What is the reading after 20 min?

(Further applications of Newton's law of cooling appear in Section 3.3.)

35. Blood plasma is stored at 40°F. Before the plasma can be used, it must be at 90°F. When the plasma is placed in an oven at 120°F, it takes 45 min for the plasma to warm to 90°F. Assume Newton's law of cooling (Problem 34) applies. How long will it take for the plasma to warm to 90°F if the oven temperature is set at **(a)** 100°F, **(b)** 140°F, and **(c)** 80°F?

36. A pot of boiling water at 100°C is removed from a stove and covered at time $t = 0$ and left to cool in the kitchen. After 5 min, the water temperature has decreased to 80°C, and another 5 min later it has dropped to 65°C. Assuming Newton's law of cooling (Problem 34) applies, determine the (constant) temperature of the kitchen.

37. Compound Interest. If $P(t)$ is the amount of dollars in a savings bank account that pays a yearly interest rate of *r% compounded continuously,* then

$$\frac{dP}{dt} = \frac{r}{100} P\,, \qquad t \text{ in years.}$$

Assume the interest is 5% annually, $P(0) = \$1000$, and no monies are withdrawn.

(a) How much will be in the account after 2 yr?
(b) When will the account reach \$4000?
(c) If \$1000 is added to the account every 12 months, how much will be in the account after $3\frac{1}{2}$ yr?

38. Free Fall. In Section 2.1, we discussed a model for an object falling toward Earth. Assuming that only air resistance and gravity are acting on the object, we found that the velocity v must satisfy the equation

$$m\frac{dv}{dt} = mg - bv\,,$$

where m is the mass, g is the acceleration due to gravity, and $b > 0$ is a constant (see Figure 2.1). If $m = 100$ kg, $g = 9.8\ \text{m/sec}^2$, $b = 5$ kg/sec, and $v(0) = 10$ m/sec, solve for $v(t)$. What is the limiting (i.e., terminal) velocity of the object?

39. Grand Prix Race. Driver A had been leading archrival B for a while by a *steady* 3 miles. Only 2 miles from the finish, driver A ran out of gas and decelerated thereafter at a rate proportional to the square of his remaining speed. One mile later, driver A's speed was exactly halved. If driver B's speed remained constant, who won the race?

40. The *atmospheric pressure* (force per unit area) on a surface at an altitude z is due to the weight of the column of air situated above the surface. Therefore, the drop in air pressure p between the top and bottom of a cylindrical volume element of height Δz and cross-section area A equals the weight of the air enclosed (density ρ times volume $V = A\Delta z$ times gravity g), per unit area:

$$p(z + \Delta z) - p(z) = -\frac{\rho(z)(A\Delta z)g}{A} = -\rho(z)g\Delta z\,.$$

Let $\Delta z \to 0$ to derive the differential equation $dp/dz = -\rho g$. To analyze this further we must postulate a formula that relates pressure and density. The *perfect gas law* relates pressure, volume, mass m, and absolute temperature T according to $pV = mRT/M$, where R is the universal gas constant and M is the molar mass (i.e., the mass of one mole) of the air. Therefore, density and pressure are related by $\rho := m/V = Mp/RT$.

(a) Derive the equation $\dfrac{dp}{dz} = -\dfrac{Mg}{RT}p$ and solve it for the "isothermal" case where T is constant to obtain the *barometric pressure equation* $p(z) = p(z_0)\exp[-Mg(z - z_0)/RT]$.

(b) If the temperature also varies with altitude $T = T(z)$, derive the solution

$$p(z) = p(z_0)\exp\left\{-\frac{Mg}{R}\int_{z_0}^{z}\frac{d\zeta}{T(\zeta)}\right\}.$$

(c) Suppose an engineer measures the barometric pressure at the top of a building to be 99,000 Pa (pascals), and 101,000 Pa at the base $(z = z_0)$. If the absolute temperature varies as $T(z) = 288 - 0.0065(z - z_0)$, determine the height of the building. Take $R = 8.31$ N-m/mol-K, $M = 0.029$ kg/mol, and $g = 9.8\ \text{m/sec}^2$. (An amusing story concerning this problem can be found at http://www.snopes.com/college/exam/barometer.asp)

2.3 Linear Equations

A type of first-order differential equation that occurs frequently in applications is the linear equation. Recall from Section 1.1 that a **linear first-order equation** is an equation that can be expressed in the form

$$\textbf{(1)} \qquad a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)\,,$$

where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable x, not on y.

For example, the equation

$$x^2\sin x - (\cos x)y = (\sin x)\frac{dy}{dx}$$

is linear, because it can be rewritten in the form

$$(\sin x)\frac{dy}{dx} + (\cos x)y = x^2 \sin x\,.$$

However, the equation

$$y\frac{dy}{dx} + (\sin x)y^3 = e^x + 1$$

is not linear; it cannot be put in the form of equation (1) due to the presence of the y^3 and $y\,dy/dx$ terms.

There are two situations for which the solution of a linear differential equation is quite immediate. The first arises if the coefficient $a_0(x)$ is identically zero, for then equation (1) reduces to

$$a_1(x)\frac{dy}{dx} = b(x)\,, \tag{2}$$

which is equivalent to

$$y(x) = \int \frac{b(x)}{a_1(x)}dx + C$$

[as long as $a_1(x)$ is not zero].

The second is less trivial. Note that if $a_0(x)$ happens to equal the derivative of $a_1(x)$—that is, $a_0(x) = a_1'(x)$—then the two terms on the left-hand side of equation (1) simply comprise the derivative of the product $a_1(x)y$:

$$a_1(x)y' + a_0(x)y = a_1(x)y' + a_1'(x)y = \frac{d}{dx}[a_1(x)y]\,.$$

Therefore equation (1) becomes

$$\frac{d}{dx}[a_1(x)y] = b(x) \tag{3}$$

and the solution is again elementary:

$$a_1(x)y = \int b(x)\,dx + C\,,$$

$$y(x) = \frac{1}{a_1(x)}\left[\int b(x)\,dx + C\right].$$

One can seldom rewrite a linear differential equation so that it reduces to a form as simple as (2). However, the form (3) can be achieved through multiplication of the original equation (1) by a well-chosen function $\mu(x)$. Such a function $\mu(x)$ is then called an "integrating factor" for equation (1). The easiest way to see this is first to divide the original equation (1) by $a_1(x)$ and put it into **standard form**

$$\boldsymbol{\frac{dy}{dx} + P(x)y = Q(x)}\,, \tag{4}$$

where $P(x) = a_0(x)/a_1(x)$ and $Q(x) = b(x)/a_1(x)$.

Next we wish to determine $\mu(x)$ so that the left-hand side of the multiplied equation

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x) \tag{5}$$

is just the derivative of the product $\mu(x)y$:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \frac{d}{dx}[\mu(x)y] = \mu(x)\frac{dy}{dx} + \mu'(x)y \, .$$

Clearly, this requires that μ satisfy

$$\mu' = \mu P \, . \tag{6}$$

To find such a function, we recognize that equation (6) is a separable differential equation, which we can write as $(1/\mu)\,d\mu = P(x)dx$. Integrating both sides gives

$$\mu(x) = e^{\int P(x)dx} \, . \tag{7}$$

With this choice† for $\mu(x)$, equation (5) becomes

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x) \, ,$$

which has the solution

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,dx + C\right] . \tag{8}$$

Here C is an arbitrary constant, so (8) gives a one-parameter family of solutions to (4). This form is known as the **general solution** to (4).

We can summarize the method for solving linear equations as follows.

Method for Solving Linear Equations

(a) Write the equation in the standard form

$$\frac{dy}{dx} + P(x)y = Q(x) \, .$$

(b) Calculate the integrating factor $\mu(x)$ by the formula

$$\mu(x) = \exp\left[\int P(x)dx\right] .$$

(c) Multiply the equation in standard form by $\mu(x)$ and, recalling that the left-hand side is just $\frac{d}{dx}[\mu(x)y]$, obtain

$$\underbrace{\mu(x)\frac{dy}{dx} + P(x)\mu(x)y}_{\frac{d}{dx}[\mu(x)y]} = \mu(x)Q(x) \, ,$$

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x) \, .$$

(d) Integrate the last equation and solve for y by dividing by $\mu(x)$ to obtain (8).

---

†Any choice of the integration constant in $\int P(x)\,dx$ will produce a suitable $\mu(x)$.

Example 1 Find the general solution to

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x\,, \qquad x > 0\,. \tag{9}$$

Solution To put this linear equation in standard form, we multiply by x to obtain

$$\frac{dy}{dx} - \frac{2}{x}y = x^2\cos x\,. \tag{10}$$

Here $P(x) = -2/x$, so

$$\int P(x)\,dx = \int \frac{-2}{x}dx = -2\ln|x|\,.$$

Thus, an integrating factor is

$$\mu(x) = e^{-2\ln|x|} = e^{\ln(x^{-2})} = x^{-2}\,.$$

Multiplying equation (10) by $\mu(x)$ yields

$$\underbrace{x^{-2}\frac{dy}{dx} - 2x^{-3}y}_{} = \cos x\,,$$

$$\frac{d}{dx}(x^{-2}y) = \cos x\,.$$

We now integrate both sides and solve for y to find

$$x^{-2}y = \int \cos x\,dx = \sin x + C$$

$$y = x^2\sin x + Cx^2\,. \tag{11}$$

It is easily checked that this solution is valid for all $x > 0$. In Figure 2.5 we have sketched solutions for various values of the constant C in (11). ◆

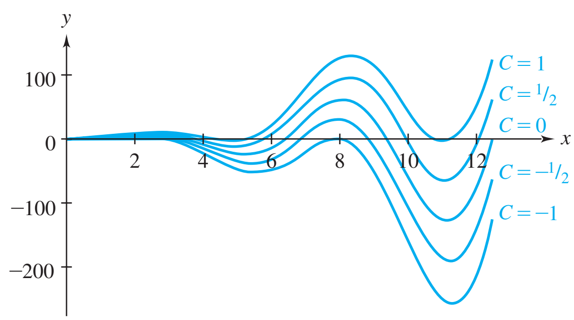


Figure 2.5 Graph of $y = x^2\sin x + Cx^2$ for five values of the constant C

In the next example, we encounter a linear equation that arises in the study of the radioactive decay of an isotope.

Example 2 A rock contains two radioactive isotopes, RA_1 and RA_2, that belong to the same radioactive series; that is, RA_1 decays into RA_2, which then decays into stable atoms. Assume that the rate at which RA_1 decays into RA_2 is $50e^{-10t}$ kg/sec. Because the rate of decay of RA_2 is proportional to the mass $y(t)$ of RA_2 present, the rate of change in RA_2 is

$$\frac{dy}{dt} = \text{rate of creation} - \text{rate of decay}\,,$$

$$\frac{dy}{dt} = 50e^{-10t} - ky\,, \tag{12}$$

where $k > 0$ is the decay constant. If $k = 2/\text{sec}$ and initially $y(0) = 40$ kg, find the mass $y(t)$ of RA_2 for $t \geq 0$.

Solution Equation (12) is linear, so we begin by writing it in standard form

$$\frac{dy}{dt} + 2y = 50e^{-10t}\,, \qquad y(0) = 40\,, \tag{13}$$

where we have substituted $k = 2$ and displayed the initial condition. We now see that $P(t) = 2$, so $\int P(t)dt = \int 2\,dt = 2t$. Thus, an integrating factor is $\mu(t) = e^{2t}$. Multiplying equation (13) by $\mu(t)$ yields

$$\underbrace{e^{2t}\frac{dy}{dt} + 2e^{2t}y}_{\frac{d}{dt}(e^{2t}y)} = 50e^{-10t+2t} = 50e^{-8t}\,,$$

$$\frac{d}{dt}(e^{2t}y) = 50e^{-8t}\,.$$

Integrating both sides and solving for y, we find

$$e^{2t}y = -\frac{25}{4}e^{-8t} + C\,,$$

$$y = -\frac{25}{4}e^{-10t} + Ce^{-2t}\,.$$

Substituting $t = 0$ and $y(0) = 40$ gives

$$40 = -\frac{25}{4}e^{0} + Ce^{0} = -\frac{25}{4} + C\,,$$

so $C = 40 + 25/4 = 185/4$. Thus, the mass $y(t)$ of RA_2 at time t is given by

$$y(t) = \left(\frac{185}{4}\right)e^{-2t} - \left(\frac{25}{4}\right)e^{-10t}\,, \qquad t \geq 0\,. \tag{14}$$ ◆

Example 3 For the initial value problem

$$y' + y = \sqrt{1 + \cos^2 x}\,, \qquad y(1) = 4\,,$$

find the value of $y(2)$.

Solution The integrating factor for the differential equation is, from equation (7),

$$\mu(x) = e^{\int 1\,dx} = e^{x}\,.$$

The general solution form (8) thus reads

$$y(x) = e^{-x}\left(\int e^x\sqrt{1+\cos^2 x}\,dx + C\right).$$

However, this indefinite integral cannot be expressed in finite terms with elementary functions (recall a similar situation in Problem 27 of Exercises 2.2). Because we *can* use numerical algorithms such as Simpson's rule (Appendix C) to perform *definite* integration, we revert to the form (5), which in this case reads

$$\frac{d}{dx}(e^x y) = e^x\sqrt{1+\cos^2 x}\,,$$

and take the definite integral from the initial value $x = 1$ to the desired value $x = 2$:

$$e^x y\Big|_{x=1}^{x=2} = e^2 y(2) - e^1 y(1) = \int_{x=1}^{x=2} e^x\sqrt{1+\cos^2 x}\,dx\,.$$

Inserting the given value of $y(1)$ and solving, we express

$$y(2) = e^{-2+1}(4) + e^{-2}\int_1^2 e^x\sqrt{1+\cos^2 x}\,dx\,.$$

Using Simpson's rule, we find that the definite integral is approximately 4.841, so

$$y(2) \approx 4e^{-1} + 4.841e^{-2} \approx 2.127\,. \quad ◆$$

In Example 3 we had no difficulty expressing the integral for the integrating factor $\mu(x) = e^{\int 1\,dx} = e^x$. Clearly, situations will arise where this integral, too, cannot be expressed with elementary functions. In such cases we must again resort to a numerical procedure such as Euler's method (Section 1.4) or to a "nested loop" implementation of Simpson's rule. You are invited to explore such a possibility in Problem 27.

Because we have established explicit formulas for the solutions to *linear* first-order differential equations, we get as a dividend a direct proof of the following theorem.

Existence and Uniqueness of Solution

Theorem 1. If $P(x)$ and $Q(x)$ are continuous on an interval (a, b) that contains the point x_0, then for any choice of initial value y_0, there exists a unique solution $y(x)$ on (a, b) to the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x)\,, \qquad y(x_0) = y_0\,. \tag{15}$$

In fact, the solution is given by (8) for a suitable value of C.

The essentials of the proof of Theorem 1 are contained in the deliberations leading to equation (8); Problem 34 provides the details. This theorem differs from Theorem 1 on page 11 in that for the *linear* initial value problem (15), we have the existence and uniqueness of the solution on the *whole* interval (a, b), rather than on some smaller unspecified interval about x_0.

The theory of linear differential equations is an important branch of mathematics not only because these equations occur in applications but also because of the elegant structure associated with them. For example, first-order linear equations always have a general solution given by equation (8). Some further properties of first-order linear equations are described in Problems 28 and 36. Higher-order linear equations are treated in Chapters 4, 6, and 8.

2.3 EXERCISES

In Problems 1–6, determine whether the given equation is separable, linear, neither, or both.

1. $x^2\dfrac{dy}{dx} + \sin x - y = 0$

2. $\dfrac{dx}{dt} + xt = e^x$

3. $(t^2+1)\dfrac{dy}{dt} = yt - y$

4. $3t = e^t\dfrac{dy}{dt} + y\ln t$

5. $x\dfrac{dx}{dt} + t^2x = \sin t$

6. $3r = \dfrac{dr}{d\theta} - \theta^3$

In Problems 7–16, obtain the general solution to the equation.

7. $\dfrac{dy}{dx} - y - e^{3x} = 0$

8. $\dfrac{dy}{dx} = \dfrac{y}{x} + 2x + 1$

9. $\dfrac{dr}{d\theta} + r\tan\theta = \sec\theta$

10. $x\dfrac{dy}{dx} + 2y = x^{-3}$

11. $(t+y+1)\,dt - dy = 0$

12. $\dfrac{dy}{dx} = x^2e^{-4x} - 4y$

13. $y\dfrac{dx}{dy} + 2x = 5y^3$

14. $x\dfrac{dy}{dx} + 3(y+x^2) = \dfrac{\sin x}{x}$

15. $(x^2+1)\dfrac{dy}{dx} + xy - x = 0$

16. $(1-x^2)\dfrac{dy}{dx} - x^2y = (1+x)\sqrt{1-x^2}$

In Problems 17–22, solve the initial value problem.

17. $\dfrac{dy}{dx} - \dfrac{y}{x} = xe^x, \qquad y(1) = e - 1$

18. $\dfrac{dy}{dx} + 4y - e^{-x} = 0, \qquad y(0) = \dfrac{4}{3}$

19. $t^2\dfrac{dx}{dt} + 3tx = t^4\ln t + 1, \qquad x(1) = 0$

20. $\dfrac{dy}{dx} + \dfrac{3y}{x} + 2 = 3x, \qquad y(1) = 1$

21. $(\cos x)\dfrac{dy}{dx} + y\sin x = 2x\cos^2 x,$

$$y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2}\pi^2}{32}$$

22. $(\sin x)\dfrac{dy}{dx} + y\cos x = x\sin x, \qquad y\left(\dfrac{\pi}{2}\right) = 2$

23. Radioactive Decay. In Example 2 assume that the rate at which RA_1 decays into RA_2 is $40e^{-20t}$ kg/sec and the decay constant for RA_2 is $k = 5$/sec. Find the mass $y(t)$ of RA_2 for $t \geq 0$ if initially $y(0) = 10$ kg.

24. In Example 2 the decay constant for isotope RA_1 was 10/sec, which expresses itself in the exponent of the rate term $50e^{-10t}$ kg/sec. When the decay constant for RA_2 is $k = 2$/sec, we see that in formula (14) for y the term $(185/4)e^{-2t}$ eventually dominates (has greater magnitude for t large).

(a) Redo Example 2 taking $k = 20$/sec. Now which term in the solution eventually dominates?

(b) Redo Example 2 taking $k = 10$/sec.

25. (a) Using definite integration, show that the solution to the initial value problem

$$\frac{dy}{dx} + 2xy = 1, \qquad y(2) = 1,$$

can be expressed as

$$y(x) = e^{-x^2}\left(e^4 + \int_2^x e^{t^2}\,dt\right).$$

(b) Use numerical integration (such as Simpson's rule, Appendix C) to approximate the solution at $x = 3$.

26. Use numerical integration (such as Simpson's rule, Appendix C) to approximate the solution, at $x = 1$, to the initial value problem

$$\frac{dy}{dx} + \frac{\sin 2x}{2(1+\sin^2 x)}y = 1, \qquad y(0) = 0.$$

Ensure your approximation is accurate to three decimal places.

27. Consider the initial value problem

$$\frac{dy}{dx} + \sqrt{1+\sin^2 x}\,y = x, \qquad y(0) = 2.$$

(a) Using definite integration, show that the integrating factor for the differential equation can be written as

$$\mu(x) = \exp\left(\int_0^x \sqrt{1+\sin^2 t}\,dt\right)$$

and that the solution to the initial value problem is

$$y(x) = \frac{1}{\mu(x)} \int_0^x \mu(s)\, s\, ds + \frac{2}{\mu(x)} .$$

(b) Obtain an approximation to the solution at $x = 1$ by using numerical integration (such as Simpson's rule, Appendix C) in a nested loop to estimate values of $\mu(x)$ and, thereby, the value of

$$\int_0^1 \mu(s) s\, ds .$$

[*Hint:* First, use Simpson's rule to approximate $\mu(x)$ at $x = 0.1, 0.2, \ldots, 1$. Then use these values and apply Simpson's rule again to approximate $\int_0^1 \mu(s)\, s\, ds$.]

(c) Use Euler's method (Section 1.4) to approximate the solution at $x = 1$, with step sizes $h = 0.1$ and 0.05.

[A direct comparison of the merits of the two numerical schemes in parts (b) and (c) is very complicated, since it should take into account the number of functional evaluations in each algorithm as well as the inherent accuracies.]

28. Constant Multiples of Solutions.

(a) Show that $y = e^{-x}$ is a solution of the linear equation

$$\frac{dy}{dx} + y = 0 , \tag{16}$$

and $y = x^{-1}$ is a solution of the nonlinear equation

$$\frac{dy}{dx} + y^2 = 0 . \tag{17}$$

(b) Show that for any constant C, the function Ce^{-x} is a solution of equation (16), while Cx^{-1} is a solution of equation (17) only when $C = 0$ or 1.

(c) Show that for any linear equation of the form

$$\frac{dy}{dx} + P(x)y = 0 ,$$

if $\hat{y}(x)$ is a solution, then for any constant C the function $C\hat{y}(x)$ is also a solution.

29. Use your ingenuity to solve the equation

$$\frac{dy}{dx} = \frac{1}{e^{4y} + 2x} .$$

[*Hint:* The roles of the independent and dependent variables may be reversed.]

30. Bernoulli Equations. The equation

$$\frac{dy}{dx} + 2y = xy^{-2} \tag{18}$$

is an example of a Bernoulli equation. (Further discussion of Bernoulli equations is in Section 2.6.)

(a) Show that the substitution $v = y^3$ reduces equation (18) to the equation

$$\frac{dv}{dx} + 6v = 3x . \tag{19}$$

(b) Solve equation (19) for v. Then make the substitution $v = y^3$ to obtain the solution to equation (18).

31. Discontinuous Coefficients. As we will see in Chapter 3, occasions arise when the coefficient $P(x)$ in a linear equation fails to be continuous because of jump discontinuities. Fortunately, we may still obtain a "reasonable" solution. For example, consider the initial value problem

$$\frac{dy}{dx} + P(x)y = x , \qquad y(0) = 1 ,$$

where

$$P(x) := \begin{cases} 1 , & 0 \le x \le 2 , \\ 3 , & x > 2 . \end{cases}$$

(a) Find the general solution for $0 \le x \le 2$.

(b) Choose the constant in the solution of part (a) so that the initial condition is satisfied.

(c) Find the general solution for $x > 2$.

(d) Now choose the constant in the general solution from part (c) so that the solution from part (b) and the solution from part (c) agree at $x = 2$. By patching the two solutions together, we can obtain a continuous function that satisfies the differential equation except at $x = 2$, where its derivative is undefined.

(e) Sketch the graph of the solution from $x = 0$ to $x = 5$.

32. Discontinuous Forcing Terms. There are occasions when the forcing term $Q(x)$ in a linear equation fails to be continuous because of jump discontinuities. Fortunately, we may still obtain a reasonable solution imitating the procedure discussed in Problem 31. Use this procedure to find the continuous solution to the initial value problem.

$$\frac{dy}{dx} + 2y = Q(x) , \qquad y(0) = 0 ,$$

where

$$Q(x) := \begin{cases} 2 , & 0 \le x \le 3 , \\ -2 , & x > 3 . \end{cases}$$

Sketch the graph of the solution from $x = 0$ to $x = 7$.

33. Singular Points. Those values of x for which $P(x)$ in equation (4) is not defined are called **singular points** of the equation. For example, $x = 0$ is a singular point of the equation $xy' + 2y = 3x$, since when the equation is written in the standard form, $y' + (2/x)y = 3$, we see that $P(x) = 2/x$ is not defined at $x = 0$. On an interval containing a singular point, the questions of the existence and uniqueness of a solution are left unanswered, since Theorem 1 on page 53 does not apply. To show the possible behavior of solutions near a singular point, consider the following equations.

(a) Show that $xy' + 2y = 3x$ has only one solution defined at $x = 0$. Then show that the initial value problem for this equation with initial condition $y(0) = y_0$ has a unique solution when $y_0 = 0$ and no solution when $y_0 \ne 0$.

(b) Show that $xy' - 2y = 3x$ has an infinite number of solutions defined at $x = 0$. Then show that the initial value problem for this equation with initial condition $y(0) = 0$ has an infinite number of solutions.

34. Existence and Uniqueness. Under the assumptions of Theorem 1, we will prove that equation (8) gives a solution to equation (4) on (a, b). We can then choose the constant C in equation (8) so that the initial value problem (15) is solved.

(a) Show that since $P(x)$ is continuous on (a, b), then $\mu(x)$ defined in (7) is a positive, continuous function satisfying $d\mu/dx = P(x)\mu(x)$ on (a, b).

(b) Since

$$\frac{d}{dx}\int \mu(x)Q(x)dx = \mu(x)Q(x) ,$$

verify that y given in equation (8) satisfies equation (4) by differentiating both sides of equation (8).

(c) Show that when we let $\int \mu(x)Q(x)\,dx$ be the antiderivative whose value at x_0 is 0 (i.e., $\int_{x_0}^{x} \mu(t)Q(t)\,dt$) and choose C to be $y_0\mu(x_0)$, the initial condition $y(x_0) = y_0$ is satisfied.

(d) Start with the assumption that $y(x)$ is a solution to the initial value problem (15) and argue that the discussion leading to equation (8) implies that $y(x)$ must obey equation (8). Then argue that the initial condition in (15) determines the constant C uniquely.

35. Mixing. Suppose a brine containing 0.2 kg of salt per liter runs into a tank initially filled with 500 L of water containing 5 kg of salt. The brine enters the tank at a rate of 5 L/min. The mixture, kept uniform by stirring, is flowing out at the rate of 5 L/min (see Figure 2.6).

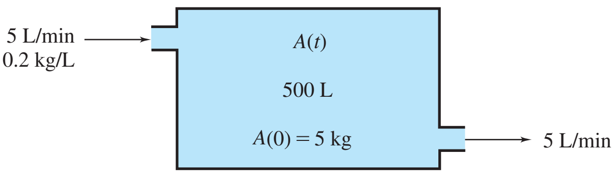


Figure 2.6 Mixing problem with equal flow rates

(a) Find the concentration, in kilograms per liter, of salt in the tank after 10 min. [*Hint:* Let A denote the number of kilograms of salt in the tank at t minutes after the process begins and use the fact that

rate of increase in A = rate of input − rate of exit.

A further discussion of mixing problems is given in Section 3.2.]

(b) After 10 min, a leak develops in the tank and an additional liter per minute of mixture flows out of the tank (see Figure 2.7). What will be the concentration, in kilograms per liter, of salt in the tank 20 min after the leak develops? [*Hint:* Use the method discussed in Problems 31 and 32.]

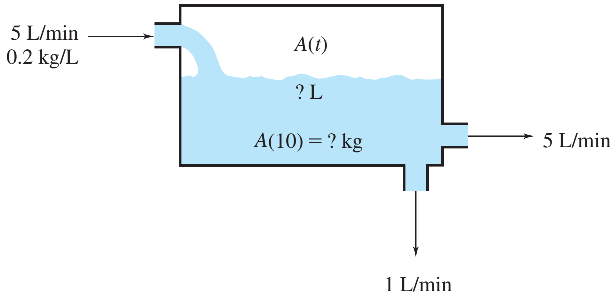


Figure 2.7 Mixing problem with unequal flow rates

36. Variation of Parameters. Here is another procedure for solving linear equations that is particularly useful for higher-order linear equations. This method is called **variation of parameters.** It is based on the idea that just by knowing the *form* of the solution, we can substitute into the given equation and solve for any unknowns. Here we illustrate the method for first-order equations (see Sections 4.6 and 6.4 for the generalization to higher-order equations).

(a) Show that the general solution to

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{20}$$

has the form

$$y(x) = Cy_h(x) + y_p(x) ,$$

where y_h ($\not\equiv 0$) is a solution to equation (20) when $Q(x) \equiv 0$, C is a constant, and $y_p(x) = v(x)y_h(x)$ for a suitable function $v(x)$. [*Hint:* Show that we can take $y_h = \mu^{-1}(x)$ and then use equation (8).]

We can in fact determine the unknown function y_h by solving a separable equation. Then direct substitution of vy_h in the original equation will give a simple equation that can be solved for v.

Use this procedure to find the general solution to

$$\frac{dy}{dx} + \frac{3}{x}y = x^2 , \qquad x > 0 , \tag{21}$$

by completing the following steps:

(b) Find a nontrivial solution y_h to the separable equation

$$\frac{dy}{dx} + \frac{3}{x}y = 0 , \qquad x > 0 . \tag{22}$$

(c) Assuming (21) has a solution of the form $y_p(x) = v(x)y_h(x)$, substitute this into equation (21), and simplify to obtain $v'(x) = x^2/y_h(x)$.

(d) Now integrate to get $v(x)$.

(e) Verify that $y(x) = Cy_h(x) + v(x)y_h(x)$ is a general solution to (21).

37. Secretion of Hormones. The secretion of hormones into the blood is often a periodic activity. If a hormone is secreted on a 24-h cycle, then the rate of change of the level of the hormone in the blood may be represented by the initial value problem

$$\frac{dx}{dt} = \alpha - \beta \cos\frac{\pi t}{12} - kx\,, \qquad x(0) = x_0\,,$$

where $x(t)$ is the amount of the hormone in the blood at time t, α is the average secretion rate, β is the amount of daily variation in the secretion, and k is a positive constant reflecting the rate at which the body removes the hormone from the blood. If $\alpha = \beta = 1, k = 2$, and $x_0 = 10$, solve for $x(t)$.

38. Use the separation of variables technique to derive the solution (7) to the differential equation (6).

39. The temperature T (in units of 100°F) of a university classroom on a cold winter day varies with time t (in hours) as

$$\frac{dT}{dt} = \begin{cases} 1 - T\,, & \text{if heating unit is ON.} \\ -T\,, & \text{if heating unit is OFF.} \end{cases}$$

Suppose $T = 0$ at 9:00 A.M., the heating unit is ON from 9–10 A.M., OFF from 10–11 A.M., ON again from 11 A.M.–noon, and so on for the rest of the day. How warm will the classroom be at noon? At 5:00 P.M.?

40. The *Nobel Prize in Physiology or Medicine* in 1963 was shared by A. L. Hodgkin and A. F. Huxley in recognition of their model for the firing of neuronal synapses. As will be discussed in Chapter 12, they proposed that the opening/closing of certain ion channels in the neuron cell was governed by a combination of probabilistic "gating variables," each satisfying a differential equation that they expressed as

$$\frac{du}{dt} = \alpha(1 - u) - \beta u \tag{23}$$

with positive parameters α, β.

(a) Use a direction field diagram (Section 1.3) to show that the solutions of equation (23) are "probabilistic" in the sense that if their initial values lie between 0 and 1, all subsequent values also lie on [0,1].

(b) Solve (23) and show that all solutions approach the value $\alpha/(\alpha + \beta)$ exponentially.

2.4 Exact Equations

Suppose the mathematical function $F(x, y)$ represents some physical quantity, such as temperature, in a region of the xy-plane. Then the level curves of F, where $F(x, y) = constant$, could be interpreted as isotherms on a weather map, as depicted in Figure 2.8.

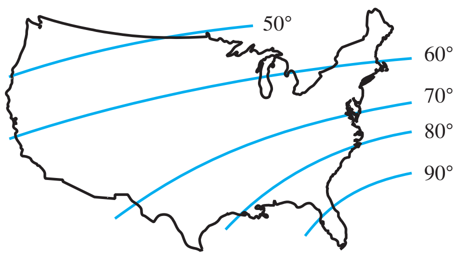


Figure 2.8 Level curves of $F(x, y)$

How does one calculate the slope of the tangent to a level curve? It is accomplished by implicit differentiation: One takes the derivative, with respect to x, of both sides of the equation $F(x, y) = C$, taking into account that y depends on x along the curve:

$$\frac{d}{dx}F(x,y) = \frac{d}{dx}(C) \qquad \text{or}$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0\,, \tag{1}$$

and solves for the slope:

(2) $$\frac{dy}{dx} = f(x, y) = -\frac{\partial F/\partial x}{\partial F/\partial y}.$$

The expression obtained by formally multiplying the left-hand member of (1) by dx is known as the *total differential* of F, written dF:

$$dF := \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy,$$

and our procedure for obtaining the equation for the slope $f(x, y)$ of the level curve $F(x, y) = C$ can be expressed as setting the total differential $dF = 0$ and solving.

Because equation (2) has the form of a differential equation, we should be able to reverse this logic and come up with a very easy technique for solving some differential equations. After all, any first-order differential equation $dy/dx = f(x, y)$ can be rewritten in the (differential) form

(3) $$M(x, y)\,dx + N(x, y)\,dy = 0$$

(in a variety of ways). Now, if the left-hand side of equation (3) can be identified as a total differential,

$$M(x, y)\,dx + N(x, y)\,dy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = dF(x, y),$$

then its solutions are given (implicitly) by the level curves

$$F(x, y) = C$$

for an arbitrary constant C.

Example 1 Solve the differential equation

$$\frac{dy}{dx} = -\frac{2xy^2 + 1}{2x^2y}.$$

Solution Some of the choices of differential forms corresponding to this equation are

$$(2xy^2 + 1)\,dx + 2x^2y\,dy = 0,$$

$$\frac{2xy^2 + 1}{2x^2y}dx + dy = 0,$$

$$dx + \frac{2x^2y}{2xy^2 + 1}dy = 0, \quad \text{etc.}$$

However, the first form is best for our purposes because it is a total differential of the function $F(x, y) = x^2y^2 + x$:

$$\begin{aligned}(2xy^2 + 1)\,dx + 2x^2y\,dy &= d[x^2y^2 + x]\\ &= \frac{\partial}{\partial x}(x^2y^2 + x)\,dx + \frac{\partial}{\partial y}(x^2y^2 + x)\,dy.\end{aligned}$$

Thus, the solutions are given implicitly by the formula $x^2y^2 + x = C$. See Figure 2.9 on page 59. ◆

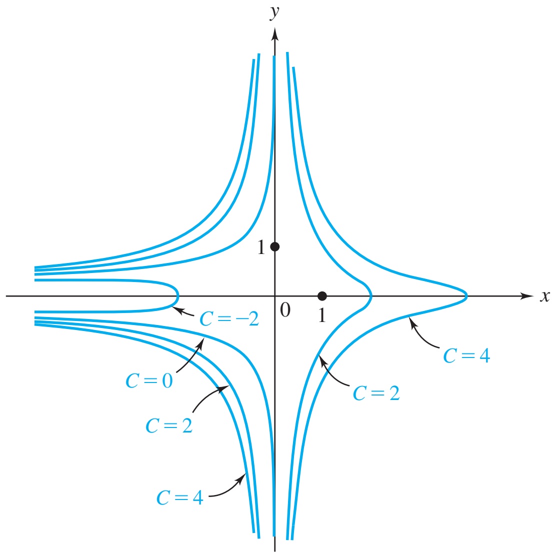


Figure 2.9 Solutions of Example 1

Next we introduce some terminology.

Exact Differential Form

Definition 2. The differential form $M(x, y)\,dx + N(x, y)\,dy$ is said to be **exact** in a rectangle R if there is a function $F(x, y)$ such that

$$\textbf{(4)} \qquad \frac{\partial F}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y) = N(x, y)$$

for all (x, y) in R. That is, the total differential of $F(x, y)$ satisfies

$$dF(x, y) = M(x, y)\,dx + N(x, y)\,dy\,.$$

If $M(x, y)\,dx + N(x, y)\,dy$ is an exact differential form, then the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is called an **exact equation.**

As you might suspect, in applications a differential equation is rarely given to us in exact differential form. However, the solution procedure is so quick and simple for such equations that we devote this section to it. From Example 1, we see that what is needed is (i) a test to determine if a differential form $M(x, y)\,dx + N(x, y)\,dy$ is exact and, if so, (ii) a procedure for finding the function $F(x, y)$ itself.

The test for exactness arises from the following observation. If

$$M(x, y)\,dx + N(x, y)\,dy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy\,,$$

then the calculus theorem concerning the equality of continuous mixed partial derivatives

$$\frac{\partial}{\partial y}\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\frac{\partial F}{\partial y}$$

would dictate a "compatibility condition" on the functions M and N:

$$\frac{\partial}{\partial y}M(x, y) = \frac{\partial}{\partial x}N(x, y) .$$

In fact, Theorem 2 states that the compatibility condition is also *sufficient* for the differential form to be exact.

Test for Exactness

Theorem 2. Suppose the first partial derivatives of $M(x, y)$ and $N(x, y)$ are continuous in a rectangle R. Then

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is an exact equation in R if and only if the compatibility condition

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y) \tag{5}$$

holds for all (x, y) in R.†

Before we address the proof of Theorem 2, note that in Example 1 the differential form that led to the total differential was

$$(2xy^2 + 1)\,dx + (2x^2y)\,dy = 0 .$$

The compatibility conditions are easily confirmed:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy^2 + 1) = 4xy ,$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x^2y) = 4xy .$$

Also clear is the fact that the other differential forms considered,

$$\frac{2xy^2 + 1}{2x^2y}dx + dy = 0 , \qquad dx + \frac{2x^2y}{2xy^2 + 1}dy = 0 ,$$

do *not* meet the compatibility conditions.

Proof of Theorem 2. There are two parts to the theorem: Exactness implies compatibility, and compatibility implies exactness. First, we have seen that if the differential equation is exact, then the two members of equation (5) are simply the mixed second partials of a function $F(x, y)$. As such, their equality is ensured by the theorem of calculus that states that mixed second partials are equal if they are continuous. Because the hypothesis of Theorem 2 guarantees the latter condition, equation (5) is validated.

---

† *Historical Footnote:* This theorem was proven by Leonhard Euler in 1734.

Rather than proceed directly with the proof of the second part of the theorem, let's derive a formula for a function $F(x, y)$ that satisfies $\partial F/\partial x = M$ and $\partial F/\partial y = N$. Integrating the first equation with respect to x yields

$$F(x, y) = \int M(x, y)\,dx + g(y)\ . \tag{6}$$

Notice that instead of using C to represent the constant of integration, we have written $g(y)$. This is because y is held fixed while integrating with respect to x, and so our "constant" may well depend on y. To determine $g(y)$, we differentiate both sides of (6) with respect to y to obtain

$$\frac{\partial F}{\partial y}(x, y) = \frac{\partial}{\partial y}\int M(x, y)\,dx + \frac{\partial}{\partial y}g(y)\ . \tag{7}$$

As g is a function of y alone, we can write $\partial g/\partial y = g'(y)$, and solving (7) for $g'(y)$ gives

$$g'(y) = \frac{\partial F}{\partial y}(x, y) - \frac{\partial}{\partial y}\int M(x, y)\,dx\ .$$

Since $\partial F/\partial y = N$, this last equation becomes

$$g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\,dx\ . \tag{8}$$

Notice that although the right-hand side of (8) indicates a possible dependence on x, *the appearances of this variable must cancel* because the left-hand side, $g'(y)$, depends only on y. By integrating (8), we can determine $g(y)$ up to a numerical constant, and therefore we can determine the function $F(x, y)$ up to a numerical constant from the functions $M(x, y)$ and $N(x, y)$.

To finish the proof of Theorem 2, we need to show that the condition (5) implies that $M\,dx + N\,dy = 0$ is an exact equation. This we do by actually exhibiting a function $F(x, y)$ that satisfies $\partial F/\partial x = M$ and $\partial F/\partial y = N$. Fortunately, we needn't look too far for such a function. The discussion in the first part of the proof suggests (6) as a candidate, where $g'(y)$ is given by (8). Namely, we *define* $F(x, y)$ by

$$F(x, y) := \int_{x_0}^{x} M(t, y)\,dt + g(y)\ , \tag{9}$$

where (x_0, y_0) is a fixed point in the rectangle R and $g(y)$ is determined, up to a numerical constant, by the equation

$$g'(y) := N(x, y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(t, y)\,dt\ . \tag{10}$$

Before proceeding we must address an extremely important question concerning the definition of $F(x, y)$. That is, how can we be sure (in this portion of the proof) that $g'(y)$, as given in equation (10), is really a function of just y alone? To show that the right-hand side of (10) is independent of x (that is, that the appearances of the variable x cancel), all we need to do is show that its partial derivative with respect to x is zero. This is where condition (5) is utilized. We leave to the reader this computation and the verification that $F(x, y)$ satisfies conditions (4) (see Problems 35 and 36). ◆

The construction in the proof of Theorem 2 actually provides an explicit procedure for solving exact equations. Let's recap and look at some examples.

Method for Solving Exact Equations

(a) If $M\,dx + N\,dy = 0$ is exact, then $\partial F/\partial x = M$. Integrate this last equation with respect to x to get

$$F(x, y) = \int M(x, y)\,dx + g(y)\,. \tag{11}$$

(b) To determine $g(y)$, take the partial derivative with respect to y of both sides of equation (11) and substitute N for $\partial F/\partial y$. We can now solve for $g'(y)$.

(c) Integrate $g'(y)$ to obtain $g(y)$ up to a numerical constant. Substituting $g(y)$ into equation (11) gives $F(x, y)$.

(d) The solution to $M\,dx + N\,dy = 0$ is given implicitly by

$$F(x, y) = C\,.$$

(Alternatively, starting with $\partial F/\partial y = N$, the implicit solution can be found by first integrating with respect to y; see Example 3.)

Example 2 Solve

$$(2xy - \sec^2 x)\,dx + (x^2 + 2y)\,dy = 0\,. \tag{12}$$

Solution Here $M(x, y) = 2xy - \sec^2 x$ and $N(x, y) = x^2 + 2y$. Because

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}\,,$$

equation (12) is exact. To find $F(x, y)$, we begin by integrating M with respect to x:

$$\begin{aligned} F(x, y) &= \int (2xy - \sec^2 x)\,dx + g(y) \\ &= x^2 y - \tan x + g(y)\,. \end{aligned} \tag{13}$$

Next we take the partial derivative of (13) with respect to y and substitute $x^2 + 2y$ for N:

$$\frac{\partial F}{\partial y}(x, y) = N(x, y)\,,$$

$$x^2 + g'(y) = x^2 + 2y\,.$$

Thus, $g'(y) = 2y$, and since the choice of the constant of integration is not important, we can take $g(y) = y^2$. Hence, from (13), we have $F(x, y) = x^2 y - \tan x + y^2$, and the solution to equation (12) is given implicitly by $x^2 y - \tan x + y^2 = C$. ◆

Remark. The procedure for solving exact equations requires several steps. As a check on our work, we observe that when we solve for $g'(y)$, we must obtain a function that is independent of x. If this is not the case, then we have erred either in our computation of $F(x, y)$ or in computing $\partial M/\partial y$ or $\partial N/\partial x$.

In the construction of $F(x, y)$, we can first integrate $N(x, y)$ with respect to y to get

$$F(x,y) = \int N(x,y)\,dy + h(x) \tag{14}$$

and then proceed to find $h(x)$. We illustrate this alternative method in the next example.

Example 3 Solve

(15) $$(1 + e^x y + xe^x y)\,dx + (xe^x + 2)\,dy = 0\,.$$

Solution Here $M = 1 + e^x y + xe^x y$ and $N = xe^x + 2$. Because

$$\frac{\partial M}{\partial y} = e^x + xe^x = \frac{\partial N}{\partial x}\,,$$

equation (15) is exact. If we now integrate $N(x, y)$ with respect to y, we obtain

$$F(x, y) = \int (xe^x + 2)\,dy + h(x) = xe^x y + 2y + h(x)\,.$$

When we take the partial derivative with respect to x and substitute for M, we get

$$\frac{\partial F}{\partial x}(x, y) = M(x, y)$$

$$e^x y + xe^x y + h'(x) = 1 + e^x y + xe^x y\,.$$

Thus, $h'(x) = 1$, so we take $h(x) = x$. Hence, $F(x, y) = xe^x y + 2y + x$, and the solution to equation (15) is given implicitly by $xe^x y + 2y + x = C$. In this case we can solve explicitly for y to obtain $y = (C - x)/(2 + xe^x)$. ◆

Remark. Since we can use either procedure for finding $F(x, y)$, it may be worthwhile to consider each of the integrals $\int M(x, y)\,dx$ and $\int N(x, y)\,dy$. If one is easier to evaluate than the other, this would be sufficient reason for us to use one method over the other. [The skeptical reader should try solving equation (15) by first integrating $M(x, y)$.]

Example 4 Show that

(16) $$(x + 3x^3 \sin y)\,dx + (x^4 \cos y)\,dy = 0$$

is *not* exact but that multiplying this equation by the factor x^{-1} yields an exact equation. Use this fact to solve (16).

Solution In equation (16), $M = x + 3x^3 \sin y$ and $N = x^4 \cos y$. Because

$$\frac{\partial M}{\partial y} = 3x^3 \cos y \not\equiv 4x^3 \cos y = \frac{\partial N}{\partial x}\,,$$

equation (16) is not exact. When we multiply (16) by the factor x^{-1}, we obtain

(17) $$(1 + 3x^2 \sin y)\,dx + (x^3 \cos y)\,dy = 0\,.$$

For this new equation, $M = 1 + 3x^2 \sin y$ and $N = x^3 \cos y$. If we test for exactness, we now find that

$$\frac{\partial M}{\partial y} = 3x^2 \cos y = \frac{\partial N}{\partial x}\,,$$

and hence (17) is exact. Upon solving (17), we find that the solution is given implicitly by $x + x^3 \sin y = C$. Since equations (16) and (17) differ only by a factor of x, then any solution to one will be a solution for the other whenever $x \neq 0$. Hence the solution to equation (16) is given implicitly by $x + x^3 \sin y = C$. ◆

In Section 2.5 we discuss methods for finding factors that, like x^{-1} in Example 4, change inexact equations into exact equations.

2.4 EXERCISES

In Problems 1–8, classify the equation as separable, linear, exact, or none of these. Notice that some equations may have more than one classification.

1. $(x^2y + x^4\cos x)\,dx - x^3\,dy = 0$
2. $(x^{10/3} - 2y)\,dx + x\,dy = 0$
3. $\sqrt{-2y - y^2}\,dx + (3 + 2x - x^2)\,dy = 0$
4. $(ye^{xy} + 2x)\,dx + (xe^{xy} - 2y)\,dy = 0$
5. $xy\,dx + dy = 0$
6. $y^2\,dx + (2xy + \cos y)\,dy = 0$
7. $[2x + y\cos(xy)]dx + [x\cos(xy) - 2y]dy = 0$
8. $\theta\,dr + (3r - \theta - 1)\,d\theta = 0$

In Problems 9–20, determine whether the equation is exact. If it is, then solve it.

9. $(2xy + 3)\,dx + (x^2 - 1)\,dy = 0$
10. $(2x + y)\,dx + (x - 2y)\,dy = 0$
11. $(e^x\sin y - 3x^2)\,dx + (e^x\cos y + y^{-2/3}/3)\,dy = 0$
12. $(\cos x\cos y + 2x)\,dx - (\sin x\sin y + 2y)\,dy = 0$
13. $e^t(y - t)\,dt + (1 + e^t)\,dy = 0$
14. $(t/y)\,dy + (1 + \ln y)\,dt = 0$
15. $\cos\theta\,dr - (r\sin\theta - e^\theta)\,d\theta = 0$
16. $(ye^{xy} - 1/y)\,dx + (xe^{xy} + x/y^2)\,dy = 0$
17. $(1/y)\,dx - (3y - x/y^2)\,dy = 0$
18. $[2x + y^2 - \cos(x + y)]\,dx$
$\quad + [2xy - \cos(x + y) - e^y]\,dy = 0$
19. $\left(2x + \dfrac{y}{1 + x^2y^2}\right)dx + \left(\dfrac{x}{1 + x^2y^2} - 2y\right)dy = 0$
20. $\left[\dfrac{2}{\sqrt{1 - x^2}} + y\cos(xy)\right]dx$
$\quad + [x\cos(xy) - y^{-1/3}]dy = 0$

In Problems 21–26, solve the initial value problem.

21. $(1/x + 2y^2x)\,dx + (2yx^2 - \cos y)\,dy = 0$,
$\quad y(1) = \pi$
22. $(ye^{xy} - 1/y)\,dx + (xe^{xy} + x/y^2)\,dy = 0$,
$\quad y(1) = 1$
23. $(e^ty + te^ty)\,dt + (te^t + 2)\,dy = 0$, $\quad y(0) = -1$
24. $(e^tx + 1)\,dt + (e^t - 1)\,dx = 0$, $\quad x(1) = 1$
25. $(y^2\sin x)\,dx + (1/x - y/x)\,dy = 0$, $\quad y(\pi) = 1$
26. $(\tan y - 2)\,dx + (x\sec^2 y + 1/y)\,dy = 0$,
$\quad y(0) = 1$

27. For each of the following equations, find the most general function $M(x, y)$ so that the equation is exact.
    (a) $M(x, y)\,dx + (\sec^2 y - x/y)\,dy = 0$
    (b) $M(x, y)\,dx + (\sin x\cos y - xy - e^{-y})\,dy = 0$
28. For each of the following equations, find the most general function $N(x, y)$ so that the equation is exact.
    (a) $[y\cos(xy) + e^x]\,dx + N(x, y)\,dy = 0$
    (b) $(ye^{xy} - 4x^3y + 2)\,dx + N(x, y)\,dy = 0$
29. Consider the equation
$$(y^2 + 2xy)\,dx - x^2\,dy = 0\ .$$
    (a) Show that this equation is not exact.
    (b) Show that multiplying both sides of the equation by y^{-2} yields a new equation that is exact.
    (c) Use the solution of the resulting exact equation to solve the original equation.
    (d) Were any solutions lost in the process?
30. Consider the equation
$$(5x^2y + 6x^3y^2 + 4xy^2)\,dx + (2x^3 + 3x^4y + 3x^2y)\,dy = 0\ .$$
    (a) Show that the equation is not exact.
    (b) Multiply the equation by x^ny^m and determine values for n and m that make the resulting equation exact.
    (c) Use the solution of the resulting exact equation to solve the original equation.
31. Argue that in the proof of Theorem 2 the function g can be taken as
$$g(y) = \int_{y_0}^{y} N(x, t)\,dt - \int_{y_0}^{y}\left[\frac{\partial}{\partial t}\int_{x_0}^{x} M(s, t)\,ds\right]dt\,,$$
    which can be expressed as
$$g(y) = \int_{y_0}^{y} N(x, t)\,dt - \int_{x_0}^{x} M(s, y)\,ds + \int_{x_0}^{x} M(s, y_0)\,ds\ .$$
    This leads ultimately to the representation
$$F(x, y) = \int_{y_0}^{y} N(x, t)\,dt + \int_{x_0}^{x} M(s, y_0)\,ds\ . \qquad (18)$$
    Evaluate this formula directly with $x_0 = 0$, $y_0 = 0$ to rework
    (a) Example 1.
    (b) Example 2.
    (c) Example 3.

32. Orthogonal Trajectories. A geometric problem occurring often in engineering is that of finding a family of curves (orthogonal trajectories) that intersects a given family of curves orthogonally at each point. For example, we may be given the lines of force of an electric field and want to find the equation for the equipotential curves. Consider the family of curves described by $F(x, y) = k$, where k is a parameter. Recall from the discussion of equation (2) that for each curve in the family, the slope is given by

$$\frac{dy}{dx} = -\frac{\partial F}{\partial x}\bigg/\frac{\partial F}{\partial y}.$$

(a) Recall that the slope of a curve that is orthogonal (perpendicular) to a given curve is just the negative reciprocal of the slope of the given curve. Using this fact, show that the curves orthogonal to the family $F(x, y) = k$ satisfy the differential equation

$$\frac{\partial F}{\partial y}(x, y)\,dx - \frac{\partial F}{\partial x}(x, y)\,dy = 0.$$

(b) Using the preceding differential equation, show that the orthogonal trajectories to the family of circles $x^2 + y^2 = k$ are just straight lines through the origin (see Figure 2.10).

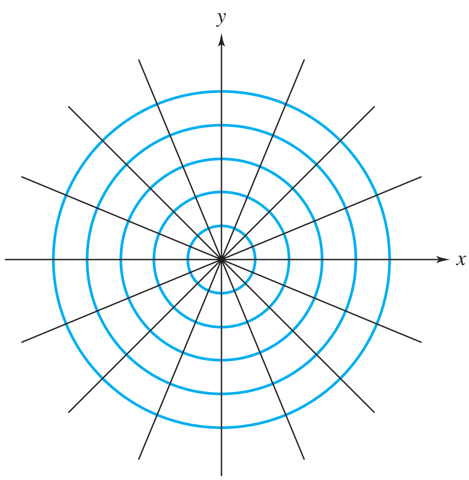


Figure 2.10 Orthogonal trajectories for concentric circles are lines through the center

(c) Show that the orthogonal trajectories to the family of hyperbolas $xy = k$ are the hyperbolas $x^2 - y^2 = k$ (see Figure 2.11).

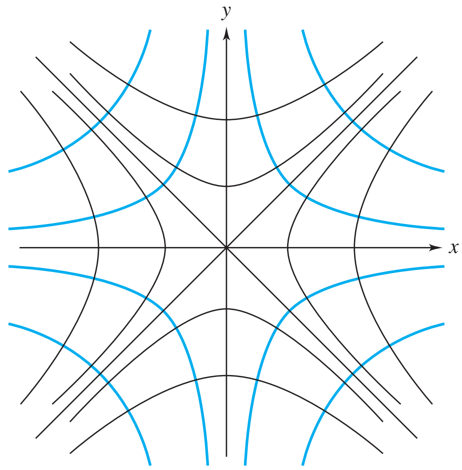


Figure 2.11 Families of orthogonal hyperbolas

33. Use the method in Problem 32 to find the orthogonal trajectories for each of the given families of curves, where k is a parameter.

(a) $2x^2 + y^2 = k$ **(b)** $y = kx^4$

(c) $y = e^{kx}$ **(d)** $y^2 = kx$

[*Hint:* First express the family in the form $F(x, y) = k$.]

34. Use the method described in Problem 32 to show that the orthogonal trajectories to the family of curves $x^2 + y^2 = kx$, k a parameter, satisfy

$$(2yx^{-1})\,dx + (y^2x^{-2} - 1)\,dy = 0.$$

Find the orthogonal trajectories by solving the above equation. Sketch the family of curves, along with their orthogonal trajectories. [*Hint:* Try multiplying the equation by $x^m y^n$ as in Problem 30.]

35. Using condition (5), show that the right-hand side of (10) is independent of x by showing that its partial derivative with respect to x is zero. [*Hint:* Since the partial derivatives of M are continuous, Leibniz's theorem allows you to interchange the operations of integration and differentiation.]

36. Verify that $F(x, y)$ as defined by (9) and (10) satisfies conditions (4).

2.5 Special Integrating Factors

If we take the standard form for the linear differential equation of Section 2.3,

$$\frac{dy}{dx} + P(x)y = Q(x) ,$$

and rewrite it in differential form by multiplying through by dx, we obtain

$$[P(x)y - Q(x)]\,dx + dy = 0 .$$

This form is certainly not exact, but it becomes exact upon multiplication by the integrating factor $\mu(x) = e^{\int P(x)\,dx}$. We have

$$[\mu(x)P(x)y - \mu(x)Q(x)]\,dx + \mu(x)\,dy = 0$$

as the form, and the compatibility condition is precisely the identity $\mu(x)P(x) = \mu'(x)$ (see Problem 20).

This leads us to generalize the notion of an integrating factor.

Integrating Factor

Definition 3. If the equation

(1) $$M(x, y)\,dx + N(x, y)\,dy = 0$$

is not exact, but the equation

(2) $$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0 ,$$

which results from multiplying equation (1) by the function $\mu(x, y)$, *is* exact, then $\mu(x, y)$ is called an **integrating factor**[†] of the equation (1).

Example 1 Show that $\mu(x, y) = xy^2$ is an integrating factor for

(3) $$(2y - 6x)\,dx + (3x - 4x^2y^{-1})\,dy = 0 .$$

Use this integrating factor to solve the equation.

Solution We leave it to you to show that (3) is not exact. Multiplying (3) by $\mu(x, y) = xy^2$, we obtain

(4) $$(2xy^3 - 6x^2y^2)\,dx + (3x^2y^2 - 4x^3y)\,dy = 0 .$$

For this equation we have $M = 2xy^3 - 6x^2y^2$ and $N = 3x^2y^2 - 4x^3y$. Because

$$\frac{\partial M}{\partial y}(x, y) = 6xy^2 - 12x^2y = \frac{\partial N}{\partial x}(x, y) ,$$

equation (4) is exact. Hence, $\mu(x, y) = xy^2$ is indeed an integrating factor of equation (3).

[†]*Historical Footnote:* A general theory of integrating factors was developed by Alexis Clairaut in 1739. Leonhard Euler also studied classes of equations that could be solved using a specific integrating factor.

Let's now solve equation (4) using the procedure of Section 2.4. To find $F(x, y)$, we begin by integrating M with respect to x:

$$F(x, y) = \int (2xy^3 - 6x^2y^2)\, dx + g(y) = x^2y^3 - 2x^3y^2 + g(y) .$$

When we take the partial derivative with respect to y and substitute for N, we find

$$\frac{\partial F}{\partial y}(x, y) = N(x, y)$$

$$3x^2y^2 - 4x^3y + g'(y) = 3x^2y^2 - 4x^3y .$$

Thus, $g'(y) = 0$, so we can take $g(y) \equiv 0$. Hence, $F(x, y) = x^2y^3 - 2x^3y^2$, and the solution to equation (4) is given implicitly by

$$x^2y^3 - 2x^3y^2 = C .$$

Although equations (3) and (4) have essentially the same solutions, *it is possible to lose or gain solutions when multiplying by* $\mu(x, y)$. In this case $y \equiv 0$ is a solution of equation (4) but not of equation (3). The extraneous solution arises because, when we multiply (3) by $\mu = xy^2$ to obtain (4), we are actually multiplying both sides of (3) by zero if $y \equiv 0$. This gives us $y \equiv 0$ as a solution to (4), but it is not a solution to (3). ◆

Generally speaking, when using integrating factors, you should check whether any solutions to $\mu(x, y) = 0$ are in fact solutions to the original differential equation.

How do we find an integrating factor? If $\mu(x, y)$ is an integrating factor of (1) with continuous first partial derivatives, then testing (2) for exactness, we must have

$$\frac{\partial}{\partial y}[\mu(x, y)M(x, y)] = \frac{\partial}{\partial x}[\mu(x, y)N(x, y)] .$$

By use of the product rule, this reduces to the equation

(5) $$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu .$$

But solving the partial differential equation (5) for μ is usually more difficult than solving the original equation (1). There are, however, two important exceptions.

Let's assume that equation (1) has an integrating factor that depends only on x; that is, $\mu = \mu(x)$. In this case equation (5) reduces to the separable equation

(6) $$\frac{d\mu}{dx} = \left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right)\mu ,$$

where $(\partial M/\partial y - \partial N/\partial x)/N$ is (presumably) just a function of x. In a similar fashion, if equation (1) has an integrating factor that depends only on y, then equation (5) reduces to the separable equation

(7) $$\frac{d\mu}{dy} = \left(\frac{\partial N/\partial x - \partial M/\partial y}{M}\right)\mu ,$$

where $(\partial N/\partial x - \partial M/\partial y)/M$ is just a function of y.

We can reverse the above argument. In particular, if $(\partial M/\partial y - \partial N/\partial x)/N$ is a function that depends only on x, then we can solve the separable equation (6) to obtain the integrating factor

$$\mu(x) = \exp\left[\int\left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right)dx\right]$$

for equation (1). We summarize these observations in the following theorem.

Special Integrating Factors

Theorem 3. If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu(x) = \exp\left[\int\left(\frac{\partial M/\partial y - \partial N/\partial x}{N}\right)dx\right] \tag{8}$$

is an integrating factor for equation (1).

If $(\partial N/\partial x - \partial M/\partial y)/M$ is continuous and depends only on y, then

$$\mu(y) = \exp\left[\int\left(\frac{\partial N/\partial x - \partial M/\partial y}{M}\right)dy\right] \tag{9}$$

is an integrating factor for equation (1).

Theorem 3 suggests the following procedure.

Method for Finding Special Integrating Factors

If $M\,dx + N\,dy = 0$ is neither separable nor linear, compute $\partial M/\partial y$ and $\partial N/\partial x$. If $\partial M/\partial y = \partial N/\partial x$, then the equation is exact. If it is not exact, consider

$$\frac{\partial M/\partial y - \partial N/\partial x}{N}. \tag{10}$$

If (10) is a function of just x, then an integrating factor is given by formula (8). If not, consider

$$\frac{\partial N/\partial x - \partial M/\partial y}{M}. \tag{11}$$

If (11) is a function of just y, then an integrating factor is given by formula (9).

Example 2 Solve

$$(2x^2 + y)\,dx + (x^2y - x)\,dy = 0. \tag{12}$$

Solution A quick inspection shows that equation (12) is neither separable nor linear. We also note that

$$\frac{\partial M}{\partial y} = 1 \neq (2xy - 1) = \frac{\partial N}{\partial x}.$$

Because (12) is not exact, we compute

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{1-(2xy-1)}{x^2y-x} = \frac{2(1-xy)}{-x(1-xy)} = \frac{-2}{x}\,.$$

We obtain a function of only x, so an integrating factor for (12) is given by formula (8). That is,

$$\mu(x) = \exp\left(\int \frac{-2}{x}dx\right) = x^{-2}\,.$$

When we multiply (12) by $\mu = x^{-2}$, we get the exact equation

$$(2+yx^{-2})\,dx + (y-x^{-1})\,dy = 0\,.$$

Solving this equation, we ultimately derive the implicit solution

$$2x - yx^{-1} + \frac{y^2}{2} = C\,. \tag{13}$$

Notice that the solution $x \equiv 0$ was lost in multiplying by $\mu = x^{-2}$. Hence, (13) and $x \equiv 0$ are solutions to equation (12). ◆

There are many differential equations that are not covered by Theorem 3 but for which an integrating factor nevertheless exists. The major difficulty, however, is in finding an explicit formula for these integrating factors, which in general will depend on both x and y.

2.5 EXERCISES

In Problems 1–6, identify the equation as separable, linear, exact, or having an integrating factor that is a function of either x alone or y alone.

1. $(2x + yx^{-1})\,dx + (xy-1)\,dy = 0$

2. $(2y^3 + 2y^2)\,dx + (3y^2x + 2xy)\,dy = 0$

3. $(2x + y)\,dx + (x - 2y)\,dy = 0$

4. $(y^2 + 2xy)\,dx - x^2\,dy = 0$

5. $(x^2 \sin x + 4y)\,dx + x\,dy = 0$

6. $(2y^2x - y)\,dx + x\,dy = 0$

In Problems 7–12, solve the equation.

7. $(2xy)\,dx + (y^2 - 3x^2)\,dy = 0$

8. $(3x^2 + y)\,dx + (x^2y - x)\,dy = 0$

9. $(x^4 - x + y)\,dx - x\,dy = 0$

10. $(2y^2 + 2y + 4x^2)\,dx + (2xy + x)\,dy = 0$

11. $(y^2 + 2xy)\,dx - x^2\,dy = 0$

12. $(2xy^3 + 1)\,dx + (3x^2y^2 - y^{-1})\,dy = 0$

In Problems 13 and 14, find an integrating factor of the form x^ny^m and solve the equation.

13. $(2y^2 - 6xy)\,dx + (3xy - 4x^2)\,dy = 0$

14. $(12 + 5xy)\,dx + (6xy^{-1} + 3x^2)\,dy = 0$

15. **(a)** Show that if $(\partial N/\partial x - \partial M/\partial y)/(xM - yN)$ depends only on the product xy, that is,

$$\frac{\partial N/\partial x - \partial M/\partial y}{xM - yN} = H(xy)\,,$$

then the equation $M(x, y)\,dx + N(x, y)\,dy = 0$ has an integrating factor of the form $\mu(xy)$. Give the general formula for $\mu(xy)$.

(b) Use your answer to part (a) to find an implicit solution to

$$(3y + 2xy^2)\,dx + (x + 2x^2y)\,dy = 0\,,$$

satisfying the initial condition $y(1) = 1$.

16. **(a)** Prove that $Mdx + Ndy = 0$ has an integrating factor that depends only on the sum $x + y$ if and only if the expression

$$\frac{\partial N/\partial x - \partial M/\partial y}{M - N}$$

depends only on $x + y$.

(b) Use part (a) to solve the equation $(3 + y + xy)dx + (3 + x + xy)dy = 0$.

17. **(a)** Find a condition on M and N that is necessary and sufficient for $Mdx + Ndy = 0$ to have an integrating factor that depends only on the product x^2y.

(b) Use part (a) to solve the equation

$$(2x + 2y + 2x^3y + 4x^2y^2)\,dx + (2x + x^4 + 2x^3y)\,dy = 0 \ .$$

18. If $xM(x, y) + yN(x, y) \equiv 0$, find the solution to the equation $M(x, y)\,dx + N(x, y)\,dy = 0$.

19. **Fluid Flow.** The streamlines associated with a certain fluid flow are represented by the family of curves $y = x - 1 + ke^{-x}$. The velocity potentials of the flow are just the orthogonal trajectories of this family.

(a) Use the method described in Problem 32 of Exercises 2.4 to show that the velocity potentials satisfy $dx + (x - y)\,dy = 0$.

[*Hint:* First express the family $y = x - 1 + ke^{-x}$ in the form $F(x, y) = k$.]

(b) Find the velocity potentials by solving the equation obtained in part (a).

20. Verify that when the linear differential equation $[P(x)y - Q(x)]\,dx + dy = 0$ is multiplied by $\mu(x) = e^{\int P(x)\,dx}$, the result is exact.

2.6 Substitutions and Transformations

When the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is not a separable, exact, or linear equation, it may still be possible to transform it into one that we know how to solve. This was in fact our approach in Section 2.5, where we used an integrating factor to transform our original equation into an exact equation.

In this section we study four types of equations that can be transformed into either a separable or linear equation by means of a suitable substitution or transformation.

Substitution Procedure

(a) Identify the type of equation and determine the appropriate substitution or transformation.
(b) Rewrite the original equation in terms of new variables.
(c) Solve the transformed equation.
(d) Express the solution in terms of the original variables.

Homogeneous Equations

Homogeneous Equation

Definition 4. If the right-hand side of the equation

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

can be expressed as a function of the ratio y/x alone, then we say the equation is **homogeneous.**

For example, the equation

$$(x - y)\,dx + x\,dy = 0 \tag{2}$$

can be written in the form

$$\frac{dy}{dx} = \frac{y - x}{x} = \frac{y}{x} - 1\,.$$

Since we have expressed $(y - x)/x$ as a function of the ratio y/x [that is, $(y - x)/x = G(y/x)$, where $G(v) \coloneqq v - 1$], then equation (2) is homogeneous.

The equation

$$(x - 2y + 1)\,dx + (x - y)\,dy = 0 \tag{3}$$

can be written in the form

$$\frac{dy}{dx} = \frac{x - 2y + 1}{y - x} = \frac{1 - 2(y/x) + (1/x)}{(y/x) - 1}\,.$$

Here the right-hand side cannot be expressed as a function of y/x alone because of the term $1/x$ in the numerator. Hence, equation (3) is not homogeneous.

One test for the homogeneity of equation (1) is to replace x by tx and y by ty. Then (1) is homogeneous if and only if

$$f(tx, ty) = f(x, y)$$

for all $t \neq 0$ [see Problem 43(a)].

To solve a homogeneous equation, we make a rather obvious substitution. Let

$$\boldsymbol{v = \frac{y}{x}}\,.$$

Our homogeneous equation now has the form

$$\frac{dy}{dx} = G(v)\,, \tag{4}$$

and all we need is to express dy/dx in terms of x and v. Since $v = y/x$, then $y = vx$. Keeping in mind that both v and y are functions of x, we use the product rule for differentiation to deduce from $y = vx$ that

$$\frac{dy}{dx} = v + x\frac{dv}{dx}\,.$$

We then substitute the above expression for dy/dx into equation (4) to obtain

$$v + x\frac{dv}{dx} = G(v)\,. \tag{5}$$

The new equation (5) is separable, and we can obtain its implicit solution from

$$\int \frac{1}{G(v) - v}\,dv = \int \frac{1}{x}\,dx\,.$$

All that remains to do is to express the solution in terms of the original variables x and y.

Example 1 Solve

$$(xy + y^2 + x^2)\,dx - x^2\,dy = 0 \,. \tag{6}$$

Solution A check will show that equation (6) is not separable, exact, or linear. If we express (6) in the derivative form

$$\frac{dy}{dx} = \frac{xy + y^2 + x^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1 \,, \tag{7}$$

then we see that the right-hand side of (7) is a function of just y/x. Thus, equation (6) is homogeneous.

Now let $v = y/x$ and recall that $dy/dx = v + x(dv/dx)$. With these substitutions, equation (7) becomes

$$v + x\frac{dv}{dx} = v + v^2 + 1 \,.$$

The above equation is separable, and, on separating the variables and integrating, we obtain

$$\int \frac{1}{v^2 + 1}\,dv = \int \frac{1}{x}\,dx \,,$$

$$\arctan v = \ln|x| + C \,.$$

Hence,

$$v = \tan(\ln|x| + C) \,.$$

Finally, we substitute y/x for v and solve for y to get

$$y = x\tan(\ln|x| + C)$$

as an explicit solution to equation (6). Also note that $x \equiv 0$ is a solution. ◆

Equations of the Form $\frac{dy}{dx} = G(ax + by)$

When the right-hand side of the equation $dy/dx = f(x, y)$ can be expressed as a function of the combination $ax + by$, where a and b are constants, that is,

$$\frac{dy}{dx} = G(ax + by) \,,$$

then the substitution

$$z = ax + by$$

transforms the equation into a separable one. The method is illustrated in the next example.

Example 2 Solve

$$\frac{dy}{dx} = y - x - 1 + (x - y + 2)^{-1} \,. \tag{8}$$

Solution The right-hand side can be expressed as a function of $x - y$, that is,

$$y - x - 1 + (x - y + 2)^{-1} = -(x - y) - 1 + [(x - y) + 2]^{-1} \,,$$

so let $z = x - y$. To solve for dy/dx, we differentiate $z = x - y$ with respect to x to obtain $dz/dx = 1 - dy/dx$, and so $dy/dx = 1 - dz/dx$. Substituting into (8) yields

$$1 - \frac{dz}{dx} = -z - 1 + (z+2)^{-1} ,$$

or

$$\frac{dz}{dx} = (z+2) - (z+2)^{-1} .$$

Solving this separable equation, we obtain

$$\int \frac{z+2}{(z+2)^2 - 1} dz = \int dx ,$$

$$\frac{1}{2} \ln|(z+2)^2 - 1| = x + C_1 ,$$

from which it follows that

$$(z+2)^2 = Ce^{2x} + 1 .$$

Finally, replacing z by $x - y$ yields

$$(x - y + 2)^2 = Ce^{2x} + 1$$

as an implicit solution to equation (8). ◆

Bernoulli Equations

Bernoulli Equation

Definition 5. A first-order equation that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n , \tag{9}$$

where $P(x)$ and $Q(x)$ are continuous on an interval (a, b) and n is a real number, is called a **Bernoulli equation.**†

Notice that when $n = 0$ or 1, equation (9) is also a linear equation and can be solved by the method discussed in Section 2.3. For other values of n, the substitution

$$v = y^{1-n}$$

transforms the Bernoulli equation into a linear equation, as we now show.

---

†*Historical Footnote:* This equation was proposed for solution by James Bernoulli in 1695. It was solved by his brother John Bernoulli. (James and John were two of eight mathematicians in the Bernoulli family.) In 1696, Gottfried Leibniz showed that the Bernoulli equation can be reduced to a linear equation by making the substitution $v = y^{1-n}$.

Dividing equation (9) by y^n yields

$$y^{-n}\frac{dy}{dx}+P(x)y^{1-n}=Q(x)\;. \tag{10}$$

Taking $v=y^{1-n}$, we find via the chain rule that

$$\frac{dv}{dx}=(1-n)y^{-n}\frac{dy}{dx}\,,$$

and so equation (10) becomes

$$\frac{1}{1-n}\frac{dv}{dx}+P(x)v=Q(x)\;.$$

Because $1/(1-n)$ is just a constant, the last equation is indeed linear.

Example 3 Solve

$$\frac{dy}{dx}-5y=-\frac{5}{2}xy^3\;. \tag{11}$$

Solution This is a Bernoulli equation with $n=3$, $P(x)=-5$, and $Q(x)=-5x/2$. To transform (11) into a linear equation, we first divide by y^3 to obtain

$$y^{-3}\frac{dy}{dx}-5y^{-2}=-\frac{5}{2}x\;.$$

Next we make the substitution $v=y^{-2}$. Since $dv/dx=-2y^{-3}\,dy/dx$, the transformed equation is

$$-\frac{1}{2}\frac{dv}{dx}-5v=-\frac{5}{2}x\,,$$

$$\frac{dv}{dx}+10v=5x\;. \tag{12}$$

Equation (12) is linear, so we can solve it for v using the method discussed in Section 2.3. When we do this, it turns out that

$$v=\frac{x}{2}-\frac{1}{20}+Ce^{-10x}\;.$$

Substituting $v=y^{-2}$ gives the solution

$$y^{-2}=\frac{x}{2}-\frac{1}{20}+Ce^{-10x}\;.$$

Not included in the last equation is the solution $y\equiv 0$ that was lost in the process of dividing (11) by y^3. ◆

A general formula for the solution to the Bernoulli equation (9) is given in Problem 48.

Equations with Linear Coefficients

We have used various substitutions for y to transform the original equation into a new equation that we could solve. In some cases we must transform *both* x and y into new variables, say, u and v. This is the situation for **equations with linear coefficients**—that is, equations of the form

$$(a_1x+b_1y+c_1)\,dx+(a_2x+b_2y+c_2)\,dy=0\;, \tag{13}$$

where the a_i's, b_i's, and c_i's are constants. We leave it as an exercise to show that when $a_1b_2 = a_2b_1$, equation (13) can be put in the form $dy/dx = G(ax + by)$, which we solved via the substitution $z = ax + by$. Moreover, if $b_1 = a_2$, then (13) is exact and can be solved using the method of Section 2.4.

Before considering the general case when $b_1 \neq a_2$ and $a_1b_2 \neq a_2b_1$, let's first look at the special situation when $c_1 = c_2 = 0$. Equation (13) then becomes

$$(a_1x + b_1y)\,dx + (a_2x + b_2y)\,dy = 0\,,$$

which can be rewritten in the form

$$\frac{dy}{dx} = -\frac{a_1x + b_1y}{a_2x + b_2y} = -\frac{a_1 + b_1(y/x)}{a_2 + b_2(y/x)}\,.$$

This equation is homogeneous, so we can solve it using the method discussed earlier in this section.

The above discussion suggests the following procedure for solving (13). If $b_1 \neq a_2$ and $a_1b_2 \neq a_2b_1$, then we seek a translation of axes of the form

$$\boldsymbol{x = u + h \qquad \text{and} \qquad y = v + k\,,}$$

where h and k are constants, that will change $a_1x + b_1y + c_1$ into $a_1u + b_1v$ and change $a_2x + b_2y + c_2$ into $a_2u + b_2v$. Some elementary algebra shows that such a transformation exists if the system of equations

$$\begin{aligned} a_1h + b_1k + c_1 &= 0\,, \\ a_2h + b_2k + c_2 &= 0 \end{aligned} \tag{14}$$

has a solution. This is ensured by the assumption $a_1b_2 \neq a_2b_1$, which is geometrically equivalent to assuming that the two lines described by the system (14) intersect. Now if (h, k) satisfies (14), then the substitutions $x = u + h$ and $y = v + k$ transform equation (13) into the homogeneous equation

$$\frac{dv}{du} = -\frac{a_1u + b_1v}{a_2u + b_2v} = -\frac{a_1 + b_1(v/u)}{a_2 + b_2(v/u)}\,, \tag{15}$$

which we know how to solve.

Example 4 Solve

$$(2x - 2y - 6)dx + (x - 3y - 5)dy = 0\,. \tag{16}$$

Solution Since $b_1 = -2 \neq 1 = a_2$ and $a_1b_2 = (2)(-3) \neq (1)(-2) = a_2b_1$, we will use the translation of axes $x = u + h$, $y = v + k$, where h and k satisfy the system

$$\begin{aligned} 2h - 2k &= 6 \\ h - 3k &= 5\,. \end{aligned}$$

Solving this system gives $h = 2$, $k = -1$. Next we substitute $x = u + 2$, $y = v - 1$ into (16), observing that $dx = du$ and $dy = dv$, and we get

$$(2u - 2v)du + (u - 3v)dv = 0$$

$$\frac{dv}{du} = \frac{2(v/u) - 2}{1 - 3(v/u)}\,.$$

The last equation is homogeneous, so we let $z = v/u$. Then $dv/du = z + u(dz/du)$, and substituting for v/u yields

$$z + u\frac{dz}{du} = \frac{2z - 2}{1 - 3z}.$$

Separating variables gives

$$\int \frac{1 - 3z}{3z^2 + z - 2}\, dz = \int \frac{1}{u}\, du\,,$$

from which, after utilizing a partial fraction expansion of the first integrand, we find

$$-\frac{4}{5}\ln|z + 1| - \frac{1}{5}\ln|3z - 2| = \ln|u| + C_1.$$

It follows after exponentiating that

$$|z + 1|^4|3z - 2| = C|u|^{-5},$$

and when we substitute back in for z, u and v, we obtain

$$\left|\frac{v}{u} + 1\right|^4\left|3\frac{v}{u} - 2\right| = C|u|^{-5},$$

$$|u + v|^4|2u - 3v| = C,$$

$$(x + y - 1)^4(2x - 3y - 7) = C.$$

The last equation gives an implicit solution to (16), with C any real constant. ◆

2.6 EXERCISES

In Problems 1–8, identify (do not solve) the equation as homogeneous, Bernoulli, linear coefficients, or of the form $y' = G(ax + by)$.

1. $2tx\, dx + (t^2 - x^2)\, dt = 0$
2. $(y - 4x - 1)^2\, dx - dy = 0$
3. $dy/dx + y/x = x^3y^2$
4. $(t + x + 2)\, dx + (3t - x - 6)\, dt = 0$
5. $\theta\, dy - y\, d\theta = \sqrt{\theta y}\, d\theta$
6. $(ye^{-2x} + y^3)\, dx - e^{-2x}\, dy = 0$
7. $\cos(x + y)\, dy = \sin(x + y)\, dx$
8. $(y^3 - \theta y^2)\, d\theta + 2\theta^2 y\, dy = 0$

Use the method discussed under "Homogeneous Equations" to solve Problems 9–16.

9. $(xy + y^2)\, dx - x^2\, dy = 0$
10. $(3x^2 - y^2)\, dx + (xy - x^3y^{-1})\, dy = 0$
11. $(y^2 - xy)\, dx + x^2\, dy = 0$
12. $(x^2 + y^2)\, dx + 2xy\, dy = 0$
13. $\dfrac{dx}{dt} = \dfrac{x^2 + t\sqrt{t^2 + x^2}}{tx}$
14. $\dfrac{dy}{d\theta} = \dfrac{\theta\sec(y/\theta) + y}{\theta}$
15. $\dfrac{dy}{dx} = \dfrac{x^2 - y^2}{3xy}$
16. $\dfrac{dy}{dx} = \dfrac{y(\ln y - \ln x + 1)}{x}$

Use the method discussed under "Equations of the Form $dy/dx = G(ax + by)$" to solve Problems 17–20.

17. $dy/dx = \sqrt{x + y} - 1$
18. $dy/dx = (x + y + 2)^2$
19. $dy/dx = (x - y + 5)^2$
20. $dy/dx = \sin(x - y)$

Use the method discussed under "Bernoulli Equations" to solve Problems 21–28.

21. $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2y^2$
22. $\dfrac{dy}{dx} - y = e^{2x}y^3$
23. $\dfrac{dy}{dx} = \dfrac{2y}{x} - x^2y^2$
24. $\dfrac{dy}{dx} + \dfrac{y}{x - 2} = 5(x - 2)y^{1/2}$
25. $\dfrac{dx}{dt} + tx^3 + \dfrac{x}{t} = 0$
26. $\dfrac{dy}{dx} + y = e^xy^{-2}$

27. $\dfrac{dr}{d\theta} = \dfrac{r^2 + 2r\theta}{\theta^2}$

28. $\dfrac{dy}{dx} + y^3 x + y = 0$

Use the method discussed under "Equations with Linear Coefficients" to solve Problems 29–32.

29. $(x + y - 1)\,dx + (y - x - 5)\,dy = 0$

30. $(-4x - y - 1)\,dx + (x + y + 3)\,dy = 0$

31. $(2x - y)\,dx + (4x + y - 3)\,dy = 0$

32. $(2x - y + 4)\,dx + (x - 2y - 2)\,dy = 0$

In Problems 33–40, solve the equation given in:

33. Problem 1.
34. Problem 2.
35. Problem 3.
36. Problem 4.
37. Problem 5.
38. Problem 6.
39. Problem 7.
40. Problem 8.

41. Use the substitution $v = x - y + 2$ to solve equation (8).

42. Use the substitution $y = vx^2$ to solve

$$\frac{dy}{dx} = \frac{2y}{x} + \cos(y/x^2)\ .$$

43. **(a)** Show that the equation $dy/dx = f(x, y)$ is homogeneous if and only if $f(tx, ty) = f(x, y)$. [*Hint:* Let $t = 1/x$.]

(b) A function $H(x, y)$ is called **homogeneous of order** $\boldsymbol{n}$ if $H(tx, ty) = t^n H(x, y)$. Show that the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is homogeneous if $M(x, y)$ and $N(x, y)$ are both homogeneous of the same order.

44. Show that equation (13) reduces to an equation of the form

$$\frac{dy}{dx} = G(ax + by)\ ,$$

when $a_1 b_2 = a_2 b_1$. [*Hint:* If $a_1 b_2 = a_2 b_1$, then $a_2/a_1 = b_2/b_1 = k$, so that $a_2 = k a_1$ and $b_2 = k b_1$.]

45. **Coupled Equations.** In analyzing coupled equations of the form

$$\frac{dy}{dt} = ax + by\ ,$$

$$\frac{dx}{dt} = \alpha x + \beta y\ ,$$

where a, b, α, and β are constants, we may wish to determine the relationship between x and y rather than the individual solutions $x(t)$, $y(t)$. For this purpose, divide the first equation by the second to obtain

$$\frac{dy}{dx} = \frac{ax + by}{\alpha x + \beta y}\ . \tag{17}$$

This new equation is homogeneous, so we can solve it via the substitution $v = y/x$. We refer to the solutions of (17) as **integral curves.** Determine the integral curves for the system

$$\frac{dy}{dt} = -4x - y\ ,$$

$$\frac{dx}{dt} = 2x - y\ .$$

46. **Magnetic Field Lines.** As described in Problem 20 of Exercises 1.3, the magnetic field lines of a dipole satisfy

$$\frac{dy}{dx} = \frac{3xy}{2x^2 - y^2}\ .$$

Solve this equation and sketch several of these lines.

47. **Riccati Equation.** An equation of the form

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x) \tag{18}$$

is called a generalized Riccati equation.†

(a) If one solution—say, $u(x)$—of (18) is known, show that the substitution $y = u + 1/v$ reduces (18) to a linear equation in v.

(b) Given that $u(x) = x$ is a solution to

$$\frac{dy}{dx} = x^3(y - x)^2 + \frac{y}{x}\ ,$$

use the result of part (a) to find all the other solutions to this equation. (The particular solution $u(x) = x$ can be found by inspection or by using a Taylor series method; see Section 8.1.)

48. Derive the following general formula for the solution to the Bernoulli equation (9):

$$y = \begin{cases} \left[\dfrac{(1-n)\int e^{(1-n)\int P(x)dx}\, Q(x)dx + C_1}{e^{(1-n)\int P(x)dx}}\right]^{1/(1-n)} & \text{for } n \neq 1 \\ C_2\, e^{\int [Q(x) - P(x)]dx} & \text{for } n = 1\ . \end{cases}$$

---

†*Historical Footnote:* Count Jacopo Riccati studied a particular case of this equation in 1724 during his investigation of curves whose radii of curvature depend only on the variable y and not the variable x.

Each time the model is used to predict the outcome of a process and hence solve a problem, it provides a test of the model that may lead to further refinements or simplifications. In many cases a model is simplified to give a quicker or less expensive answer—provided, of course, that sufficient accuracy is maintained.

One should always keep in mind that a model is *not* reality but only a representation of reality. The more refined models *may* provide an understanding of the underlying processes of nature. For this reason applied mathematicians strive for better, more refined models. Still, the real test of a model is its ability to find an acceptable answer for the posed problem.

In this chapter we discuss various models that involve differential equations. Section 3.2, Compartmental Analysis, studies mixing problems and population models. Sections 3.3 through 3.5 are physics-based and examine heating and cooling, Newtonian mechanics, and electrical circuits. Finally Sections 3.6 and 3.7 introduce some numerical methods for solving first-order initial value problems. This will enable us to consider more realistic models that cannot be solved using the methods of Chapter 2.

3.2 Compartmental Analysis

Many complicated processes can be broken down into distinct stages and the entire system modeled by describing the interactions between the various stages. Such systems are called **compartmental** and are graphically depicted by **block diagrams**. In this section we study the basic unit of these systems, a single compartment, and analyze some simple processes that can be handled by such a model.

The basic one-compartment system consists of a function $x(t)$ that represents the amount of a substance in the compartment at time t, an input rate at which the substance enters the compartment, and an output rate at which the substance leaves the compartment (see Figure 3.1).

Because the derivative of x with respect to t can be interpreted as the rate of change in the amount of the substance in the compartment with respect to time, the one-compartment system suggests

$$\textbf{(1)} \qquad \frac{dx}{dt} = \textbf{input rate} - \textbf{output rate}$$

as a mathematical model for the process.

Mixing Problems

A problem for which the one-compartment system provides a useful representation is the mixing of fluids in a tank. Let $x(t)$ represent the amount of a substance in a tank (compartment) at time t. To use the compartmental analysis model, we must be able to determine the rates at which this substance enters and leaves the tank. In mixing problems one is often given the rate at which a fluid containing the substance flows into the tank, along with the concentration of the substance in that fluid. Hence, multiplying the flow rate (volume/time) by the concentration (amount/volume) yields the input rate (amount/time).

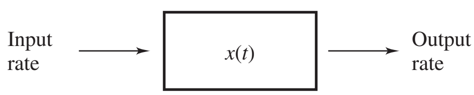


Figure 3.1 Schematic representation of a one-compartment system

The output rate of the substance is usually more difficult to determine. If we are given the exit rate of the mixture of fluids in the tank, then how do we determine the concentration of the substance in the mixture? One simplifying assumption that we might make is that the concentration is kept uniform in the mixture. Then we can compute the concentration of the substance in the mixture by dividing the amount $x(t)$ by the volume of the mixture in the tank at time t. Multiplying this concentration by the exit rate of the mixture then gives the desired output rate of the substance. This model is used in Examples 1 and 2.

Example 1 Consider a large tank holding 1000 L of pure water into which a brine solution of salt begins to flow at a constant rate of 6 L/min. The solution inside the tank is kept well stirred and is flowing out of the tank at a rate of 6 L/min. If the concentration of salt in the brine entering the tank is 0.1 kg/L, determine when the concentration of salt in the tank will reach 0.05 kg/L (see Figure 3.2).

Solution We can view the tank as a compartment containing salt. If we let $x(t)$ denote the mass of salt in the tank at time t, we can determine the concentration of salt in the tank by dividing $x(t)$ by the volume of fluid in the tank at time t. We use the mathematical model described by equation (1) to solve for $x(t)$.

First we must determine the rate at which salt enters the tank. We are given that brine flows into the tank at a rate of 6 L/min. Since the concentration is 0.1 kg/L, we conclude that the input rate of salt into the tank is

$$(6\ \text{L/min})(0.1\ \text{kg/L}) = 0.6\ \text{kg/min}\,. \tag{2}$$

We must now determine the output rate of salt from the tank. The brine solution in the tank is kept well stirred, so let's assume that the concentration of salt in the tank is uniform. That is, the concentration of salt in any part of the tank at time t is just $x(t)$ divided by the volume of fluid in the tank. Because the tank initially contains 1000 L and the rate of flow into the tank is the same as the rate of flow out, the volume is a constant 1000 L. Hence, the output rate of salt is

$$(6\ \text{L/min})\left[\frac{x(t)}{1000}\ \text{kg/L}\right] = \frac{3x(t)}{500}\ \text{kg/min}\,. \tag{3}$$

The tank initially contained pure water, so we set $x(0) = 0$. Substituting the rates in (2) and (3) into equation (1) then gives the initial value problem

$$\frac{dx}{dt} = 0.6 - \frac{3x}{500}\,, \qquad x(0) = 0\,, \tag{4}$$

as a mathematical model for the mixing problem.

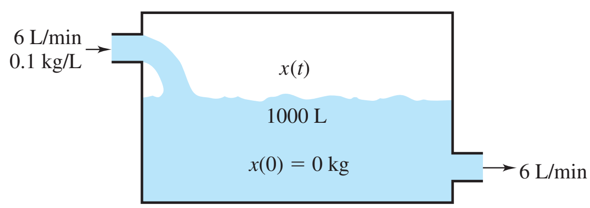


Figure 3.2 Mixing problem with equal flow rates

The equation in (4) is separable (and linear) and easy to solve. Using the initial condition $x(0) = 0$ to evaluate the arbitrary constant, we obtain

$$x(t) = 100(1 - e^{-3t/500}) . \tag{5}$$

Thus, the concentration of salt in the tank at time t is

$$\frac{x(t)}{1000} = 0.1\left(1 - e^{-3t/500}\right) \text{ kg/L} .$$

To determine when the concentration of salt is 0.05 kg/L, we set the right-hand side equal to 0.05 and solve for t. This gives

$$0.1\left(1 - e^{-3t/500}\right) = 0.05 \quad \text{or} \quad e^{-3t/500} = 0.5 ,$$

and hence

$$t = \frac{500 \ln 2}{3} \approx 115.52 \text{ min} .$$

Consequently the concentration of salt in the tank will be 0.05 kg/L after 115.52 min. ◆

From equation (5), we observe that the mass of salt in the tank steadily increases and has the limiting value

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} 100(1 - e^{-3t/500}) = 100 \text{ kg} .$$

Thus, the limiting concentration of salt in the tank is 0.1 kg/L, which is the same as the concentration of salt in the brine flowing into the tank. This certainly agrees with our expectations!

It might be interesting to see what would happen to the concentration if the flow rate into the tank is greater than the flow rate out.

Example 2 For the mixing problem described in Example 1, assume now that the brine leaves the tank at a rate of 5 L/min instead of 6 L/min, with all else being the same (see Figure 3.3). Determine the concentration of salt in the tank as a function of time.

Solution The difference between the rate of flow into the tank and the rate of flow out is $6 - 5 = 1$ L/min, so the volume of fluid in the tank after t minutes is $(1000 + t)$ L. Hence, the rate at which salt leaves the tank is

$$(5 \text{ L/min})\left[\frac{x(t)}{1000 + t} \text{ kg/L}\right] = \frac{5x(t)}{1000 + t} \text{ kg/min} .$$

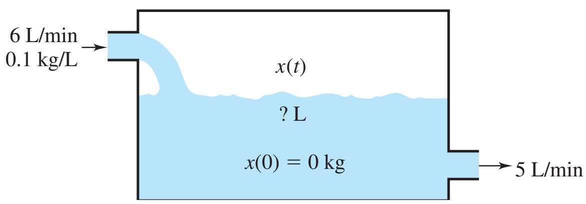


Figure 3.3 Mixing problem with unequal flow rates

Using this in place of (3) for the output rate gives the initial value problem

$$\frac{dx}{dt} = 0.6 - \frac{5x}{1000 + t}, \qquad x(0) = 0, \tag{6}$$

as a mathematical model for the mixing problem.

The differential equation in (6) is linear, so we can use the procedure outlined on page 50 to solve for $x(t)$. The integrating factor is $\mu(t) = (1000 + t)^5$. Thus,

$$\begin{aligned} \frac{d}{dt}\left[(1000 + t)^5 x\right] &= 0.6(1000 + t)^5 \\ (1000 + t)^5 x &= 0.1(1000 + t)^6 + c \\ x(t) &= 0.1(1000 + t) + c(1000 + t)^{-5}. \end{aligned}$$

Using the initial condition $x(0) = 0$, we find $c = -0.1(1000)^6$, and thus the solution to (6) is

$$x(t) = 0.1\left[(1000 + t) - (1000)^6(1000 + t)^{-5}\right].$$

Hence, the concentration of salt in the tank at time t is

$$\frac{x(t)}{1000 + t} = 0.1\left[1 - (1000)^6(1000 + t)^{-6}\right] \text{ kg/L}. \tag{7}$$ ◆

As in Example 1, the concentration given by (7) approaches 0.1 kg/L as $t \to \infty$. However, in Example 2 the volume of fluid in the tank becomes unbounded, and when the tank begins to overflow, the model in (6) is no longer appropriate.

Population Models

How does one predict the growth of a population? If we are interested in a single population, we can think of the species as being contained in a compartment (a petri dish, an island, a country, etc.) and study the growth process as a one-compartment system.

Let $p(t)$ be the population at time t. While the population is always an integer, it is usually large enough so that very little error is introduced in assuming that $p(t)$ is a continuous function. We now need to determine the growth (input) rate and the death (output) rate for the population.

Let's consider a population of bacteria that reproduce by simple cell division. In our model, we assume that the growth rate is proportional to the population present. This assumption is consistent with observations of bacterial growth. As long as there are sufficient space and ample food supply for the bacteria, we can also assume that the death rate is zero. (Remember that in cell division, the parent cell does not die, but becomes two new cells.) Hence, a mathematical model for a population of bacteria is

$$\frac{dp}{dt} = k_1 p, \qquad p(0) = p_0, \tag{8}$$

where $k_1 > 0$ is the proportionality constant for the growth rate and p_0 is the population at time $t = 0$. For human populations the assumption that the death rate is zero is certainly wrong! However, if we assume that people die only of natural causes, we might expect the death rate also to be proportional to the size of the population. That is, we revise (8) to read

$$\frac{dp}{dt} = k_1 p - k_2 p = (k_1 - k_2)p = kp, \tag{9}$$

where $k := k_1 - k_2$ and k_2 is the proportionality constant for the death rate. Let's assume that $k_1 > k_2$ so that $k > 0$. This gives the mathematical model

$$\frac{dp}{dt} = kp\,, \qquad p(0) = p_0\,, \tag{10}$$

which is called the **Malthusian,**† or **exponential, law** of population growth. This equation is separable, and solving the initial value problem for $p(t)$ gives

$$p(t) = p_0 e^{kt}\,. \tag{11}$$

To test the Malthusian model, let's apply it to the demographic history of the United States.

Example 3 In 1790 the population of the United States was 3.93 million, and in 1890 it was 62.98 million. Using the Malthusian model, estimate the U.S. population as a function of time.

Solution If we set $t = 0$ to be the year 1790, then by formula (11) we have

$$p(t) = (3.93)e^{kt}\,, \tag{12}$$

where $p(t)$ is the population in millions. One way to obtain a value for k would be to make the model fit the data for some specific year, such as 1890 (t = 100 years).†† We have

$$p(100) = 62.98 = (3.93)e^{100k}\,.$$

Solving for k yields

$$k = \frac{\ln(62.98) - \ln(3.93)}{100} \approx 0.027742\,.$$

Substituting this value in equation (12), we find

$$p(t) = (3.93)e^{(0.027742)t}\,. \tag{13}$$ ◆

In Table 3.1 on page 97 we list the U.S. population as given by the U.S. Bureau of the Census and the population predicted by the Malthusian model using equation (13). From Table 3.1 we see that the predictions based on the Malthusian model are in reasonable agreement with the census data until about 1900. After 1900 the predicted population is too large, and the Malthusian model is unacceptable.

We remark that a Malthusian model can be generated using the census data for any two different years. We selected 1790 and 1890 for purposes of comparison with the logistic model that we now describe.

The Malthusian model considered only death by natural causes. What about premature deaths due to malnutrition, inadequate medical supplies, communicable diseases, violent crimes, etc.? These factors involve a competition within the population, so we might

†*Historical Footnote:* Thomas R. Malthus (1766–1834) was a British economist who studied population models.

††The choice of the year 1890 is purely arbitrary, of course; a more democratic (and better) way of extracting parameters from data is described after Example 4.

TABLE 3.1 **A Comparison of the Malthusian and Logistic Models with U.S. Census Data (Population is given in Millions)**

Year	U.S. Census	Malthusian (Example 3)	Logistic (Example 4)	$\frac{1}{p}\frac{dp}{dt}$	Logistic (Least Squares)
1790	3.93	3.93	3.93		4.11
1800	5.31	5.19	5.30	0.0312	5.42
1810	7.24	6.84	7.13	0.0299	7.14
1820	9.64	9.03	9.58	0.0292	9.39
1830	12.87	11.92	12.82	0.0289	12.33
1840	17.07	15.73	17.07	0.0302	16.14
1850	23.19	20.76	22.60	0.0310	21.05
1860	31.44	27.40	29.70	0.0265	27.33
1870	39.82	36.16	38.66	0.0235	35.28
1880	50.19	47.72	49.71	0.0231	45.21
1890	62.98	62.98	62.98	0.0207	57.41
1900	76.21	83.12	78.42	0.0192	72.11
1910	92.23	109.69	95.73	0.0162	89.37
1920	106.02	144.76	114.34	0.0146	109.10
1930	123.20	191.05	133.48	0.0106	130.92
1940	132.16	252.13	152.26	0.0106	154.20
1950	151.33	333.74	169.90	0.0156	178.12
1960	179.32	439.12	185.76	0.0145	201.75
1970	203.30	579.52	199.50	0.0116	224.21
1980	226.54	764.80	211.00	0.0100	244.79
1990	248.71	1009.33	220.38	0.0110	263.01
2000	281.42	1332.03	227.84	0.0107	278.68
2010	308.75	1757.91	233.68		291.80
2020	?	2319.95	238.17		302.56

assume that another component of the death rate is proportional to the number of two-party interactions. There are $p(p-1)/2$ such possible interactions for a population of size p. Thus, if we combine the birth rate (8) with the death rate and rearrange constants, we get the **logistic model**

$$\frac{dp}{dt} = k_1 p - k_3 \frac{p(p-1)}{2}$$

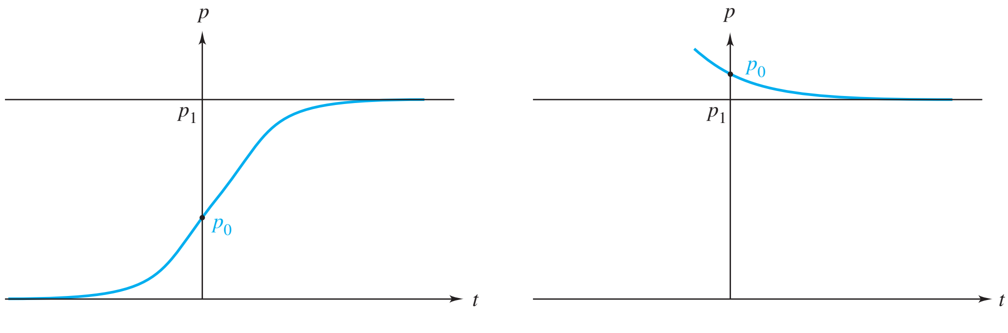
or

$$\textbf{(14)} \qquad \frac{dp}{dt} = -Ap(p-p_1)\,, \qquad p(0) = p_0\,,$$

where $A = k_3/2$ and $p_1 = (2k_1/k_3) + 1$.

Equation (14) has two equilibrium (constant) solutions: $p(t) \equiv p_1$ and $p(t) \equiv 0$. The nonequilibrium solutions can be found by separating variables and using the integral table at the back of the book:

$$\int \frac{dp}{p(p-p_1)} = -A\int dt \quad \text{or} \quad \frac{1}{p_1}\ln\left|\frac{p-p_1}{p}\right| = -At + c_1 \quad \text{or} \quad \left|1-\frac{p_1}{p}\right| = c_2 e^{-Ap_1 t}\,.$$



(a) $0 < p_0 < p_1$

(b) $p_0 > p_1$

Figure 3.4 The logistic curves

If $p(t) = p_0$ at $t = 0$, and $c_3 = 1 - p_1/p_0$, then solving for $p(t)$, we find

$$p(t) = \frac{p_1}{1 - c_3 e^{-Ap_1 t}} = \frac{p_0 p_1}{p_0 + (p_1 - p_0)e^{-Ap_1 t}} . \tag{15}$$

The function $p(t)$ given in (15) is called the **logistic function,** and graphs of logistic curves are displayed in Figure 3.4.† Note that for $A > 0$ and $p_0 > 0$, the limit population as $t \to \infty$, is p_1.

Let's test the logistic model on the population growth of the United States.

Example 4 Taking the 1790 population of 3.93 million as the initial population and given the 1840 and 1890 populations of 17.07 and 62.98 million, respectively, use the logistic model to estimate the population at time t.

Solution With $t = 0$ corresponding to the year 1790, we know that $p_0 = 3.93$. We must now determine the parameters A, p_1 in equation (15). For this purpose, we use the given facts that $p(50) = 17.07$ and $p(100) = 62.98$; that is,

$$17.07 = \frac{3.93\, p_1}{3.93 + (p_1 - 3.93)e^{-50Ap_1}} , \tag{16}$$

$$62.98 = \frac{3.93\, p_1}{3.93 + (p_1 - 3.93)e^{-100Ap_1}} . \tag{17}$$

Equations (16) and (17) are two nonlinear equations in the two unknowns A, p_1. To solve such a system, we would usually resort to a numerical approximation scheme such as Newton's method. However, for the case at hand, it is possible to find the solutions directly because the data are given at times t_a and t_b with $t_b = 2t_a$ (see Problem 12). Carrying out the algebra described in Problem 12, we ultimately find that

$$p_1 \approx 251.7812 \quad \text{and} \quad A \approx 0.0001210 . \tag{18}$$

Thus, the logistic model for the given data is

$$p(t) = \frac{989.50}{3.93 + (247.85)e^{-(0.030463)t}} . \tag{19}$$ ◆

†*Historical Footnote:* The logistic model for population growth was first developed by P. F. Verhulst around 1840.

The population data predicted by (19) are displayed in column 4 of Table 3.1 on page 97. As you can see, these predictions are in better agreement with the census data than the Malthusian model is. And, of course, the agreement is perfect in the years 1790, 1840, and 1890. However, the choice of these particular years for estimating the parameters p_0, A, and p_1 is quite arbitrary, and we would expect that a more robust model would use *all* of the data, in some way, for the estimation. One way to implement this idea is the following.

Observe from equation (14) that the logistic model predicts a linear relationship between $(dp/dt)/p$ and p:

$$\frac{1}{p}\frac{dp}{dt} = Ap_1 - Ap \,,$$

with Ap_1 as the intercept and $-A$ as the slope. In column five of Table 3.1, we list values of $(dp/dt)/p$, which are estimated from centered differences according to

$$\frac{1}{p(t)}\frac{dp}{dt}(t) \approx \frac{1}{p(t)}\frac{p(t+10)-p(t-10)}{20} \tag{20}$$

(see Problem 16). In Figure 3.5 these estimated values of $(dp/dt)/p$ are plotted against p in what is called a *scatter diagram*. The linear relationship predicted by the logistic model suggests that we approximate the plot by a straight line. A standard technique for doing this is the so-called *least-squares linear fit,* which is discussed in Appendix E. This yields the straight line

$$\frac{1}{p}\frac{dp}{dt} \approx 0.0280960 - 0.00008231p \,,$$

which is also depicted in Figure 3.5. Now with $A = 0.00008231$ and $p_1 = (0.0280960/A) \approx 341.4$, we can solve equation (15) for p_0:

$$p_0 = \frac{p(t)p_1e^{-Ap_1t}}{p_1 - p(t)\left[1 - e^{-Ap_1t}\right]} \,. \tag{21}$$

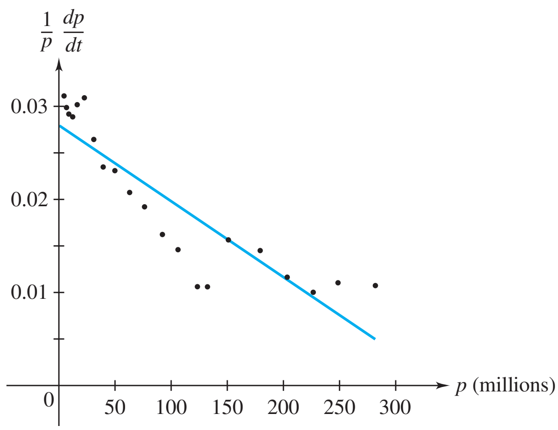


Figure 3.5 Scatter data and straight line fit

By averaging the right-hand side of (21) over all the data, we obtain the estimate $p_0 \approx 4.107$. Finally, the insertion of these estimates for the parameters in equation (15) leads to the predictions listed in column six of Table 3.1.

Note that this model yields $p_1 \approx 341.4$ million as the limit on the future population of the United States.

3.2 EXERCISES

1. A brine solution of salt flows at a constant rate of 8 L/min into a large tank that initially held 100 L of brine solution in which was dissolved 0.5 kg of salt. The solution inside the tank is kept well stirred and flows out of the tank at the same rate. If the concentration of salt in the brine entering the tank is 0.05 kg/L, determine the mass of salt in the tank after t min. When will the concentration of salt in the tank reach 0.02 kg/L?

2. A brine solution of salt flows at a constant rate of 6 L/min into a large tank that initially held 50 L of brine solution in which was dissolved 0.5 kg of salt. The solution inside the tank is kept well stirred and flows out of the tank at the same rate. If the concentration of salt in the brine entering the tank is 0.05 kg/L, determine the mass of salt in the tank after t min. When will the concentration of salt in the tank reach 0.03 kg/L?

3. A nitric acid solution flows at a constant rate of 6 L/min into a large tank that initially held 200 L of a 0.5% nitric acid solution. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 8 L/min. If the solution entering the tank is 20% nitric acid, determine the volume of nitric acid in the tank after t min. When will the percentage of nitric acid in the tank reach 10%?

4. A brine solution of salt flows at a constant rate of 4 L/min into a large tank that initially held 100 L of pure water. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 3 L/min. If the concentration of salt in the brine entering the tank is 0.2 kg/L, determine the mass of salt in the tank after t min. When will the concentration of salt in the tank reach 0.1 kg/L?

5. A swimming pool whose volume is 10,000 gal contains water that is 0.01% chlorine. Starting at $t = 0$, city water containing 0.001% chlorine is pumped into the pool at a rate of 5 gal/min. The pool water flows out at the same rate. What is the percentage of chlorine in the pool after 1 h? When will the pool water be 0.002% chlorine?

6. The air in a small room 12 ft by 8 ft by 8 ft is 3% carbon monoxide. Starting at $t = 0$, fresh air containing no carbon monoxide is blown into the room at a rate of 100 ft^3/min. If air in the room flows out through a vent at the same rate, when will the air in the room be 0.01% carbon monoxide?

7. Beginning at time $t = 0$, fresh water is pumped at the rate of 3 gal/min into a 60-gal tank initially filled with brine. The resulting less-and-less salty mixture overflows at the same rate into a second 60-gal tank that initially contained only pure water, and from there it eventually spills onto the ground. Assuming perfect mixing in both tanks, when will the water in the second tank taste saltiest? And exactly how salty will it then be, compared with the original brine?

8. A tank initially contains s_0 lb of salt dissolved in 200 gal of water, where s_0 is some positive number. Starting at time $t = 0$, water containing 0.5 lb of salt per gallon enters the tank at a rate of 4 gal/min, and the well-stirred solution leaves the tank at the same rate. Letting $c(t)$ be the concentration of salt in the tank at time t, show that the limiting concentration—that is, $\lim_{t\to\infty} c(t)$—is 0.5 lb/gal.

9. In 1990 the Department of Natural Resources released 1000 splake (a crossbreed of fish) into a lake. In 1997 the population of splake in the lake was estimated to be 3000. Using the Malthusian law for population growth, estimate the population of splake in the lake in the year 2020.

10. Use a sketch of the phase line (see Project B, Chapter 1, page 33) to argue that any solution to the mixing problem model

$$\frac{dx}{dt} = a - bx\,; \qquad a, b > 0\,,$$

approaches the equilibrium solution $x(t) \equiv a/b$ as t approaches $+\infty$; that is, a/b is a sink.

11. Use a sketch of the phase line (see Project B, Chapter 1) to argue that any solution to the logistic model

$$\frac{dp}{dt} = (a - bp)p\,; \qquad p(t_0) = p_0\,,$$

where a, b, and p_0 are positive constants, approaches the equilibrium solution $p(t) \equiv a/b$ as t approaches $+\infty$.

12. For the logistic curve (15), assume $p_a := p(t_a)$ and $p_b := p(t_b)$ are given with $t_b = 2t_a$ $(t_a > 0)$. Show that

$$p_1 = \left[\frac{p_a p_b - 2p_0 p_b + p_0 p_a}{p_a^2 - p_0 p_b}\right] p_a ,$$

$$A = \frac{1}{p_1 t_a} \ln\left[\frac{p_b(p_a - p_0)}{p_0(p_b - p_a)}\right] .$$

[*Hint:* Equate the expressions (21) for p_0 at times t_a and t_b. Set $\chi = \exp(-Ap_1 t_a)$ and $\chi^2 = \exp(-Ap_1 t_b)$ and solve for χ. Insert into one of the earlier expressions and solve for p_1.]

13. In Problem 9, suppose we have the additional information that the population of splake in 2004 was estimated to be 5000. Use a logistic model to estimate the population of splake in the year 2020. What is the predicted limiting population? [*Hint:* Use the formulas in Problem 12.]

14. In 1980 the population of alligators on the Kennedy Space Center grounds was estimated to be 1500. In 2006 the population had grown to an estimated 6000. Using the Malthusian law for population growth, estimate the alligator population on the Kennedy Space Center grounds in the year 2020.

15. In Problem 14, suppose we have the additional information that the population of alligators on the grounds of the Kennedy Space Center in 1993 was estimated to be 4100. Use a logistic model to estimate the population of alligators in the year 2020. What is the predicted limiting population? [*Hint:* Use the formulas in Problem 12.]

16. Show that for a differentiable function $p(t)$, we have

$$\lim_{h \to 0} \frac{p(t+h) - p(t-h)}{2h} = p'(t) ,$$

which is the basis of the centered difference approximation used in (20).

17. **(a)** For the U.S. census data, use the forward difference approximation to the derivative, that is,

$$\frac{1}{p(t)} \frac{dp}{dt}(t) \approx \frac{1}{p(t)} \frac{p(t+10) - p(t)}{10} ,$$

to recompute column 5 of Table 3.1 on page 97.

(b) Using the data from part (a), determine the constants A, p_1 in the least-squares fit

$$\frac{1}{p} \frac{dp}{dt} \approx Ap_1 - Ap$$

(see Appendix E).

(c) With the values for A and p_1 found in part (b), determine p_0 by averaging formula (21) over the data.

(d) Substitute A, p_1, and p_0 as determined above into the logistic formula (15) and calculate the populations predicted for each of the years listed in Table 3.1.

(e) Compare this model with that of the centered difference-based model in column 6 of Table 3.1.

18. Using the U.S. census data in Table 3.1 for 1900, 1920, and 1940 to determine parameters in the logistic equation model, what populations does the model predict for 2000 and 2010? Compare your answers with the census data for those years.

19. The initial mass of a certain species of fish is 7 million tons. The mass of fish, if left alone, would increase at a rate proportional to the mass, with a proportionality constant of 2/yr. However, commercial fishing removes fish mass at a constant rate of 15 million tons per year. When will all the fish be gone? If the fishing rate is changed so that the mass of fish remains constant, what should that rate be?

20. From theoretical considerations, it is known that light from a certain star should reach Earth with intensity I_0. However, the path taken by the light from the star to Earth passes through a dust cloud, with absorption coefficient 0.1/light-year. The light reaching Earth has intensity $1/2\, I_0$. How thick is the dust cloud? (The rate of change of light intensity with respect to thickness is proportional to the intensity. One light-year is the distance traveled by light during 1 yr.)

21. A snowball melts in such a way that the rate of change in its volume is proportional to its surface area. If the snowball was initially 4 in. in diameter and after 30 min its diameter is 3 in., when will its diameter be 2 in.? Mathematically speaking, when will the snowball disappear?

22. Suppose the snowball in Problem 21 melts so that the rate of change in its *diameter* is proportional to its surface area. Using the same given data, determine when its diameter will be 2 in. Mathematically speaking, when will the snowball disappear?

In Problems 23–27, assume that the rate of decay of a radioactive substance is proportional to the amount of the substance present. The half-life of a radioactive substance is the time it takes for one-half of the substance to disintegrate.

23. If initially there are 50 g of a radioactive substance and after 3 days there are only 10 g remaining, what percentage of the original amount remains after 4 days?

24. If initially there are 300 g of a radioactive substance and after 5 yr there are 200 g remaining, how much time must elapse before only 10 g remain?

25. Carbon dating is often used to determine the age of a fossil. For example, a humanoid skull was found in a cave in South Africa along with the remains of a campfire. Archaeologists believe the age of the skull to be the same age as the campfire. It is determined that only 2% of the original amount of carbon-14 remains in the burnt wood of the campfire. Estimate the age of the skull if the half-life of carbon-14 is about 5600 years.

change of current produces a high dI/dt and, in accordance with the formula $E_L = LdI/dt$, the inductor generates a voltage surge sufficient to cause a spark across the terminals—thus igniting the gasoline.

If an inductor and a capacitor *both* appear in a circuit, the governing differential equation will be second order. We'll return to *RLC* circuits in Section 5.7.

3.5 EXERCISES

1. An *RL* circuit with a 5-Ω resistor and a 0.05-H inductor carries a current of 1 A at $t = 0$, at which time a voltage source $E(t) = 5\cos 120t$ V is added. Determine the subsequent inductor current and voltage.
2. An *RC* circuit with a 1-Ω resistor and a 0.000001-F capacitor is driven by a voltage $E(t) = \sin 100t\ V$. If the initial capacitor voltage is zero, determine the subsequent resistor and capacitor voltages and the current.
3. The pathway for a binary electrical signal between gates in an integrated circuit can be modeled as an *RC* circuit, as in Figure 3.13(b); the voltage source models the transmitting gate, and the capacitor models the receiving gate. Typically, the resistance is 100 Ω, and the capacitance is very small, say, 10^{-12} F (1 picofarad, pF). If the capacitor is initially uncharged and the transmitting gate changes instantaneously from 0 to 5 V, how long will it take for the voltage at the receiving gate to reach (say) 3 V? (This is the time it takes to transmit a logical "1.")
4. If the resistance in the *RL* circuit of Figure 3.13(a) is zero, show that the current $I(t)$ is directly proportional to the integral of the applied voltage $E(t)$. Similarly, show that if the resistance in the *RC* circuit of Figure 3.13(b) is zero, the current is directly proportional to the derivative of the applied voltage.
5. The *power* generated or dissipated by a circuit element equals the voltage across the element times the current through the element. Show that the power dissipated by a resistor equals I^2R, the power associated with an inductor equals the derivative of $(1/2)LI^2$, and the power associated with a capacitor equals the derivative of $(1/2)CE_C^2$.
6. Derive a power balance equation for the *RL* and *RC* circuits. (See Problem 5.) Discuss the significance of the signs of the three power terms.
7. An industrial electromagnet can be modeled as an *RL* circuit, while it is being energized with a voltage source. If the inductance is 10 H and the wire windings contain 3 Ω of resistance, how long does it take a constant applied voltage to energize the electromagnet to within 90% of its final value (that is, the current equals 90% of its asymptotic value)?
8. A 10^{-8}-F capacitor (10 nanofarads) is charged to 50 V and then disconnected. One can model the charge leakage of the capacitor with a *RC* circuit with no voltage source and the resistance of the air between the capacitor plates. On a cold dry day, the resistance of the air gap is $5 \times 10^{13}\ \Omega$; on a humid day, the resistance is $7 \times 10^6\ \Omega$. How long will it take the capacitor voltage to dissipate to half its original value on each day?

3.6 Numerical Methods: A Closer Look At Euler's Algorithm

Although the analytical techniques presented in Chapter 2 were useful for the variety of mathematical models presented earlier in this chapter, *the majority of the differential equations encountered in applications cannot be solved either implicitly or explicitly.* This is especially true of higher-order equations and systems of equations, which we study in later chapters. In this section and the next, we discuss methods for obtaining a numerical *approximation* of the solution to an initial value problem for a first-order differential equation. Our goal is to develop algorithms that you can use with a calculator or computer.[†] These algorithms also extend naturally

[†]Appendix G describes various websites and commercial software that sketch direction fields and automate most of the differential equation algorithms discussed in this book.

to higher-order equations (see Section 5.3). We describe the rationale behind each method but leave the more detailed discussion to texts on numerical analysis.[†]

Consider the initial value problem

$$\textbf{(1)} \qquad y' = f(x, y) \,, \qquad y(x_0) = y_0 \,.$$

To guarantee that (1) has a unique solution, we assume that f and $\partial f/\partial y$ are continuous in a rectangle $R := \{(x, y): a < x < b, c < y < d\}$ containing (x_0, y_0). It follows from Theorem 1 in Chapter 1 (page 11) that the initial value problem (1) has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number. Because δ is not known *a priori*, there is no assurance that the solution will exist at a particular point $x\ (\neq x_0)$, even if x is in the interval (a, b). However, if $\partial f/\partial y$ is continuous and *bounded*[††] on the vertical strip

$$S := \{(x, y): a < x < b, -\infty < y < \infty\} \,,$$

then it turns out that (1) has a unique solution on the whole interval (a, b). In describing numerical methods, we assume that this last condition is satisfied and that f possesses as many continuous partial derivatives as needed.

In Section 1.4 we used the concept of direction fields to motivate a scheme for approximating the solution to the initial value problem (1). This scheme, called **Euler's method,** is one of the most basic, so it is worthwhile to discuss its advantages, disadvantages, and possible improvements. We begin with a derivation of Euler's method that is somewhat different from that presented in Section 1.4.

Let $h > 0$ be fixed (h is called the **step size**) and consider the equally spaced points

$$\textbf{(2)} \qquad x_n := x_0 + nh \,, \qquad n = 0, 1, 2, \ldots \,.$$

Our goal is to obtain an approximation to the solution $\phi(x)$ of the initial value problem (1) at those points x_n that lie in the interval (a, b). Namely, we will describe a method that generates values $y_0, y_1, y_2, \ldots$ that approximate $\phi(x)$ at the respective points $x_0, x_1, x_2, \ldots$; that is,

$$y_n \approx \phi(x_n) \,, \qquad n = 0, 1, 2, \ldots \,.$$

Of course, the first "approximant" y_0 is exact, since $y_0 = \phi(x_0)$ is given. Thus, we must describe how to compute $y_1, y_2, \ldots$.

For Euler's method we begin by integrating both sides of equation (1) from x_n to x_{n+1} to obtain

$$\phi(x_{n+1}) - \phi(x_n) = \int_{x_n}^{x_{n+1}} \phi'(t)\, dt = \int_{x_n}^{x_{n+1}} f(t, \phi(t))dt \,,$$

where we have substituted $\phi(x)$ for y. Solving for $\phi(x_{n+1})$, we have

$$\textbf{(3)} \qquad \phi(x_{n+1}) = \phi(x_n) + \int_{x_n}^{x_{n+1}} f(t, \phi(t))dt \,.$$

Without knowing $\phi(t)$, we cannot integrate $f(t, \phi(t))$. Hence, we must approximate the integral in (3). Assuming we have already found $y_n \approx \phi(x_n)$, the simplest approach is to

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[†] See, for example, *A First Course in the Numerical Analysis of Differential Equations,* 2nd ed., by A. Iserles (Cambridge University Press, 2008), or *Numerical Analysis*, 9th ed., by R. L. Burden and J. D. Faires (Cengage Learning, Independence, KY, 2011).

[††] A function $g(x, y)$ is bounded on S if there exists a number M such that $|g(x, y)| \le M$ for all (x, y) in S.

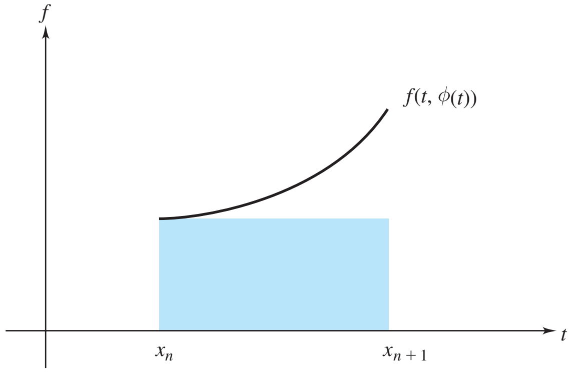

Figure 3.14 Approximation by a rectangle

approximate the area under the function $f(t, \phi(t))$ by the rectangle with base $[x_n, x_{n+1}]$ and height $f(x_n, \phi(x_n))$ (see Figure 3.14). This gives

$$\phi(x_{n+1}) \approx \phi(x_n) + (x_{n+1} - x_n)f(x_n, \phi(x_n)) .$$

Substituting h for $x_{n+1} - x_n$ and the approximation y_n for $\phi(x_n)$, we arrive at the numerical scheme

(4) $$y_{n+1} = y_n + hf(x_n, y_n) , \qquad n = 0, 1, 2, \ldots ,$$

which is Euler's method.

Starting with the given value y_0, we use (4) to compute $y_1 = y_0 + hf(x_0, y_0)$ and then use y_1 to compute $y_2 = y_1 + hf(x_1, y_1)$, and so on. Several examples of Euler's method can be found in Section 1.4. (Compare page 24.)

As discussed in Section 1.4, if we wish to use Euler's method to approximate the solution to the initial value problem (1) at a particular value of x, say, $x = c$, then we must first determine a suitable step size h so that $x_0 + Nh = c$ for some integer N. For example, we can take $N = 1$ and $h = c - x_0$ in order to arrive at the approximation after just one step:

$$\phi(c) = \phi(x_0 + h) \approx y_1 .$$

If, instead, we wish to take 10 steps in Euler's method, we choose $h = (c - x_0)/10$ and ultimately obtain

$$\phi(c) = \phi(x_0 + 10h) = \phi(x_{10}) \approx y_{10} .$$

In general, depending on the size of h, we will get different approximations to $\phi(c)$. It is reasonable to expect that as h gets smaller (or, equivalently, as N gets larger), the Euler approximations approach the exact value $\phi(c)$. On the other hand, as h gets smaller, the number (and cost) of computations increases and hence so do machine errors that arise from round-off. Thus, it is important to analyze how the error in the approximation scheme varies with h.

If Euler's method is used to approximate the solution $\phi(x) = e^x$ to the problem

(5) $$y' = y , \qquad y(0) = 1 ,$$

at $x = 1$, then we obtain approximations to the constant $e = \phi(1)$. It turns out that these approximations take a particularly simple form that enables us to compare the error in the approximation with the step size h. Indeed, setting $f(x, y) = y$ in (4) yields

$$y_{n+1} = y_n + hy_n = (1 + h)y_n , \qquad n = 0, 1, 2, \ldots .$$

TABLE 3.4 Euler's Approximations to $e = 2.71828\ldots$

h	Euler's Approximation $(1+h)^{1/h}$	Error $e-(1+h)^{1/h}$	Error$/h$
1	2.00000	0.71828	0.71828
10^{-1}	2.59374	0.12454	1.24539
10^{-2}	2.70481	0.01347	1.34680
10^{-3}	2.71692	0.00136	1.35790
10^{-4}	2.71815	0.00014	1.35902

Since $y_0 = 1$, we get

$$y_1 = (1+h)y_0 = 1+h\,,$$
$$y_2 = (1+h)y_1 = (1+h)(1+h) = (1+h)^2\,,$$
$$y_3 = (1+h)y_2 = (1+h)(1+h)^2 = (1+h)^3\,,$$

and, in general,

$$y_n = (1+h)^n\,, \qquad n = 0, 1, 2, \ldots\,. \tag{6}$$

For the problem in (5) we have $x_0 = 0$, so to obtain approximations at $x = 1$, we must set $nh = 1$. That is, h must be the reciprocal of an integer $(h = 1/n)$. Replacing n by $1/h$ in (6), we see that Euler's method gives the (familiar) approximation $(1+h)^{1/h}$ to the constant e. In Table 3.4, we list this approximation for $h = 1, 10^{-1}, 10^{-2}, 10^{-3}$, and 10^{-4}, along with the corresponding errors

$$e - (1+h)^{1/h}\,.$$

From the second and third columns in Table 3.4, we see that the approximation gains roughly one decimal place in accuracy as h decreases by a factor of 10; that is, the error is roughly proportional to h. This observation is further confirmed by the entries in the last column of Table 3.4. In fact, using methods of calculus (see Exercises 1.4, Problem 13), it can be shown that

$$\lim_{h\to 0}\frac{\text{error}}{h} = \lim_{h\to 0}\frac{e-(1+h)^{1/h}}{h} = \frac{e}{2} \approx 1.35914\,. \tag{7}$$

The general situation is similar: When Euler's method is used to approximate the solution to the initial value problem (1), the error in the approximation is at worst a constant times the step size h. Moreover, in view of (7), this is the best one can say.

Numerical analysts have a convenient notation for describing the convergence behavior of a numerical scheme. For fixed x we denote by $y(x;h)$ the approximation to the solution $\phi(x)$ of (1) obtained via the scheme when using a step size of h. We say that the numerical scheme **converges** at x if

$$\lim_{h\to 0} y(x;h) = \phi(x)\,.$$

In other words, as the step size h decreases to zero, the approximations for a convergent scheme approach the exact value $\phi(x)$. The rate at which $y(x;h)$ tends to $\phi(x)$ is often expressed in terms of a suitable power of h. If the error $\phi(x) - y(x;h)$ tends to zero like a constant times h^p, we write

$$\boldsymbol{\phi(x) - y(x;h) = O(h^p)}$$

and say that the **method is of order p.** Of course, the higher the power p, the faster is the rate of convergence as $h \to 0$.

As seen from our earlier discussion, the rate of convergence of Euler's method is $O(h)$; that is, *Euler's method is of order* $p = 1$. In fact, the limit in (7) shows that for equation (5), the error is roughly $1.36h$ for small h. This means that to have an error less than 0.01 requires $h < 0.01/1.36$, or $n = 1/h > 136$ computation steps. Thus Euler's method converges too slowly to be of practical use.

How can we improve Euler's method? To answer this, let's return to the derivation expressed in formulas (3) and (4) and analyze the "errors" that were introduced to get the approximation. The crucial step in the process was to approximate the integral

$$\int_{x_n}^{x_{n+1}} f\big(t, \phi(t)\big)\, dt$$

by using a rectangle (recall Figure 3.14 on page 123). This step gives rise to what is called the *local truncation error* in the method. From calculus we know that a better (more accurate) approach to approximating the integral is to use a trapezoid—that is, to apply the trapezoidal rule (see Figure 3.15). This gives

$$\int_{x_n}^{x_{n+1}} f\big(t, \phi(t)\big)\, dt \approx \frac{h}{2}\Big[f\big(x_n, \phi(x_n)\big) + f\big(x_{n+1}, \phi(x_{n+1})\big)\Big] ,$$

which leads to the numerical scheme

$$\textbf{(8)} \qquad y_{n+1} = y_n + \frac{h}{2}\big[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\big] , \qquad n = 0, 1, 2, \ldots .$$

We call equation (8) the **trapezoid scheme.** It is an example of an **implicit method;** that is, unlike Euler's method, equation (8) gives only an implicit formula for y_{n+1}, since y_{n+1} appears as an argument of f. Assuming we have already computed y_n, some root-finding technique such as Newton's method (see Appendix B) might be needed to compute y_{n+1}. Despite the inconvenience of working with an implicit method, the trapezoid scheme has two advantages over Euler's method. First, it is a method of order $p = 2$; that is, it converges at a rate that is proportional to h^2 and hence is faster than Euler's method. Second, as described in Project F, page 148, the trapezoid scheme has the desirable feature of being **stable.**

Can we somehow modify the trapezoid scheme in order to obtain an explicit method? One idea is first to get an estimate, say, y^*_{n+1}, of the value y_{n+1} using Euler's method and then use formula (8) with y_{n+1} replaced by y^*_{n+1} on the right-hand side. This two-step process is an example of a **predictor–corrector method.** That is, we predict y_{n+1} using Euler's

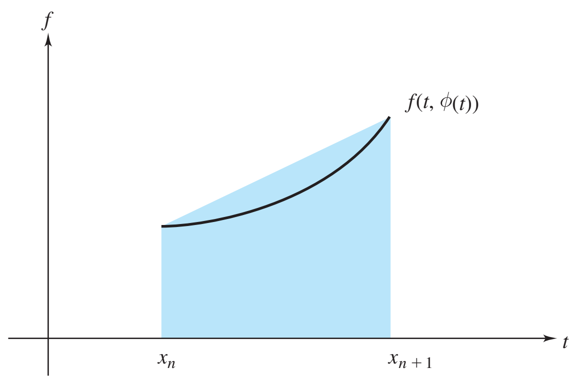

Figure 3.15 Approximation by a trapezoid

method and then use that value in (8) to obtain a "more correct" approximation. Setting $y_{n+1} = y_n + hf(x_n, y_n)$ in the right-hand side of (8), we obtain

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\big(x_n + h, y_n + hf(x_n, y_n)\big)\right], \qquad n = 0, 1, \ldots, \tag{9}$$

where $x_{n+1} = x_n + h$. This explicit scheme is known as the **improved Euler's method.**

Example 1 Compute the improved Euler's method approximation to the solution $\phi(x) = e^x$ of

$$y' = y, \qquad y(0) = 1$$

at $x = 1$ using step sizes of $h = 1, 10^{-1}, 10^{-2}, 10^{-3}$, and 10^{-4}.

Solution The starting values are $x_0 = 0$ and $y_0 = 1$. Since $f(x, y) = y$, formula (9) becomes

$$y_{n+1} = y_n + \frac{h}{2}\left[y_n + (y_n + hy_n)\right] = y_n + hy_n + \frac{h^2}{2}y_n ;$$

that is,

$$y_{n+1} = \left(1 + h + \frac{h^2}{2}\right)y_n . \tag{10}$$

Since $y_0 = 1$, we see inductively that

$$y_n = \left(1 + h + \frac{h^2}{2}\right)^n, \qquad n = 0, 1, 2, \ldots .$$

To obtain approximations at $x = 1$, we must have $1 = x_0 + nh = nh$, and so $n = 1/h$. Hence, the improved Euler's approximations to $e = \phi(1)$ are just

$$\left(1 + h + \frac{h^2}{2}\right)^{1/h} .$$

In Table 3.5 on page 127 we have computed this approximation for the specified values of h, along with the corresponding errors

$$e - \left(1 + h + \frac{h^2}{2}\right)^{1/h} .$$

Comparing the entries of this table with those of Table 3.4 on page 124, we observe that the improved Euler's method converges much more rapidly than the original Euler's method. In fact, from the first few entries in the second and third columns of Table 3.5, it appears that the approximation gains two decimal places in accuracy each time h is decreased by a factor of 10. In other words, the error is roughly proportional to h^2 (see the last column of the table and also Problem 4). The entries in the last row of the table must be regarded with caution. Indeed, when $h = 10^{-3}$, the true error is so small that our calculator rounded it to zero, to five decimal places. The entry in color in the last column may be inaccurate due to the loss of significant figures in the calculator arithmetic. ◆

As Example 1 suggests, the improved Euler's method converges at the rate $O(h^2)$, and indeed it can be proved that in general *this method is of order* $p = 2$.

TABLE 3.5 Improved Euler's Approximation to $e = 2.71828...$

h	Approximation $\left(1 + h + \frac{h^2}{2}\right)^{1/h}$	Error	Error$/h^2$
1	2.50000	0.21828	0.21828
10^{-1}	2.71408	0.00420	0.42010
10^{-2}	2.71824	0.00004	0.44966
10^{-3}	2.71828	0.00000	0.45271

A step-by-step outline for a subroutine that implements the improved Euler's method over a given interval $[x_0, c]$ is described below. For programming purposes it is usually more convenient to input the number of steps N in the interval rather than the step size h itself. For an interval starting at $x = x_0$ and ending at $x = c$, the relation between h and N is

$$Nh = c - x_0 . \tag{11}$$

(Note that the subroutine includes an option for printing x and y.) Of course, the implementation of this algorithm with N steps on the interval $[x_0, c]$ only produces approximations to the actual solution at $N + 1$ equally spaced points. If we wish to use these points to help graph an approximate solution over the whole interval $[x_0, c]$, then we must somehow "fill in" the gaps between these points. A crude method is to simply join the points by straight-line segments producing a polygonal line approximation to $\phi(x)$. More sophisticated techniques for prescribing the intermediate points are used in professional codes.

Improved Euler's Method Subroutine

Purpose To approximate the solution $\phi(x)$ to the initial value problem

$$y' = f(x, y) , \qquad y(x_0) = y_0 ,$$

for $x_0 \le x \le c$.

INPUT x_0, y_0, c, N (number of steps), PRNTR (= 1 to print a table)

Step 1 Set step size $h = (c - x_0)/N$, $x = x_0$, $y = y_0$

Step 2 For $i = 1$ to N, do Steps 3–5

Step 3 Set

$$F = f(x, y)$$
$$G = f(x + h, y + hF)$$

Step 4 Set

$$x = x + h$$
$$y = y + h(F + G)/2$$

Step 5 If PRNTR = 1, print x, y

Now we want to devise a program that will compute $\phi(c)$ *to a desired accuracy*. As we have seen, the accuracy of the approximation depends on the step size h. Our strategy, then, will be to estimate $\phi(c)$ for a given step size and then halve the step size and recompute the estimate, halve again, and so on. When two consecutive estimates of $\phi(c)$ differ by less than some prescribed tolerance ε, we take the final estimate as our approximation to $\phi(c)$. Admittedly, this does not guarantee that $\phi(c)$ is known to within ϵ, but it provides a reasonable stopping procedure in practice.† The following procedure also contains a safeguard to stop if the desired tolerance is not reached after M halvings of h.

Improved Euler's Method With Tolerance

Purpose To approximate the solution to the initial value problem

$$y' = f(x, y) , \qquad y(x_0) = y_0 ,$$

at $x = c$, with tolerance ε

INPUT x_0, y_0, c, ε ,
M (maximum number of halvings of step size)

Step 1 Set $z = y_0$, PRNTR $= 0$
Step 2 For $m = 0$ to M, do Steps 3–7††
Step 3 Set $N = 2^m$
Step 4 Call IMPROVED EULER'S METHOD SUBROUTINE
Step 5 Print h, y
Step 6 If $|y - z| < \varepsilon$, go to Step 10
Step 7 Set $z = y$
Step 8 Print "$\phi(c)$ is approximately"; y; "but may not be within the tolerance"; ε
Step 9 Go to Step 11
Step 10 Print "$\phi(c)$ is approximately"; y; "with tolerance"; ε
Step 11 STOP

OUTPUT Approximations of the solution to the initial value problem at $x = c$ using 2^m steps

If one desires a stopping procedure that simulates the **relative error**

$$\left| \frac{\text{approximation} - \text{true value}}{\text{true value}} \right| ,$$

then replace Step 6 by

$$\text{Step 6}'. \qquad \text{If} \quad \left| \frac{z - y}{y} \right| < \varepsilon , \quad \text{go to Step 10} .$$

Example 2 Use the improved Euler's method with tolerance to approximate the solution to the initial value problem

$$y' = x + 2y , \qquad y(0) = 0.25 , \tag{12}$$

at $x = 2$. For a tolerance of $\epsilon = 0.001$, use a stopping procedure based on the absolute error.

†Professional codes monitor accuracy much more carefully and vary step size in an adaptive fashion for this purpose.

††To save time, one can start with $m = K < M$ rather than with $m = 0$.

Solution The starting values are $x_0 = 0$, $y_0 = 0.25$. Because we are computing the approximations for $c = 2$, the initial value for h is

$$h = (2-0)2^{-0} = 2 .$$

For equation (12), we have $f(x, y) = x + 2y$, so the numbers F and G in the subroutine are

$$F = x + 2y ,$$
$$G = (x+h) + 2(y+hF) = x + 2y + h(1 + 2x + 4y) ,$$

and we find

$$x = x + h ,$$
$$y = y + \frac{h}{2}(F+G) = y + \frac{h}{2}(2x+4y) + \frac{h^2}{2}(1 + 2x + 4y) .$$

Thus, with $x_0 = 0$, $y_0 = 0.25$, and $h = 2$, we get for the first approximation

$$y = 0.25 + (0+1) + 2(1+1) = 5.25 .$$

To describe the further outputs of the algorithm, we use the notation $y(2; h)$ for the approximation obtained with step size h. Thus, $y(2; 2) = 5.25$, and we find from the algorithm

$$\begin{aligned} y(2;1) &= 11.25000 & \qquad y(2;2^{-5}) &= 25.98132 \\ y(2;2^{-1}) &= 18.28125 & y(2;2^{-6}) &= 26.03172 \\ y(2;2^{-2}) &= 23.06067 & y(2;2^{-7}) &= 26.04468 \\ y(2;2^{-3}) &= 25.12012 & y(2;2^{-8}) &= 26.04797 \\ y(2;2^{-4}) &= 25.79127 & y(2;2^{-9}) &= 26.04880 . \end{aligned}$$

Since $|y(2; 2^{-9}) - y(2; 2^{-8})| = 0.00083$, which is less than $\varepsilon = 0.001$, we stop.

The exact solution of (12) is $\phi(x) = \frac{1}{2}\left(e^{2x} - x - \frac{1}{2}\right)$, so we have determined that

$$\phi(2) = \frac{1}{2}\left(e^4 - \frac{5}{2}\right) \approx 26.04880 . \ \blacklozenge$$

In the next section, we discuss methods with higher rates of convergence than either Euler's or the improved Euler's methods.

3.6 EXERCISES

In many of the following problems, it will be essential to have a calculator or computer available. You may use a software package† or write a program for solving initial value problems using the improved Euler's method algorithms on pages 127 and 128. (Remember, all trigonometric calculations are done in radians.)

1. Show that when Euler's method is used to approximate the solution of the initial value problem
$$y' = 5y , \qquad y(0) = 1 ,$$
at $x = 1$, then the approximation with step size h is $(1 + 5h)^{1/h}$.

2. Show that when Euler's method is used to approximate the solution of the initial value problem
$$y' = -\frac{1}{2}y , \qquad y(0) = 3 ,$$
at $x = 2$, then the approximation with step size h is
$$3\left(1 - \frac{h}{2}\right)^{2/h} .$$

†Appendix G describes various websites and commercial software that sketch direction fields and automate most of the differential equation algorithms discussed in this book.

3. Show that when the trapezoid scheme given in formula (8) is used to approximate the solution $\phi(x) = e^x$ of

$$y' = y\,, \qquad y(0) = 1\,,$$

at $x = 1$, then we get

$$y_{n+1} = \left(\frac{1 + h/2}{1 - h/2}\right) y_n\,, \qquad n = 0, 1, 2, \ldots,$$

which leads to the approximation

$$\left(\frac{1 + h/2}{1 - h/2}\right)^{1/h}$$

for the constant e. Compute this approximation for $h = 1$, 10^{-1}, 10^{-2}, 10^{-3}, and 10^{-4} and compare your results with those in Tables 3.4 and 3.5.

4. In Example 1, page 126, the improved Euler's method approximation to e with step size h was shown to be

$$\left(1 + h + \frac{h^2}{2}\right)^{1/h}.$$

First prove that the error $:= e - (1 + h + h^2/2)^{1/h}$ approaches zero as $h \to 0$. Then use L'Hôpital's rule to show that

$$\lim_{h \to 0} \frac{\text{error}}{h^2} = \frac{e}{6} \approx 0.45305\,.$$

Compare this constant with the entries in the last column of Table 3.5.

5. Show that when the improved Euler's method is used to approximate the solution of the initial value problem

$$y' = 4y\,, \qquad y(0) = \frac{1}{3}\,,$$

at $x = 1/2$, then the approximation with step size h is

$$\frac{1}{3}(1 + 4h + 8h^2)^{1/(2h)}\,.$$

6. Since the integral $y(x) := \int_0^x f(t)\,dt$ with variable upper limit satisfies (for continuous f) the initial value problem

$$y' = f(x)\,, \qquad y(0) = 0\,,$$

any numerical scheme that is used to approximate the solution at $x = 1$ will give an approximation to the definite integral

$$\int_0^1 f(t)\,dt\,.$$

Derive a formula for this approximation of the integral using

(a) Euler's method.

(b) the trapezoid scheme.

(c) the improved Euler's method.

7. Use the improved Euler's method subroutine with step size $h = 0.1$ to approximate the solution to the initial value problem

$$y' = x - y^2\,, \qquad y(1) = 0\,,$$

at the points $x = 1.1$, 1.2, 1.3, 1.4, and 1.5. (Thus, input $N = 5$.) Compare these approximations with those obtained using Euler's method (see Exercises 1.4, Problem 5, page 28).

8. Use the improved Euler's method subroutine with step size $h = 0.2$ to approximate the solution to the initial value problem

$$y' = \frac{1}{x}(y^2 + y)\,, \qquad y(1) = 1\,,$$

at the points $x = 1.2$, 1.4, 1.6, and 1.8. (Thus, input $N = 4$.) Compare these approximations with those obtained using Euler's method (see Exercises 1.4, Problem 6, page 28).

9. Use the improved Euler's method subroutine with step size $h = 0.2$ to approximate the solution to

$$y' = x + 3\cos(xy)\,, \qquad y(0) = 0\,,$$

at the points $x = 0, 0.2, 0.4, \ldots, 2.0$. Use your answers to make a rough sketch of the solution on $[0, 2]$.

10. Use the improved Euler's method subroutine with step size $h = 0.1$ to approximate the solution to

$$y' = 4\cos(x + y)\,, \qquad y(0) = 1\,,$$

at the points $x = 0, 0.1, 0.2, \ldots, 1.0$. Use your answers to make a rough sketch of the solution on $[0, 1]$.

11. Use the improved Euler's method with tolerance to approximate the solution to

$$\frac{dx}{dt} = 1 + t\sin(tx)\,, \qquad x(0) = 0\,,$$

at $t = 1$. For a tolerance of $\varepsilon = 0.01$, use a stopping procedure based on the absolute error.

12. Use the improved Euler's method with tolerance to approximate the solution to

$$y' = 1 - \sin y\,, \qquad y(0) = 0\,,$$

at $x = \pi$. For a tolerance of $\varepsilon = 0.01$, use a stopping procedure based on the absolute error.

13. Use the improved Euler's method with tolerance to approximate the solution to

$$y' = 1 - y + y^3\,, \qquad y(0) = 0\,,$$

at $x = 1$. For a tolerance of $\varepsilon = 0.003$, use a stopping procedure based on the absolute error.

14. By experimenting with the improved Euler's method subroutine, find the maximum value over the interval $[0, 2]$ of the solution to the initial value problem

$$y' = \sin(x + y), \qquad y(0) = 2.$$

Where does this maximum value occur? Give answers to two decimal places.

15. The solution to the initial value problem

$$\frac{dy}{dx} = (x + y + 2)^2, \qquad y(0) = -2$$

crosses the x-axis at a point in the interval $[0, 1.4]$. By experimenting with the improved Euler's method subroutine, determine this point to two decimal places.

16. The solution to the initial value problem

$$\frac{dy}{dx} + \frac{y}{x} = x^3y^2, \qquad y(1) = 3$$

has a vertical asymptote ("blows up") at some point in the interval $[1, 2]$. By experimenting with the improved Euler's method subroutine, determine this point to two decimal places.

17. Use Euler's method (4) with $h = 0.1$ to approximate the solution to the initial value problem

$$y' = -20y, \qquad y(0) = 1,$$

on the interval $0 \le x \le 1$ (that is, at $x = 0, 0.1, \ldots, 1.0$). Compare your answers with the actual solution $y = e^{-20x}$. What went wrong? Next, try the step size $h = 0.025$ and also $h = 0.2$. What conclusions can you draw concerning the choice of step size?

18. Local versus Global Error. In deriving formula (4) for Euler's method, a rectangle was used to approximate the area under a curve (see Figure 3.14). With $g(t) := f(t, \phi(t))$, this approximation can be written as

$$\int_{x_n}^{x_{n+1}} g(t)\,dt \approx hg(x_n), \quad \text{where} \quad h = x_{n+1} - x_n.$$

(a) Show that if g has a continuous derivative that is bounded in absolute value by B, then the rectangle approximation has error $O(h^2)$; that is, for some constant M,

$$\left| \int_{x_n}^{x_{n+1}} g(t)\,dt - hg(x_n) \right| \le Mh^2.$$

This is called the *local truncation error* of the scheme. [*Hint:* Write

$$\int_{x_n}^{x_{n+1}} g(t)dt - hg(x_n) = \int_{x_n}^{x_{n+1}} [g(t) - g(x_n)]\,dt.$$

Next, using the mean value theorem, show that $|g(t) - g(x_n)| \le B|t - x_n|$. Then integrate to obtain the error bound $(B/2)h^2$.]

(b) In applying Euler's method, local truncation errors occur in each step of the process and are propagated throughout the further computations. Show that the *sum* of the local truncation errors in part (a) that arise after n steps is $O(h)$. This is the *global error,* which is the same as the convergence rate of Euler's method.

19. Building Temperature. In Section 3.3 we modeled the temperature inside a building by the initial value problem

$$\frac{dT}{dt} = K[M(t) - T(t)] + H(t) + U(t) \tag{13}$$
$$T(t_0) = T_0,$$

where M is the temperature outside the building, T is the temperature inside the building, H is the additional heating rate, U is the furnace heating or air conditioner cooling rate, K is a positive constant, and T_0 is the initial temperature at time t_0. In a typical model, $t_0 = 0$ (midnight), $T_0 = 65°F$, $H(t) = 0.1$, $U(t) = 1.5[70 - T(t)]$, and

$$M(t) = 75 - 20\cos(\pi t/12).$$

The constant K is usually between $1/4$ and $1/2$, depending on such things as insulation. To study the effect of insulating this building, consider the typical building described above and use the improved Euler's method subroutine with $h = 2/3$ to approximate the solution to (13) on the interval $0 \le t \le 24$ (1 day) for $K = 0.2, 0.4$, and 0.6.

20. Falling Body. In Example 1 of Section 3.4, page 110, we modeled the velocity of a falling body by the initial value problem

$$m\frac{dv}{dt} = mg - bv, \qquad v(0) = v_0,$$

under the assumption that the force due to air resistance is $-bv$. However, in certain cases the force due to air resistance behaves more like $-bv^r$, where $r(>1)$ is some constant. This leads to the model

$$m\frac{dv}{dt} = mg - bv^r, \qquad v(0) = v_0. \tag{14}$$

To study the effect of changing the parameter r in (14), take $m = 1$, $g = 9.81$, $b = 2$, and $v_0 = 0$. Then use the improved Euler's method subroutine with $h = 0.2$ to approximate the solution to (14) on the interval $0 \le t \le 5$ for $r = 1.0, 1.5$, and 2.0. What is the relationship between these solutions and the constant solution $v(t) \equiv (9.81/2)^{1/r}$?

3.7 Higher-Order Numerical Methods: Taylor and Runge–Kutta

In Sections 1.4 and 3.6, we discussed a simple numerical procedure, Euler's method, for obtaining a numerical approximation of the solution $\phi(x)$ to the initial value problem

$$\textbf{(1)} \qquad y' = f(x, y) , \qquad y(x_0) = y_0 .$$

Euler's method is easy to implement because it involves only linear approximations to the solution $\phi(x)$. But it suffers from slow convergence, being a method of order 1; that is, the error is $O(h)$. Even the improved Euler's method discussed in Section 3.6 has order of only 2. In this section we present numerical methods that have faster rates of convergence. These include **Taylor methods,** which are natural extensions of the Euler procedure, and **Runge–Kutta methods,** which are the more popular schemes for solving initial value problems because they have fast rates of convergence and are easy to program.

As in the previous section, we assume that f and $\partial f/\partial y$ are continuous and bounded on the vertical strip $\{(x, y): a < x < b, -\infty < y < \infty\}$ and that f possesses as many continuous partial derivatives as needed.

To derive the Taylor methods, let $\phi_n(x)$ be the *exact* solution of the related initial value problem

$$\textbf{(2)} \qquad \phi_n' = f(x, \phi_n) , \qquad \phi_n(x_n) = y_n .$$

The Taylor series for $\phi_n(x)$ about the point x_n is

$$\phi_n(x) = \phi_n(x_n) + h\phi_n'(x_n) + \frac{h^2}{2!}\phi_n''(x_n) + \cdots ,$$

where $h = x - x_n$. Since ϕ_n satisfies (2), we can write this series in the form

$$\textbf{(3)} \qquad \phi_n(x) = y_n + hf(x_n, y_n) + \frac{h^2}{2!}\phi_n''(x_n) + \cdots .$$

Observe that the recursive formula for y_{n+1} in Euler's method is obtained by truncating the Taylor series after the linear term. For a better approximation, we will use more terms in the Taylor series. This requires that we express the higher-order derivatives of the solution in terms of the function $f(x, y)$.

If y satisfies $y' = f(x, y)$, we can compute y'' by using the chain rule:

$$\textbf{(4)} \qquad \begin{aligned} y'' &= \frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y)\, y' \\ &= \frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y)\, f(x, y) \\ &=: f_2(x, y) . \end{aligned}$$

In a similar fashion, define $f_3, f_4, \ldots$, that correspond to the expressions $y'''(x), y^{(4)}(x)$, etc. If we truncate the expansion in (3) after the h^p term, then, with the above notation, the recursive formulas for the **Taylor method of order p** are

$$\textbf{(5)} \qquad \boldsymbol{x_{n+1} = x_n + h ,}$$

$$\textbf{(6)} \qquad \boldsymbol{y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2!}f_2(x_n, y_n) + \cdots + \frac{h^p}{p!}f_p(x_n, y_n) .}$$

As before, $y_n \approx \phi(x_n)$, where $\phi(x)$ is the solution to the initial value problem (1). It can be shown[†] that **the Taylor method of order p has the rate of convergence $O(h^p)$.**

Example 1 Determine the recursive formulas for the Taylor method of order 2 for the initial value problem

$$y' = \sin(xy)\,, \qquad y(0) = \pi\,. \tag{7}$$

Solution We must compute $f_2(x, y)$ as defined in (4). Since $f(x, y) = \sin(xy)$,

$$\frac{\partial f}{\partial x}(x, y) = y\cos(xy)\,, \qquad \frac{\partial f}{\partial y}(x, y) = x\cos(xy)\,.$$

Substituting into (4), we have

$$\begin{aligned} f_2(x, y) &= \frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y)\, f(x, y) \\ &= y\cos(xy) + x\cos(xy)\sin(xy) \\ &= y\cos(xy) + \frac{x}{2}\sin(2xy)\,, \end{aligned}$$

and the recursive formulas (5) and (6) become

$$\begin{aligned} x_{n+1} &= x_n + h\,, \\ y_{n+1} &= y_n + h\sin(x_n y_n) + \frac{h^2}{2}\left[y_n\cos(x_n y_n) + \frac{x_n}{2}\sin(2x_n y_n)\right], \end{aligned}$$

where $x_0 = 0$, $y_0 = \pi$ are the starting values. ◆

The convergence rate, $O(h^p)$, of the pth-order Taylor method raises an interesting question: If we could somehow let p go to infinity, would we obtain **exact** solutions for the interval $[x_0, x_0 + h]$? This possibility is explored in depth in Chapter 8. Of course, a practical difficulty in employing high-order Taylor methods is the tedious computation of the partial derivatives needed to determine f_p (typically these computations grow exponentially with p). One way to circumvent this difficulty is to use one of the **Runge–Kutta methods.**[††]

Observe that the general Taylor method has the form

$$y_{n+1} = y_n + hF(x_n, y_n; h)\,, \tag{8}$$

where the choice of F depends on p. In particular [compare (6)], for

$$p = 1\,, \qquad F = T_1(x, y; h) := f(x, y)\,,$$

$$p = 2\,, \qquad F = T_2(x, y; h) := f(x, y) + \frac{h}{2}\left[\frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y)\, f(x, y)\right]. \tag{9}$$

The idea behind the Runge–Kutta method of order 2 is to choose F in (8) of the form

$$F = K_2(x, y; h) := f\big(x + \alpha h, y + \beta h f(x, y)\big)\,, \tag{10}$$

where the constants α, β are to be selected so that (8) has the rate of convergence $O(h^2)$. The advantage here is that K_2 is computed by two evaluations of the original function $f(x, y)$ and does not involve the derivatives of $f(x, y)$.

[†] See *Introduction to Numerical Analysis* by J. Stoer and R. Bulirsch (Springer-Verlag, New York, 2002).

[††] *Historical Footnote:* These methods were developed by C. Runge in 1895 and W. Kutta in 1901.

To ensure $O(h^2)$ convergence, we compare this new scheme with the Taylor method of order 2 and require

$$T_2(x, y; h) - K_2(x, y; h) = O(h^2) \,, \quad \text{as} \quad h \to 0 \,.$$

That is, we choose α, β so that the Taylor expansions for T_2 and K_2 agree through terms of order h. For (x, y) fixed, when we expand $K_2 = K_2(h)$ as given in (10) about $h = 0$, we find

$$\textbf{(11)} \qquad \begin{aligned} K_2(h) &= K_2(0) + \frac{dK_2}{dh}(0)h + O(h^2) \\ &= f(x, y) + \left[\alpha\frac{\partial f}{\partial x}(x, y) + \beta\frac{\partial f}{\partial y}(x, y)\, f(x, y)\right]h + O(h^2) \,, \end{aligned}$$

where the expression in brackets for dK_2/dh, evaluated at $h = 0$, follows from the chain rule. Comparing (11) with (9), we see that for T_2 and K_2 to agree through terms of order h, we must have $\alpha = \beta = 1/2$. Thus,

$$K_2(x, y; h) = f\left(x + \frac{h}{2},\ y + \frac{h}{2}f(x, y)\right) .$$

The Runge–Kutta method we have derived is called the **midpoint method** and it has the recursive formulas

$$\textbf{(12)} \qquad x_{n+1} = x_n + h \,,$$

$$\textbf{(13)} \qquad y_{n+1} = y_n + hf\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2}f(x_n, y_n)\right) .$$

By construction, the midpoint method has the same rate of convergence as the Taylor method of order 2; that is, $O(h^2)$. This is the same rate as the improved Euler's method.

In a similar fashion, one can work with the Taylor method of order 4 and, after some elaborate calculations, obtain the **classical fourth-order Runge–Kutta method.** The recursive formulas for this method are

$$\textbf{(14)} \qquad \begin{aligned} x_{n+1} &= x_n + h \,, \\ y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \,, \end{aligned}$$

where

$$\begin{aligned} k_1 &= hf(x_n, y_n) \,, \\ k_2 &= hf\left(x_n + \frac{h}{2},\ y_n + \frac{k_1}{2}\right) , \\ k_3 &= hf\left(x_n + \frac{h}{2},\ y_n + \frac{k_2}{2}\right) , \\ k_4 &= hf(x_n + h,\ y_n + k_3) \,. \end{aligned}$$

The classical fourth-order Runge–Kutta method is one of the more popular methods because its rate of convergence is $O(h^4)$ and it is easy to program. Typically, it produces very accurate approximations even when the number of iterations is reasonably small. However, as the number of iterations becomes large, other types of errors may creep in.

Program outlines for the fourth-order Runge–Kutta method are given below. Just as with the algorithms for the improved Euler's method, the first program (the Runge–Kutta subroutine) is useful for approximating the solution over an interval $[x_0, c]$ and takes the number of steps in the interval as input. As in Section 3.6, the number of steps N is related to the step size h and the interval $[x_0, c]$ by

$$Nh = c - x_0 .$$

The subroutine has the option to print out a table of values of x and y. The second algorithm (Runge–Kutta with tolerance) on page 136 is used to approximate, for a given tolerance, the solution at an inputted value $x = c$. This algorithm[†] automatically halves the step sizes successively until the two approximations $y(c; h)$ and $y(c; h/2)$ differ by less than the prescribed tolerance ε. For a stopping procedure based on the relative error, Step 6 of the algorithm should be replaced by

Step 6′ If $\left|\dfrac{y - z}{y}\right| < \varepsilon$, go to Step 10 .

Classical Fourth-Order Runge–Kutta Subroutine

Purpose To approximate the solution to the initial value problem

$$y' = f(x, y) , \qquad y(x_0) = y_0$$

for $x_0 \le x \le c$

INPUT x_0, y_0, c, N (number of steps), PRNTR ($= 1$ to print a table)

Step 1 Set step size $h = (c - x_0)/N$, $x = x_0$, $y = y_0$

Step 2 For $i = 1$ to N, do Steps 3–5

Step 3 Set

$$k_1 = hf(x, y)$$

$$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x + \frac{h}{2}, y + \frac{k_2}{2}\right)$$

$$k_4 = hf(x + h, y + k_3)$$

Step 4 Set

$$x = x + h$$

$$y = y + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Step 5 If PRNTR $= 1$, print x, y

[†]Note that the form of the algorithm on page 136 is the same as that for the improved Euler's method on page 128 except for Step 4, where the Runge–Kutta subroutine is called. More sophisticated stopping procedures are used in production-grade codes.

Classical Fourth-Order Runge–Kutta Algorithm with Tolerance

Purpose To approximate the solution to the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

at $x = c$, with tolerance ε

INPUT	x_0, y_0, c, ε, M (maximum number of iterations)
Step 1	Set $z = y_0$, PRNTR $= 0$
Step 2	For $m = 0$ to M, do Steps 3–7 (or, to save time, start with $m > 0$)
Step 3	Set $N = 2^m$
Step 4	Call FOURTH-ORDER RUNGE–KUTTA SUBROUTINE
Step 5	Print h, y
Step 6	If $\lvert z - y \rvert < \varepsilon$, go to Step 10
Step 7	Set $z = y$
Step 8	Print "$\phi(c)$ is approximately"; y; "but may not be within the tolerance"; ε
Step 9	Go to Step 11
Step 10	Print "$\phi(c)$ is approximately"; y; "with tolerance"; ε
Step 11	STOP
OUTPUT	Approximations of the solution to the initial value problem at $x = c$, using 2^m steps.

Example 2 Use the classical fourth-order Runge–Kutta algorithm to approximate the solution $\phi(x)$ of the initial value problem

$$y' = y, \qquad y(0) = 1,$$

at $x = 1$ with a tolerance of 0.001.

Solution The inputs are $x_0 = 0$, $y_0 = 1$, $c = 1$, $\varepsilon = 0.001$, and $M = 25$ (say). Since $f(x, y) = y$, the formulas in Step 3 of the subroutine become

$$k = hy, \qquad k_2 = h\left(y + \frac{k_1}{2}\right), \qquad k_3 = h\left(y + \frac{k_2}{2}\right), \qquad k_4 = h(y + k_3).$$

The initial value for N in this algorithm is $N = 1$, so

$$h = (1 - 0)/1 = 1.$$

Thus, in Step 3 of the subroutine, we compute

$$\begin{aligned} k_1 &= (1)(1) = 1, & k_2 &= (1)(1 + 0.5) = 1.5, \\ k_3 &= (1)(1 + 0.75) = 1.75, & k_4 &= (1)(1 + 1.75) = 2.75, \end{aligned}$$

and, in Step 4 of the subroutine, we get for the first approximation

$$\begin{aligned} y &= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ &= 1 + \frac{1}{6}\left[1 + 2(1.5) + 2(1.75) + 2.75\right] \\ &= 2.70833, \end{aligned}$$

where we have rounded to five decimal places. Because

$$|z-y| = |y_0 - y| = |1 - 2.70833| = 1.70833 > \varepsilon\,,$$

we start over and reset $N = 2$, $h = 0.5$.

Doing Steps 3 and 4 for $i = 1$ and 2, we ultimately obtain (for $i = 2$) the approximation

$$y = 2.71735\,.$$

Since $|z-y| = |2.70833 - 2.71735| = 0.00902 > \varepsilon$, we again start over and reset $N = 4$, $h = 0.25$. This leads to the approximation

$$y = 2.71821\,,$$

so that

$$|z-y| = |2.71735 - 2.71821| = 0.00086\,,$$

which is less than $\varepsilon = 0.001$. Hence $\phi(1) = e \approx 2.71821$. ◆

In Example 2 we were able to obtain a better approximation for $\phi(1) = e$ with $h = 0.25$ than we obtained in Section 3.6 using Euler's method with $h = 0.001$ (see Table 3.4, page 124) and roughly the same accuracy as we obtained in Section 3.6 using the improved Euler's method with $h = 0.01$ (see Table 3.5, page 127).

Example 3 Use the fourth-order Runge–Kutta subroutine to approximate the solution $\phi(x)$ of the initial value problem

$$y' = y^2\,, \qquad y(0) = 1\,, \tag{15}$$

on the interval $0 \le x \le 2$ using $N = 8$ steps (i.e., $h = 0.25$).

Solution Here the starting values are $x_0 = 0$ and $y_0 = 1$. Since $f(x, y) = y^2$, the formulas in Step 3 of the subroutine are

$$k_1 = hy^2\,, \qquad k_2 = h\left(y + \frac{k_1}{2}\right)^2,$$

$$k_3 = h\left(y + \frac{k_2}{2}\right)^2, \qquad k_4 = h(y + k_3)^2\,.$$

From the output, we find

$$\begin{aligned} x &= 0.25 & y &= 1.33322\,,\\ x &= 0.50 & y &= 1.99884\,,\\ x &= 0.75 & y &= 3.97238\,,\\ x &= 1.00 & y &= 32.82820\,,\\ x &= 1.25 & y &= 4.09664 * 10^{11}\,,\\ x &= 1.50 & y &= \text{overflow}\,. \end{aligned}$$

What happened? Fortunately, the equation in (15) is separable, and, solving for $\phi(x)$, we obtain $\phi(x) = (1-x)^{-1}$. It is now obvious where the problem lies: The true solution $\phi(x)$ is not defined at $x = 1$. If we had been more cautious, we would have realized that $\partial f/\partial y = 2y$ is *not* bounded for all y. Hence, the existence of a unique solution is not guaranteed for all x between 0 and 2, and in this case, the method does *not* give meaningful approximations for x near (or greater than) 1. ◆

Example 4 Use the fourth-order Runge–Kutta algorithm to approximate the solution $\phi(x)$ of the initial value problem

$$y' = x - y^2 , \qquad y(0) = 1 ,$$

at $x = 2$ with a tolerance of 0.0001.

Solution This time we check to see whether $\partial f/\partial y$ is bounded. Here $\partial f/\partial y = -2y$, which is certainly unbounded in any vertical strip. However, let's consider the qualitative behavior of the solution $\phi(x)$. The solution curve starts at (0, 1), where $\phi'(0) = 0 - 1 < 0$, so $\phi(x)$ begins decreasing and continues to decrease until it crosses the curve $y = \sqrt{x}$. After crossing this curve, $\phi(x)$ begins to increase, since $\phi'(x) = x - \phi^2(x) > 0$. As $\phi(x)$ increases, it remains below the curve $y = \sqrt{x}$. This is because if the solution were to get "close" to the curve $y = \sqrt{x}$, then the derivative of $\phi(x)$ would approach zero, so that overtaking the function $\sqrt{x}$ is impossible.

Therefore, although the existence-uniqueness theorem does not guarantee a solution, we are inclined to try the algorithm anyway. The above argument shows that $\phi(x)$ probably exists for $x > 0$, so we feel reasonably sure the fourth-order Runge–Kutta method will give a good approximation of the true solution $\phi(x)$. Proceeding with the algorithm, we use the starting values $x_0 = 0$ and $y_0 = 1$. Since $f(x, y) = x - y^2$, the formulas in Step 3 of the subroutine become

$$k_1 = h(x - y^2) , \qquad k_2 = h\left[\left(x + \frac{h}{2}\right) - \left(y + \frac{k_1}{2}\right)^2\right] ,$$

$$k_3 = h\left[\left(x + \frac{h}{2}\right) - \left(y + \frac{k_2}{2}\right)^2\right] , \qquad k_4 = h\left[(x + h) - (y + k_3)^2\right] .$$

In Table 3.6, we give the approximations $y(2; 2^{-m+1})$ for $\phi(2)$ for $m = 0, 1, 2, 3$, and 4. The algorithm stops at $m = 4$, since

$$|y(2; 0.125) - y(2; 0.25)| = 0.00000 .$$

Hence, $\phi(2) \approx 1.25132$ with a tolerance of 0.0001. ◆

TABLE 3.6 Classical Fourth-Order Runge–Kutta Approximation for $\phi(2)$

m	h	Approximation for $\phi(2)$	$\lvert y(2; h) - y(2; 2h) \rvert$
0	2.0	−8.33333	
1	1.0	1.27504	9.60837
2	0.5	1.25170	0.02334
3	0.25	1.25132	0.00038
4	0.125	1.25132	0.00000

3.7 Exercises

As in Exercises 3.6, for some problems you will find it essential to have a calculator or computer available.[†] *For Problems 1–17, note whether or not $\partial f/\partial y$ is bounded.*

1. Determine the recursive formulas for the Taylor method of order 2 for the initial value problem

$$y' = \cos(x+y), \qquad y(0) = \pi.$$

2. Determine the recursive formulas for the Taylor method of order 2 for the initial value problem

$$y' = xy - y^2, \qquad y(0) = -1.$$

3. Determine the recursive formulas for the Taylor method of order 4 for the initial value problem

$$y' = x - y, \qquad y(0) = 0.$$

4. Determine the recursive formulas for the Taylor method of order 4 for the initial value problem

$$y' = x^2 + y, \qquad y(0) = 0.$$

5. Use the Taylor methods of orders 2 and 4 with $h = 0.25$ to approximate the solution to the initial value problem

$$y' = x + 1 - y, \qquad y(0) = 1,$$

at $x = 1$. Compare these approximations to the actual solution $y = x + e^{-x}$ evaluated at $x = 1$.

6. Use the Taylor methods of orders 2 and 4 with $h = 0.25$ to approximate the solution to the initial value problem

$$y' = 1 - y, \qquad y(0) = 0,$$

at $x = 1$. Compare these approximations to the actual solution $y = 1 - e^{-x}$ evaluated at $x = 1$.

7. Use the fourth-order Runge–Kutta subroutine with $h = 0.25$ to approximate the solution to the initial value problem

$$y' = 2y - 6, \qquad y(0) = 1,$$

at $x = 1$. (Thus, input $N = 4$.) Compare this approximation to the actual solution $y = 3 - 2e^{2x}$ evaluated at $x = 1$.

8. Use the fourth-order Runge–Kutta subroutine with $h = 0.25$ to approximate the solution to the initial value problem

$$y' = 1 - y, \qquad y(0) = 0,$$

at $x = 1$. Compare this approximation with the one obtained in Problem 6 using the Taylor method of order 4.

9. Use the fourth-order Runge–Kutta subroutine with $h = 0.25$ to approximate the solution to the initial value problem

$$y' = x + 1 - y, \qquad y(0) = 1,$$

at $x = 1$. Compare this approximation with the one obtained in Problem 5 using the Taylor method of order 4.

10. Use the fourth-order Runge–Kutta algorithm to approximate the solution to the initial value problem

$$y' = 1 - xy, \qquad y(1) = 1,$$

at $x = 2$. For a tolerance of $\varepsilon = 0.001$, use a stopping procedure based on the absolute error.

11. The solution to the initial value problem

$$y' = \frac{2}{x^4} - y^2, \qquad y(1) = -0.414$$

crosses the x-axis at a point in the interval $[1, 2]$. By experimenting with the fourth-order Runge–Kutta subroutine, determine this point to two decimal places.

12. By experimenting with the fourth-order Runge–Kutta subroutine, find the maximum value over the interval $[1, 2]$ of the solution to the initial value problem

$$y' = \frac{1.8}{x^4} - y^2, \qquad y(1) = -1.$$

Where does this maximum occur? Give your answers to two decimal places.

13. The solution to the initial value problem

$$\frac{dy}{dx} = y^2 - 2e^x y + e^{2x} + e^x, \qquad y(0) = 3$$

has a vertical asymptote ("blows up") at some point in the interval $[0, 2]$. By experimenting with the fourth-order Runge–Kutta subroutine, determine this point to two decimal places.

14. Use the fourth-order Runge–Kutta algorithm to approximate the solution to the initial value problem

$$y' = y\cos x, \qquad y(0) = 1,$$

at $x = \pi$. For a tolerance of $\varepsilon = 0.01$, use a stopping procedure based on the absolute error.

[†]Appendix G describes various websites and commercial software that sketch direction fields and automate most of the differential equation algorithms discussed in this book.

15. Use the fourth-order Runge–Kutta subroutine with $h = 0.1$ to approximate the solution to

$$y' = \cos(5y) - x\,, \qquad y(0) = 0\,,$$

at the points $x = 0, 0.1, 0.2, \ldots, 3.0$. Use your answers to make a rough sketch of the solution on $[0, 3]$.

16. Use the fourth-order Runge–Kutta subroutine with $h = 0.1$ to approximate the solution to

$$y' = 3\cos(y - 5x)\,, \qquad y(0) = 0\,,$$

at the points $x = 0, 0.1, 0.2, \ldots, 4.0$. Use your answers to make a rough sketch of the solution on [0, 4].

17. The Taylor method of order 2 can be used to approximate the solution to the initial value problem

$$y' = y\,, \qquad y(0) = 1\,,$$

at $x = 1$. Show that the approximation y_n obtained by using the Taylor method of order 2 with the step size $1/n$ is given by the formula

$$y_n = \left(1 + \frac{1}{n} + \frac{1}{2n^2}\right)^n, \qquad n = 1, 2, \ldots.$$

The solution to the initial value problem is $y = e^x$, so y_n is an approximation to the constant e.

18. If the Taylor method of order p is used in Problem 17, show that

$$y_n = \left(1 + \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{6n^3} + \cdots + \frac{1}{p!n^p}\right)^n,$$
$$n = 1, 2, \ldots.$$

19. Fluid Flow. In the study of the nonisothermal flow of a Newtonian fluid between parallel plates, the equation

$$\frac{d^2y}{dx^2} + x^2e^y = 0\,, \qquad x > 0\,,$$

was encountered. By a series of substitutions, this equation can be transformed into the first-order equation

$$\frac{dv}{du} = u\left(\frac{u}{2} + 1\right)v^3 + \left(u + \frac{5}{2}\right)v^2.$$

Use the fourth-order Runge–Kutta algorithm to approximate $v(3)$ if $v(u)$ satisfies $v(2) = 0.1$. For a tolerance of $\varepsilon = 0.0001$, use a stopping procedure based on the relative error.

20. Chemical Reactions. The reaction between nitrous oxide and oxygen to form nitrogen dioxide is given by the balanced chemical equation $2NO + O_2 = 2NO_2$. At high temperatures the dependence of the rate of this reaction on the concentrations of NO, O_2, and NO_2 is complicated. However, at 25°C the rate at which NO_2 is formed obeys the law of mass action and is given by the rate equation

$$\frac{dx}{dt} = k(\alpha - x)^2\left(\beta - \frac{x}{2}\right),$$

where $x(t)$ denotes the concentration of NO_2 at time t, k is the rate constant, α is the initial concentration of NO, and β is the initial concentration of O_2. At 25°C, the constant k is 7.13×10^3 (liter)^2/(mole)^2(second). Let $\alpha = 0.0010$ mole/L, $\beta = 0.0041$ mole/L, and $x(0) = 0$ mole/L. Use the fourth-order Runge–Kutta algorithm to approximate $x(10)$. For a tolerance of $\varepsilon = 0.000001$, use a stopping procedure based on the relative error.

21. Transmission Lines. In the study of the electric field that is induced by two nearby transmission lines, an equation of the form

$$\frac{dz}{dx} + g(x)z^2 = f(x)$$

arises. Let $f(x) = 5x + 2$ and $g(x) = x^2$. If $z(0) = 1$, use the fourth-order Runge–Kutta algorithm to approximate $z(1)$. For a tolerance of $\varepsilon = 0.0001$, use a stopping procedure based on the absolute error.

CHAPTER 4

Linear Second-Order Equations

4.1 Introduction: The Mass–Spring Oscillator

A damped mass–spring oscillator consists of a mass m attached to a spring fixed at one end, as shown in Figure 4.1. Devise a differential equation that governs the motion of this oscillator, taking into account the forces acting on it due to the spring elasticity, damping friction, and possible external influences.

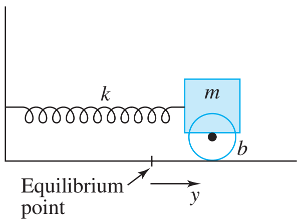

Figure 4.1 Damped mass–spring oscillator

Newton's second law—force equals mass times acceleration $(F = ma)$—is without a doubt the most commonly encountered differential equation in practice. It is an ordinary differential equation of the *second order* since acceleration is the second derivative of position (y) with respect to time $\left(a = d^2y/dt^2\right)$.

When the second law is applied to a mass–spring oscillator, the resulting motions are common experiences of everyday life, and we can exploit our familiarity with these vibrations to obtain a qualitative description of the solutions of more general second-order equations.

We begin by referring to Figure 4.1, which depicts the mass–spring oscillator. When the spring is unstretched and the inertial mass m is still, the system is at equilibrium; we measure the coordinate y of the mass by its displacement from the equilibrium position. When the mass m is displaced from equilibrium, the spring is stretched or compressed and it exerts a force that resists the displacement. For most springs this force is directly proportional to the displacement y and is thus given by

(1) $$F_{\text{spring}} = -ky\,,$$

where the positive constant k is known as the *stiffness* and the negative sign reflects the opposing nature of the force. **Hooke's law,** as equation (1) is commonly known, is only valid for sufficiently small displacements; if the spring is compressed so strongly that the coils press against each other, the opposing force obviously becomes much stronger.

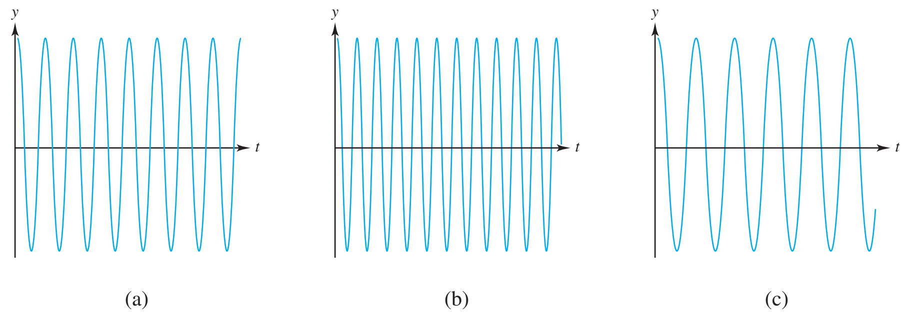

Figure 4.2 (a) Sinusoidal oscillation, (b) stiffer spring, and (c) heavier mass

Practically all mechanical systems also experience friction, and for vibrational motion this force is usually modeled accurately by a term proportional to velocity:

$$F_{\text{friction}} = -b\frac{dy}{dt} = -by' , \tag{2}$$

where $b\ (\geq 0)$ is the *damping coefficient* and the negative sign has the same significance as in equation (1).

The other forces on the oscillator are usually regarded as *external* to the system. Although they may be gravitational, electrical, or magnetic, commonly the most important external forces are transmitted to the mass by shaking the supports holding the system. For the moment we lump all the external forces into a single, *known* function $F_{\text{ext}}(t)$. Newton's law then provides the differential equation for the mass–spring oscillator:

$$my'' = -ky - by' + F_{\text{ext}}(t)$$

or

$$\boldsymbol{my'' + by' + ky = F_{\text{ext}}(t)} \ . \tag{3}$$

What do mass–spring motions look like? From our everyday experience with weak auto suspensions, musical gongs, and bowls of jelly, we expect that when there is no friction $(b = 0)$ or external force, the (idealized) motions would be perpetual vibrations like the ones depicted in Figure 4.2. These vibrations resemble sinusoidal functions, with their amplitude depending on the initial displacement and velocity. The frequency of the oscillations increases for stiffer springs but decreases for heavier masses.

In Section 4.3 we will show how to find these solutions. Example 1 demonstrates a quick calculation that confirms our intuitive predictions.

Example 1 Verify that if $b = 0$ and $F_{\text{ext}}(t) = 0$, equation (3) has a solution of the form $y(t) = \cos\omega t$ and that the angular frequency ω increases with k and decreases with m.

Solution Under the conditions stated, equation (3) simplifies to

$$my'' + ky = 0 \ . \tag{4}$$

The second derivative of $y(t)$ is $-\omega^2\cos\omega t$, and if we insert it into (4), we find

$$my'' + ky = -m\omega^2\cos\omega t + k\cos\omega t ,$$

which is indeed zero if $\omega = \sqrt{k/m}$. This ω increases with k and decreases with m, as predicted. ◆

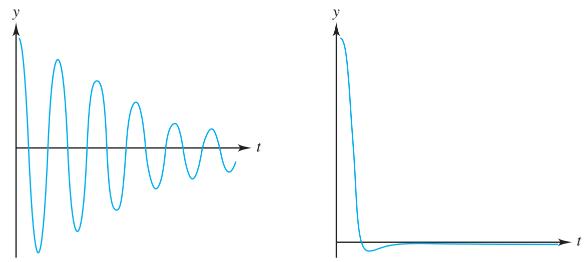

Figure 4.3 (a) Low damping and (b) high damping

When damping is present, the oscillations die out, and the motions resemble Figure 4.3. In Figure 4.3(a) the graph displays a damped oscillation; damping has slowed the frequency, and the amplitude appears to diminish exponentially with time. In Figure 4.3(b) the damping is so dominant that it has prevented the system from oscillating at all. Devices that are *supposed* to vibrate, like tuning forks or crystal oscillators, behave like Figure 4.3(a), and the damping effect is usually regarded as an undesirable loss mechanism. Good automotive suspension systems, on the other hand, behave like Figure 4.3(b); they exploit damping to *suppress* the oscillations.

The procedures for solving (unforced) mass–spring systems with damping are also described in Section 4.3, but as Examples 2 and 3 below show, the calculations are more complex. Example 2 has a relatively low damping coefficient $(b = 6)$ and illustrates the solutions for the "underdamped" case in Figure 4.3(a). In Example 3 the damping is more severe $(b = 10)$, and the solution is "overdamped" as in Figure 4.3(b).

Example 2 Verify that the exponentially damped sinusoid given by $y(t) = e^{-3t}\cos 4t$ is a solution to equation (3) if $F_{\text{ext}} = 0$, $m = 1$, $k = 25$, and $b = 6$.

Solution The derivatives of y are

$$\begin{aligned} y'(t) &= -3e^{-3t}\cos 4t - 4e^{-3t}\sin 4t\,, \\ y''(t) &= 9e^{-3t}\cos 4t + 12e^{-3t}\sin 4t + 12e^{-3t}\sin 4t - 16e^{-3t}\cos 4t \\ &= -7e^{-3t}\cos 4t + 24e^{-3t}\sin 4t\,, \end{aligned}$$

and insertion into (3) gives

$$\begin{aligned} my'' + by' + ky &= (1)y'' + 6y' + 25y \\ &= -7e^{-3t}\cos 4t + 24e^{-3t}\sin 4t + 6(-3e^{-3t}\cos 4t - 4e^{-3t}\sin t) \\ &\quad + 25e^{-3t}\cos 4t \\ &= 0\,. \end{aligned}$$ ◆

Example 3 Verify that the simple exponential function $y(t) = e^{-5t}$ is a solution to equation (3) if $F_{\text{ext}} = 0$, $m = 1$, $k = 25$, and $b = 10$.

Solution The derivatives of y are $y'(t) = -5e^{-5t}$, $y''(t) = 25e^{-5t}$ and insertion into (3) produces

$$my'' + by' + ky = (1)y'' + 10y' + 25y = 25e^{-5t} + 10(-5e^{-5t}) + 25e^{-5t} = 0\,.$$ ◆

Now if a mass–spring system is driven by an external force that is sinusoidal at the angular frequency ω, our experiences indicate that although the initial response of the system may be

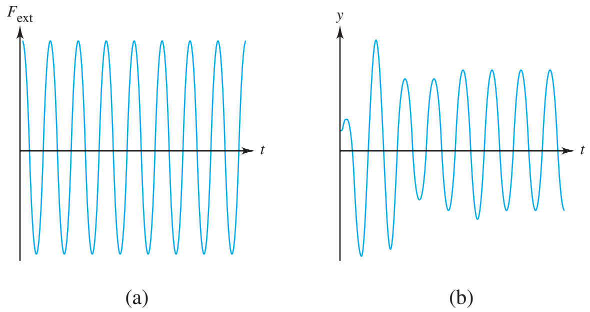

Figure 4.4 (a) Driving force and (b) response

somewhat erratic, eventually it will respond in "sync" with the driver and oscillate at the same frequency, as illustrated in Figure 4.4.

Common examples of systems vibrating in synchronization with their drivers are sound system speakers, cyclists bicycling over railroad tracks, electronic amplifier circuits, and ocean tides (driven by the periodic pull of the moon). However, there is more to the story than is revealed above. Systems can be enormously sensitive to the particular frequency ω at which they are driven. Thus, accurately tuned musical notes can shatter fine crystal, wind-induced vibrations at the right (wrong?) frequency can bring down a bridge, and a dripping faucet can cause inordinate headaches. These "resonance" responses (for which the responses have maximum amplitudes) may be quite destructive, and structural engineers have to be very careful to ensure that their products will not resonate with any of the vibrations likely to occur in the operating environment. Radio engineers, on the other hand, *do* want their receivers to resonate selectively to the desired broadcasting channel.

The calculation of these forced solutions is the subject of Sections 4.4 and 4.5. The next example illustrates some of the features of synchronous response and resonance.

Example 4 Find the synchronous response of the mass–spring oscillator with $m = 1, b = 1, k = 25$ to the force $\sin \Omega t$.

Solution We seek solutions of the differential equation

$$y'' + y' + 25y = \sin \Omega t \tag{5}$$

that are sinusoids in sync with $\sin \Omega t$; so let's try the form $y(t) = A \cos \Omega t + B \sin \Omega t$. Since

$$\begin{aligned} y' &= -\Omega A \sin \Omega t + \Omega B \cos \Omega t , \\ y'' &= -\Omega^2 A \cos \Omega t - \Omega^2 B \sin \Omega t , \end{aligned}$$

we can simply insert these forms into equation (5), collect terms, and match coefficients to obtain a solution:

$$\begin{aligned} \sin \Omega t &= y'' + y' + 25y \\ &= -\Omega^2 A \cos \Omega t - \Omega^2 B \sin \Omega t + [-\Omega A \sin \Omega t + \Omega B \cos \Omega t] \\ &\quad +25[A \cos \Omega t + B \sin \Omega t] \\ &= [-\Omega^2 B - \Omega A + 25B] \sin \Omega t + [-\Omega^2 A + \Omega B + 25A] \cos \Omega t , \end{aligned}$$

so

$$\begin{aligned} -\Omega A + (-\Omega^2 + 25)B &= 1 \\ (-\Omega^2 + 25)A + \Omega B &= 0 . \end{aligned}$$

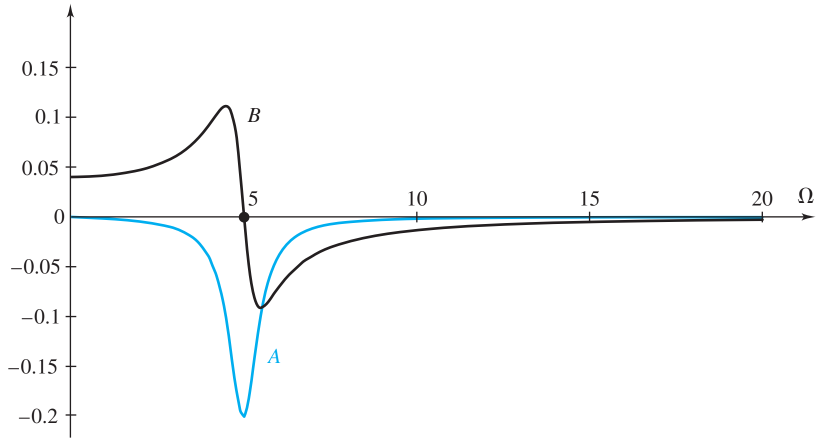

Figure 4.5 Vibration amplitudes around resonance

We find

$$A = \frac{-\Omega}{\Omega^2 + (\Omega^2 - 25)^2}, \qquad B = \frac{-\Omega^2 + 25}{\Omega^2 + (\Omega^2 - 25)^2}.$$

Figure 4.5 displays A and B as functions of the driving frequency Ω. A resonance clearly occurs around $\Omega \approx 5$. ◆

In most of this chapter, we are going to restrict our attention to differential equations of the form

$$ay'' + by' + cy = f(t)\,, \tag{6}$$

where $y(t)$ [or $y(x)$, or $x(t)$, etc.] is the unknown function that we seek; a, b, and c are constants; and $f(t)$ [or $f(x)$] is a *known* function. The proper nomenclature for (6) is the *linear, second-order ordinary differential equation with constant coefficients*. In Sections 4.7 and 4.8, we will generalize our focus to equations with nonconstant coefficients, as well as to nonlinear equations. However, (6) is an excellent starting point because we are able to obtain explicit solutions and observe, in concrete form, the theoretical properties that are predicted for more general equations. For motivation of the mathematical procedures and theory for solving (6), we will consistently compare it with the mass–spring paradigm:

$$[\textbf{inertia}] \times y'' + [\textbf{damping}] \times y' + [\textbf{stiffness}] \times y = F_{\text{ext}}\,.$$

4.1 EXERCISES

1. Verify that for $b = 0$ and $F_{\text{ext}}(t) = 0$, equation (3) has a solution of the form

$$y(t) = \cos\omega t, \text{ where } \omega = \sqrt{k/m}\,.$$

2. If $F_{\text{ext}}(t) = 0$, equation (3) becomes

$$my'' + by' + ky = 0\,.$$

For this equation, verify the following:

(a) If $y(t)$ is a solution, so is $cy(t)$, for any constant c.

(b) If $y_1(t)$ and $y_2(t)$ are solutions, so is their sum $y_1(t) + y_2(t)$.

3. Show that if $F_{\text{ext}}(t) = 0$, $m = 1$, $k = 9$, and $b = 6$, then equation (3) has the "critically damped" solutions $y_1(t) = e^{-3t}$ and $y_2(t) = te^{-3t}$. What is the limit of these solutions as $t \to \infty$?

4. Verify that $y = \sin 3t + 2\cos 3t$ is a solution to the initial value problem

$$2y'' + 18y = 0\,; \qquad y(0) = 2\,, \qquad y'(0) = 3\,.$$

Find the maximum of $|y(t)|$ for $-\infty < t < \infty$.

5. Verify that the exponentially damped sinusoid $y(t) = e^{-3t}\sin(\sqrt{3}\,t)$ is a solution to equation (3) if $F_{\text{ext}}(t) = 0$, $m = 1$, $b = 6$, and $k = 12$. What is the limit of this solution as $t \to \infty$?

6. An external force $F(t) = 2\cos 2t$ is applied to a mass–spring system with $m = 1$, $b = 0$, and $k = 4$, which is initially at rest; i.e., $y(0) = 0$, $y'(0) = 0$. Verify that $y(t) = \frac{1}{2}t\sin 2t$ gives the motion of this spring. What will eventually (as t increases) happen to the spring?

In Problems 7–9, find a synchronous solution of the form $A\cos\Omega t + B\sin\Omega t$ to the given forced oscillator equation using the method of Example 4 to solve for A and B.

7. $y'' + 2y' + 4y = 5\sin 3t, \quad \Omega = 3$

8. $y'' + 2y' + 5y = -50\sin 5t, \quad \Omega = 5$

9. $y'' + 2y' + 4y = 6\cos 2t + 8\sin 2t, \quad \Omega = 2$

10. Undamped oscillators that are driven at resonance have unusual (and nonphysical) solutions.

(a) To investigate this, find the synchronous solution $A\cos\Omega t + B\sin\Omega t$ to the generic forced oscillator equation

$$my'' + by' + ky = \cos\Omega t\,. \tag{7}$$

(b) Sketch graphs of the coefficients A and B, as functions of Ω, for $m = 1$, $b = 0.1$, and $k = 25$.

(c) Now set $b = 0$ in your formulas for A and B and resketch the graphs in part (b), with $m = 1$, and $k = 25$. What happens at $\Omega = 5$? Notice that the amplitudes of the synchronous solutions grow without bound as Ω approaches 5.

(d) Show directly, by substituting the form $A\cos\Omega t + B\sin\Omega t$ into equation (7), that when $b = 0$ there are *no* synchronous solutions if $\Omega = \sqrt{k/m}$.

(e) Verify that $(2m\Omega)^{-1}t\sin\Omega t$ solves equation (7) when $b = 0$ and $\Omega = \sqrt{k/m}$. Notice that this *non*synchronous solution grows in time, without bound.

Clearly one cannot neglect damping in analyzing an oscillator forced at resonance, because otherwise the solutions, as shown in part (e), are nonphysical. This behavior will be studied later in this chapter.

4.2 Homogeneous Linear Equations: The General Solution

We begin our study of the linear second-order constant-coefficient differential equation

$$ay'' + by' + cy = f(t) \qquad (a \neq 0) \tag{1}$$

with the special case where the function $f(t)$ is zero:

$$\boldsymbol{ay'' + by' + cy = 0\,.} \tag{2}$$

This case arises when we consider mass–spring oscillators vibrating freely—that is, without external forces applied. Equation (2) is called the *homogeneous form* of equation (1); $f(t)$ is the "nonhomogeneity" in (1). (This nomenclature is not related to the way we used the term for first-order equations in Section 2.6.)

A look at equation (2) tells us that a solution of (2) must have the property that its second derivative is expressible as a linear combination of its first and zeroth derivatives.† This suggests that we try to find a solution of the form $y = e^{rt}$, since derivatives of e^{rt} are just constants times e^{rt}. If we substitute $y = e^{rt}$ into (2), we obtain

$$\begin{aligned} ar^2e^{rt} + bre^{rt} + ce^{rt} &= 0\,, \\ e^{rt}(ar^2 + br + c) &= 0\,. \end{aligned}$$

†The zeroth derivative of a function is the function itself.

Because e^{rt} is never zero, we can divide by it to obtain

$$ar^2 + br + c = 0 . \tag{3}$$

Consequently, $y = e^{rt}$ is a solution to (2) if and only if r satisfies equation (3). Equation (3) is called the **auxiliary equation** (also known as the **characteristic equation**) associated with the homogeneous equation (2).

Now the auxiliary equation is just a quadratic, and its roots are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} .$$

When the discriminant, $b^2 - 4ac$, is positive, the roots r_1 and r_2 are real and distinct. If $b^2 - 4ac = 0$, the roots are real and equal. And when $b^2 - 4ac < 0$, the roots are complex conjugate numbers. We consider the first two cases in this section; the complex case is deferred to Section 4.3.

Example 1 Find a pair of solutions to

$$y'' + 5y' - 6y = 0 . \tag{4}$$

Solution The auxiliary equation associated with (4) is

$$r^2 + 5r - 6 = (r - 1)(r + 6) = 0 ,$$

which has the roots $r_1 = 1$, $r_2 = -6$. Thus, e^t and e^{-6t} are solutions. ◆

Notice that the identically zero function, $y(t) \equiv 0$, is always a solution to (2). Furthermore, when we have a pair of solutions $y_1(t)$ and $y_2(t)$ to this equation, as in Example 1, we can construct an infinite number of other solutions by forming linear combinations:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{5}$$

for any choice of the constants c_1 and c_2. The fact that (5) is a solution to (2) can be seen by direct substitution and rearrangement:

$$\begin{aligned} ay'' + by' + cy &= a(c_1y_1 + c_2y_2)'' + b(c_1y_1 + c_2y_2)' + c(c_1y_1 + c_2y_2) \\ &= a(c_1y_1'' + c_2y_2'') + b(c_1y_1' + c_2y_2') + c(c_1y_1 + c_2y_2) \\ &= c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2) \\ &= 0 + 0 . \end{aligned}$$

The two "degrees of freedom" c_1 and c_2 in the combination (5) suggest that solutions to the differential equation (2) can be found meeting additional conditions, such as the initial conditions for the first-order equations in Chapter 1. But the presence of c_1 *and* c_2 leads one to anticipate that *two* such conditions, rather than just one, can be imposed. This is consistent with the mass–spring interpretation of equation (2), since predicting the motion of a mechanical system requires knowledge not only of the forces but also of the initial position $y(0)$ and velocity $y'(0)$ of the mass. A typical *initial value problem* for these second-order equations is given in the following example.

Example 2 Solve the initial value problem

$$y'' + 2y' - y = 0 ; \qquad y(0) = 0 , \qquad y'(0) = -1 . \tag{6}$$

Solution We will first find a pair of solutions as in the previous example. Then we will adjust the constants c_1 and c_2 in (5) to obtain a solution that matches the initial conditions on $y(0)$ and $y'(0)$. The auxiliary equation is

$$r^2 + 2r - 1 = 0 .$$

Using the quadratic formula, we find that the roots of this equation are

$$r_1 = -1 + \sqrt{2} \qquad \text{and} \qquad r_2 = -1 - \sqrt{2} .$$

Consequently, the given differential equation has solutions of the form

$$y(t) = c_1 e^{(-1+\sqrt{2})t} + c_2 e^{(-1-\sqrt{2})t} . \tag{7}$$

To find the specific solution that satisfies the initial conditions given in (6), we first differentiate y as given in (7), then plug y and y' into the initial conditions of (6). This gives

$$\begin{aligned} y(0) &= c_1 e^0 + c_2 e^0 , \\ y'(0) &= (-1+\sqrt{2}) c_1 e^0 + (-1-\sqrt{2}) c_2 e^0 , \end{aligned}$$

or

$$\begin{aligned} 0 &= c_1 + c_2 , \\ -1 &= (-1+\sqrt{2}) c_1 + (-1-\sqrt{2}) c_2 . \end{aligned}$$

Solving this system yields $c_1 = -\sqrt{2}/4$ and $c_2 = \sqrt{2}/4$. Thus,

$$y(t) = -\frac{\sqrt{2}}{4} e^{(-1+\sqrt{2})t} + \frac{\sqrt{2}}{4} e^{(-1-\sqrt{2})t}$$

is the desired solution. ◆

To gain more insight into the significance of the two-parameter solution form (5), we need to look at some of the properties of the second-order equation (2). First of all, there is an existence-and-uniqueness theorem for solutions to (2); it is somewhat like the corresponding Theorem 1 in Section 1.2 for first-order equations but updated to reflect the fact that *two* initial conditions are appropriate for *second*-order equations. As motivation for the theorem, suppose the differential equation (2) were *really easy*, with $b = 0$ and $c = 0$. Then $y'' = 0$ would merely say that the graph of $y(t)$ is simply a *straight line*, so it is uniquely determined by specifying a point on the line,

$$y(t_0) = Y_0 , \tag{8}$$

and the slope of the line,

$$y'(t_0) = Y_1 . \tag{9}$$

Theorem 1 states that conditions (8) and (9) suffice to determine the solution uniquely for the more general equation (2).

Existence and Uniqueness: Homogeneous Case

Theorem 1. For any real numbers $a\ (\neq 0)$, b, c, t_0, Y_0, and Y_1, there exists a unique solution to the initial value problem

$$ay'' + by' + cy = 0 ; \qquad y(t_0) = Y_0 , \qquad y'(t_0) = Y_1 . \tag{10}$$

The solution is valid for all t in $(-\infty, +\infty)$.

Note in particular that if a solution $y(t)$ and its derivative vanish simultaneously at a point t_0 (i.e., $Y_0 = Y_1 = 0$), then $y(t)$ must be the identically zero solution.

In this section and the next, we will construct explicit solutions to (10), so the question of *existence* of a solution is not really an issue. It is extremely valuable to know, however, that the solution is *unique*. The proof of uniqueness is rather different from anything else in this chapter, so we defer it to Chapter 13.[†]

Now we want to use this theorem to show that, given two solutions $y_1(t)$ and $y_2(t)$ to equation (2), we can always find values of c_1 and c_2 so that $c_1y_1(t) + c_2y_2(t)$ meets specified initial conditions in (10) and therefore is the (unique) solution to the initial value problem. But we need to be a little more precise; if, for example, $y_2(t)$ is simply the identically zero solution, then $c_1y_1(t) + c_2y_2(t) = c_1y_1(t)$ actually has only *one* constant and cannot be expected to satisfy *two* conditions. Furthermore, if $y_2(t)$ is simply a constant multiple of $y_1(t)$—say, $y_2(t) = \kappa y_1(t)$—then again $c_1y_1(t) + c_2y_2(t) = (c_1 + \kappa c_2)y_1(t) = Cy_1(t)$ actually has only one constant. The condition we need is *linear independence*.

Linear Independence of Two Functions

Definition 1. A pair of functions $y_1(t)$ and $y_2(t)$ is said to be **linearly independent on the interval *I*** if and only if neither of them is a constant multiple of the other on all of I.[††] We say that y_1 and y_2 are **linearly dependent on *I*** if one of them *is* a constant multiple of the other on all of I.

Representation of Solutions to Initial Value Problem

Theorem 2. If $y_1(t)$ and $y_2(t)$ are any two solutions to the differential equation (2) that are linearly independent on $(-\infty, \infty)$, then unique constants c_1 and c_2 can always be found so that $c_1y_1(t) + c_2y_2(t)$ satisfies the initial value problem (10) on $(-\infty, \infty)$.

The proof of Theorem 2 will be easy once we establish the following technical lemma.

A Condition for Linear Dependence of Solutions

Lemma 1. For any real numbers $a\ (\neq 0)$, b, and c, if $y_1(t)$ and $y_2(t)$ are any two solutions to the differential equation (2) on $(-\infty, \infty)$ and if the equality

$$y_1(\tau)y_2'(\tau) - y_1'(\tau)y_2(\tau) = 0 \tag{11}$$

holds at *any* point τ, then y_1 and y_2 are linearly dependent on $(-\infty, \infty)$. (The expression on the left-hand side of (11) is called the *Wronskian* of y_1 and y_2 at the point τ; see Problem 34 on page 164.)

[†]All references to Chapters 11–13 refer to the expanded text, *Fundamentals of Differential Equations and Boundary Value Problems,* 7th ed.

[††]This definition will be generalized to three or more functions in Problem 35 and Chapter 6.

Proof of Lemma 1. *Case 1*. If $y_1(\tau) \neq 0$, then let κ equal $y_2(\tau)/y_1(\tau)$ and consider the solution to (2) given by $y(t) = \kappa y_1(t)$. It satisfies the same "initial conditions" at $t = \tau$ as does $y_2(t)$:

$$y(\tau) = \frac{y_2(\tau)}{y_1(\tau)} y_1(\tau) = y_2(\tau) \, ; \qquad y'(\tau) = \frac{y_2(\tau)}{y_1(\tau)} y_1'(\tau) = y_2'(\tau) \, ,$$

where the last equality follows from (11). By uniqueness, $y_2(t)$ must be the same function as $\kappa y_1(t)$ on I.

Case 2. If $y_1(\tau) = 0$ but $y_1'(\tau) \neq 0$, then (11) implies $y_2(\tau) = 0$. Let $\kappa = y_2'(\tau)/y_1'(\tau)$. Then the solution to (2) given by $y(t) = \kappa y_1(t)$ (again) satisfies the same "initial conditions" at $t = \tau$ as does $y_2(t)$:

$$y(\tau) = \frac{y_2'(\tau)}{y_1'(\tau)} y_1(\tau) = 0 = y_2(\tau) \, ; \qquad y'(\tau) = \frac{y_2'(\tau)}{y_1'(\tau)} y_1'(\tau) = y_2'(\tau) \, .$$

By uniqueness, then, $y_2(t) = \kappa y_1(t)$ on I.

Case 3. If $y_1(\tau) = y_1'(\tau) = 0$, then $y_1(t)$ is a solution to the differential equation (2) satisfying the initial conditions $y_1(\tau) = y_1'(\tau) = 0$; but $y(t) \equiv 0$ is the *unique* solution to this initial value problem. Thus, $y_1(t) \equiv 0$ [and is a constant multiple of $y_2(t)$]. ◆

Proof of Theorem 2. We already know that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution to (2); we must show that c_1 and c_2 can be chosen so that

$$y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) = Y_0$$

and

$$y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) = Y_1 \, .$$

But simple algebra shows these equations have the solution†

$$c_1 = \frac{Y_0 y_2'(t_0) - Y_1 y_2(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)} \quad \text{and} \quad c_2 = \frac{Y_1 y_1(t_0) - Y_0 y_1'(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$

as long as the denominator is nonzero, and the technical lemma assures us that this condition is met. ◆

Now we can honestly say that if y_1 and y_2 are linearly independent solutions to (2) on $(-\infty, +\infty)$, then (5) is a **general solution,** since *any* solution $y_g(t)$ of (2) can be expressed in this form; simply pick c_1 and c_2 so that $c_1 y_1 + c_2 y_2$ matches the value and the derivative of y_g *at any point*. By uniqueness, $c_1 y_1 + c_2 y_2$ and y_g have to be the same function. See Figure 4.6 on page 162.

How do we *find* a general solution for the differential equation (2)? We already know the answer if the roots of the auxiliary equation (3) are real and distinct because clearly $y_1(t) = e^{r_1 t}$ is not a constant multiple of $y_2(t) = e^{r_2 t}$ if $r_1 \neq r_2$.

Distinct Real Roots

If the auxiliary equation (3) has distinct real roots r_1 and r_2, then both $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are solutions to (2) and $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is a general solution.

†To solve for c_1, for example, multiply the first equation by $y_2'(t_0)$ and the second by $y_2(t_0)$ and subtract.

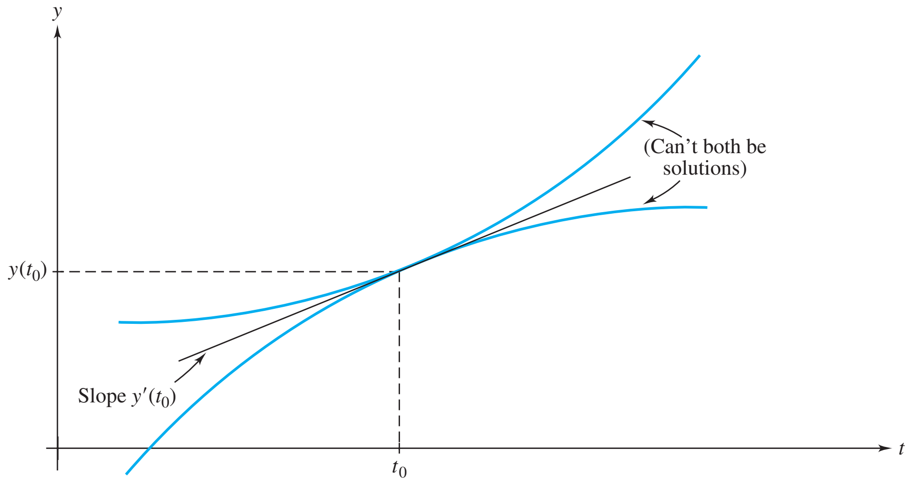

Figure 4.6 $y(t_0)$, $y'(t_0)$ determine a *unique* solution.

When the roots of the auxiliary equation are equal, we only get one nontrivial solution, $y_1 = e^{rt}$. To satisfy *two* initial conditions, $y(t_0)$ and $y'(t_0)$, then we will need a second, linearly independent solution. The following rule is the key to finding a second solution.

> **Repeated Root**
>
> If the auxiliary equation (3) has a repeated root r, then both $y_1(t) = e^{rt}$ and $y_2(t) = te^{rt}$ are solutions to (2), and $y(t) = c_1 e^{rt} + c_2 te^{rt}$ is a general solution.

We illustrate this result before giving its proof.

Example 3 Find a solution to the initial value problem

$$y'' + 4y' + 4y = 0\,; \qquad y(0) = 1\,, \qquad y'(0) = 3\,. \tag{12}$$

Solution The auxiliary equation for (12) is

$$r^2 + 4r + 4 = (r+2)^2 = 0\,.$$

Because $r = -2$ is a double root, the rule says that (12) has solutions $y_1 = e^{-2t}$ and $y_2 = te^{-2t}$. Let's confirm that $y_2(t)$ is a solution:

$$\begin{aligned}
y_2(t) &= te^{-2t}\,,\\
y_2'(t) &= e^{-2t} - 2te^{-2t}\,,\\
y_2''(t) &= -2e^{-2t} - 2e^{-2t} + 4te^{-2t} = -4e^{-2t} + 4te^{-2t}\,,\\
y_2'' + 4y_2' + 4y_2 &= -4e^{-2t} + 4te^{-2t} + 4(e^{-2t} - 2te^{-2t}) + 4te^{-2t} = 0\,.
\end{aligned}$$

Further observe that e^{-2t} and te^{-2t} are linearly independent since neither is a constant multiple of the other on $(-\infty, \infty)$. Finally, we insert the general solution $y(t) = c_1 e^{-2t} + c_2 te^{-2t}$ into the initial conditions,

$$\begin{aligned}
y(0) &= c_1 e^0 + c_2(0)e^0 = 1\,,\\
y'(0) &= -2c_1 e^0 + c_2 e^0 - 2c_2(0)e^0 = 3\,,
\end{aligned}$$

and solve to find $c_1 = 1$, $c_2 = 5$. Thus $y = e^{-2t} + 5te^{-2t}$ is the desired solution. ◆

Why is it that $y_2(t) = te^{rt}$ is a solution to the differential equation (2) when r is a double root (and not otherwise)? In later chapters we will see a theoretical justification of this rule in very general circumstances; for present purposes, though, simply note what happens if we substitute y_2 into the differential equation (2):

$$\begin{aligned}
y_2(t) &= te^{rt}\,,\\
y_2'(t) &= e^{rt} + rte^{rt}\,,\\
y''_2(t) &= re^{rt} + re^{rt} + r^2te^{rt} = 2re^{rt} + r^2te^{rt}\,,\\
ay_2'' + by_2' + cy_2 &= [2ar + b]e^{rt} + [ar^2 + br + c]te^{rt}\,.
\end{aligned}$$

Now if r is a root of the auxiliary equation (3), the expression in the second brackets is zero. However, *if r is a double root, the expression in the first brackets is zero also*:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm (0)}{2a}\,; \tag{13}$$

hence, $2ar + b = 0$ for a double root. In such a case, then, y_2 is a solution.

The method we have described for solving homogeneous linear second-order equations with constant coefficients applies to any order (even first-order) homogeneous linear equations with constant coefficients. We give a detailed treatment of such higher-order equations in Chapter 6. For now, we will be content to illustrate the method by means of an example. We remark briefly that a homogeneous linear nth-order equation has a general solution of the form

$$y(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t)\,,$$

where the individual solutions $y_i(t)$ are "linearly independent." By this we mean that no y_i is expressible as a linear combination of the others; see Problem 35 on page 164.

Example 4 Find a general solution to

$$y''' + 3y'' - y' - 3y = 0\,. \tag{14}$$

Solution If we try to find solutions of the form $y = e^{rt}$, then, as with second-order equations, we are led to finding roots of the auxiliary equation

$$r^3 + 3r^2 - r - 3 = 0\,. \tag{15}$$

We observe that $r = 1$ is a root of the above equation, and dividing the polynomial on the left-hand side of (15) by $r - 1$ leads to the factorization

$$(r-1)(r^2 + 4r + 3) = (r-1)(r+1)(r+3) = 0\,.$$

Hence, the roots of the auxiliary equation are 1, -1, and -3, and so three solutions of (14) are e^t, e^{-t}, and e^{-3t}. The linear independence of these three exponential functions is proved in Problem 36. A general solution to (14) is then

$$y(t) = c_1e^t + c_2e^{-t} + c_3e^{-3t}\,. \quad \blacklozenge \tag{16}$$

So far we have seen only exponential solutions to the linear second-order constant coefficient equation. You may wonder where the vibratory solutions that govern mass–spring oscillators are. In the next section, it will be seen that they arise when the solutions to the auxiliary equation are complex.

4.2 EXERCISES

In Problems 1–12, find a general solution to the given differential equation.

1. $2y'' + 7y' - 4y = 0$

2. $y'' + 6y' + 9y = 0$

3. $y'' + 5y' + 6y = 0$

4. $y'' - y' - 2y = 0$

5. $y'' + 8y' + 16y = 0$

6. $y'' - 5y' + 6y = 0$

7. $6y'' + y' - 2y = 0$

8. $z'' + z' - z = 0$

9. $4y'' - 4y' + y = 0$

10. $y'' - y' - 11y = 0$

11. $4w'' + 20w' + 25w = 0$

12. $3y'' + 11y' - 7y = 0$

In Problems 13–20, solve the given initial value problem.

13. $y'' + 2y' - 8y = 0\,; \quad y(0) = 3\,, \quad y'(0) = -12$

14. $y'' + y' = 0\,; \quad y(0) = 2\,, \quad y'(0) = 1$

15. $y'' - 4y' + 3y = 0\,; \quad y(0) = 1\,, \quad y'(0) = 1/3$

16. $y'' - 4y' - 5y = 0\,; \quad y(-1) = 3\,, \quad y'(-1) = 9$

17. $y'' - 6y' + 9y = 0\,; \quad y(0) = 2\,, \quad y'(0) = 25/3$

18. $z'' - 2z' - 2z = 0\,; \quad z(0) = 0\,, \quad z'(0) = 3$

19. $y'' + 2y' + y = 0\,; \quad y(0) = 1\,, \quad y'(0) = -3$

20. $y'' - 4y' + 4y = 0\,; \quad y(1) = 1\,, \quad y'(1) = 1$

21. First-Order Constant-Coefficient Equations.

(a) Substituting $y = e^{rt}$, find the auxiliary equation for the first-order linear equation

$$ay' + by = 0\,,$$

where a and b are constants with $a \neq 0$.

(b) Use the result of part (a) to find the general solution.

In Problems 22–25, use the method described in Problem 21 to find a general solution to the given equation.

22. $3y' - 7y = 0$

23. $5y' + 4y = 0$

24. $3z' + 11z = 0$

25. $6w' - 13w = 0$

26. Boundary Value Problems. When the values of a solution to a differential equation are specified at two different points, these conditions are called **boundary conditions.** (In contrast, initial conditions specify the values of a function and its derivative at the same point.) The purpose of this exercise is to show that for boundary value problems there is no existence–uniqueness theorem that is analogous to Theorem 1. Given that every solution to

$$y'' + y = 0 \tag{17}$$

is of the form

$$y(t) = c_1 \cos t + c_2 \sin t\,,$$

where c_1 and c_2 are arbitrary constants, show that

(a) There is a unique solution to (17) that satisfies the boundary conditions $y(0) = 2$ and $y(\pi/2) = 0$.

(b) There is no solution to (17) that satisfies $y(0) = 2$ and $y(\pi) = 0$.

(c) There are infinitely many solutions to (17) that satisfy $y(0) = 2$ and $y(\pi) = -2$.

In Problems 27–32, use Definition 1 to determine whether the functions y_1 and y_2 are linearly dependent on the interval $(0, 1)$.

27. $y_1(t) = \cos t \sin t\,, \quad y_2(t) = \sin 2t$

28. $y_1(t) = e^{3t}\,, \quad y_2(t) = e^{-4t}$

29. $y_1(t) = te^{2t}\,, \quad y_2(t) = e^{2t}$

30. $y_1(t) = t^2 \cos(\ln t)\,, \quad y_2(t) = t^2 \sin(\ln t)$

31. $y_1(t) = \tan^2 t - \sec^2 t\,, \quad y_2(t) \equiv 3$

32. $y_1(t) \equiv 0\,, \quad y_2(t) = e^t$

33. Explain why two functions are linearly dependent on an interval I if and only if there exist constants c_1 and c_2, *not both zero*, such that

$$c_1 y_1(t) + c_2 y_2(t) = 0 \quad \text{for all } t \text{ in } I\,.$$

34. Wronskian. For any two differentiable functions y_1 and y_2, the function

$$W[y_1, y_2](t) = y_1(t) y_2'(t) - y_1'(t) y_2(t) \tag{18}$$

is called the *Wronskian*[†] of y_1 and y_2. This function plays a crucial role in the proof of Theorem 2.

(a) Show that $W[y_1, y_2]$ can be conveniently expressed as the 2×2 determinant

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}\,.$$

(b) Let $y_1(t), y_2(t)$ be a pair of solutions to the homogeneous equation $ay'' + by' + cy = 0$ (with $a \neq 0$) on an open interval I. Prove that $y_1(t)$ and $y_2(t)$ are linearly independent on I if and only if their Wronskian is never zero on I. [*Hint:* This is just a reformulation of Lemma 1.]

(c) Show that if $y_1(t)$ and $y_2(t)$ are any two differentiable functions that are linearly dependent on I, then their Wronskian is identically zero on I.

35. Linear Dependence of Three Functions. Three functions $y_1(t)$, $y_2(t)$, and $y_3(t)$ are said to be linearly dependent on an interval I if, on I, at least one of these functions is a linear combination of the remaining two [e.g., if $y_1(t) = c_1 y_2(t) + c_2 y_3(t)$]. Equivalently (compare Problem 33), y_1, y_2, and y_3 are linearly dependent on I if there exist constants C_1, C_2, and C_3, *not all zero*, such that

$$C_1 y_1(t) + C_2 y_2(t) + C_3 y_3(t) = 0 \quad \text{for all } t \text{ in } I.$$

Otherwise, we say that these functions are linearly independent on I.

[†]*Historical Footnote:* The Wronskian was named after the Polish mathematician H. Wronski (1778–1863).

For each of the following, determine whether the given three functions are linearly dependent or linearly independent on $(-\infty, \infty)$:

(a) $y_1(t) = 1\,, \quad y_2(t) = t\,, \quad y_3(t) = t^2\,.$

(b) $y_1(t) = -3\,, \quad y_2(t) = 5\sin^2 t\,, \quad y_3(t) = \cos^2 t\,.$

(c) $y_1(t) = e^t\,, \quad y_2(t) = te^t\,, \quad y_3(t) = t^2e^t\,.$

(d) $y_1(t) = e^t\,, \quad y_2(t) = e^{-t}\,, \quad y_3(t) = \cosh t\,.$

36. Using the definition in Problem 35, prove that if r_1, r_2, and r_3 are distinct real numbers, then the functions e^{r_1t}, e^{r_2t}, and e^{r_3t} are linearly independent on $(-\infty, \infty)$. [*Hint*: Assume to the contrary that, say, $e^{r_1t} = c_1e^{r_2t} + c_2e^{r_3t}$ for all t. Divide by e^{r_2t} to get $e^{(r_1-r_2)t} = c_1 + c_2e^{(r_3-r_2)t}$ and then differentiate to deduce that $e^{(r_1-r_2)t}$ and $e^{(r_3-r_2)t}$ are linearly dependent, which is a contradiction. (Why?)]

In Problems 37–41, find three linearly independent solutions (see Problem 35) of the given third-order differential equation and write a general solution as an arbitrary linear combination of these.

37. $y''' + y'' - 6y' + 4y = 0$

38. $y''' - 6y'' - y' + 6y = 0$

39. $z''' + 2z'' - 4z' - 8z = 0$

40. $y''' - 7y'' + 7y' + 15y = 0$

41. $y''' + 3y'' - 4y' - 12y = 0$

42. (True or False): If f_1, f_2, f_3 are three functions defined on $(-\infty, \infty)$ that are *pairwise* linearly independent on $(-\infty, \infty)$, then f_1, f_2, f_3 form a linearly independent set on $(-\infty, \infty)$. Justify your answer.

43. Solve the initial value problem:

$$y''' - y' = 0\,; \qquad y(0) = 2\,,$$
$$y'(0) = 3\,, \qquad y''(0) = -1\,.$$

44. Solve the initial value problem:

$$y''' - 2y'' - y' + 2y = 0\,;$$
$$y(0) = 2\,, \quad y'(0) = 3\,, \quad y''(0) = 5\,.$$

45. By using Newton's method or some other numerical procedure to approximate the roots of the auxiliary equation, find general solutions to the following equations:

(a) $3y''' + 18y'' + 13y' - 19y = 0\,.$

(b) $y^{\text{iv}} - 5y'' + 5y = 0\,.$

(c) $y^{\text{v}} - 3y^{\text{iv}} - 5y''' + 15y'' + 4y' - 12y = 0\,.$

46. One way to define hyperbolic functions is by means of differential equations. Consider the equation $y'' - y = 0$. The *hyperbolic cosine*, $\cosh t$, is defined as the solution of this equation subject to the initial values: $y(0) = 1$ and $y'(0) = 0$. The *hyperbolic sine*, $\sinh t$, is defined as the solution of this equation subject to the initial values: $y(0) = 0$ and $y'(0) = 1$.

(a) Solve these initial value problems to derive explicit formulas for $\cosh t$ and $\sinh t$. Also show that $\frac{d}{dt}\cosh t = \sinh t$ and $\frac{d}{dt}\sinh t = \cosh t$.

(b) Prove that a general solution of the equation $y'' - y = 0$ is given by $y = c_1\cosh t + c_2\sinh t$.

(c) Suppose a, b, and c are given constants for which $ar^2 + br + c = 0$ has two distinct real roots. If the two roots are expressed in the form $\alpha - \beta$ and $\alpha + \beta$, show that a general solution of the equation $ay'' + by' + cy = 0$ is $y = c_1e^{\alpha t}\cosh(\beta t) + c_2e^{\alpha t}\sinh(\beta t)$.

(d) Use the result of part (c) to solve the initial value problem: $y'' + y' - 6y = 0$, $y(0) = 2$, $y'(0) = -17/2$.

4.3 Auxiliary Equations with Complex Roots

The *simple harmonic equation* $y'' + y = 0$, so called because of its relation to the fundamental vibration of a musical tone, has as solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$. Notice, however, that the auxiliary equation associated with the harmonic equation is $r^2 + 1 = 0$, which has imaginary roots $r = \pm i$, where i denotes $\sqrt{-1}$.[†] In the previous section, we expressed the solutions to a linear second-order equation with constant coefficients in terms of exponential functions. It would appear, then, that one might be able to attribute a meaning to the forms e^{it} and e^{-it} and that these "functions" should be related to $\cos t$ and $\sin t$. This matchup is accomplished by Euler's formula, which is discussed in this section.

When $b^2 - 4ac < 0$, the roots of the auxiliary equation

$$ar^2 + br + c = 0 \tag{1}$$

[†]Electrical engineers frequently use the symbol j to denote $\sqrt{-1}$.

associated with the homogeneous equation

(2) $$ay'' + by' + cy = 0$$

are the complex conjugate numbers

$$r_1 = \alpha + i\beta \quad \text{and} \quad r_2 = \alpha - i\beta \quad (i = \sqrt{-1})\,,$$

where α, β are the real numbers

(3) $$\alpha = -\frac{b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}\,.$$

As in the previous section, we would like to assert that the functions $e^{r_1 t}$ and $e^{r_2 t}$ are solutions to the equation (2). This is in fact the case, but before we can proceed, we need to address some fundamental questions. For example, if $r_1 = \alpha + i\beta$ is a complex number, what do we mean by the expression $e^{(\alpha+i\beta)t}$? If we assume that the law of exponents applies to complex numbers, then

(4) $$e^{(\alpha+i\beta)t} = e^{\alpha t + i\beta t} = e^{\alpha t}e^{i\beta t}\,.$$

We now need only clarify the meaning of $e^{i\beta t}$.

For this purpose, let's assume that the Maclaurin series for e^z is the same for complex numbers z as it is for real numbers. Observing that $i^2 = -1$, then for θ real we have

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \cdots + \frac{(i\theta)^n}{n!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right). \end{aligned}$$

Now recall the Maclaurin series for $\cos\theta$ and $\sin\theta$:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\,,$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\,.$$

Recognizing these expansions in the proposed series for $e^{i\theta}$, we make the identification

(5) $$\boldsymbol{e^{i\theta} = \cos\theta + i\sin\theta\,,}$$

which is known as **Euler's formula.**†

When Euler's formula (with $\theta = \beta t$) is used in equation (4), we find

(6) $$e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)\,,$$

which expresses the complex function $e^{(\alpha+i\beta)t}$ in terms of familiar real functions. Having made sense out of $e^{(\alpha+i\beta)t}$, we can now show (see Problem 30 on page 172) that

(7) $$\frac{d}{dt}e^{(\alpha+i\beta)t} = (\alpha + i\beta)e^{(\alpha+i\beta)t}\,,$$

†*Historical Footnote:* This formula first appeared in Leonhard Euler's monumental two-volume *Introductio in Analysin Infinitorum* (1748).

and, with the choices of α and β as given in (3), the complex function $e^{(\alpha+i\beta)t}$ is indeed a solution to equation (2), as is $e^{(\alpha-i\beta)t}$, and a general solution is given by

$$\textbf{(8)} \qquad \begin{aligned} y(t) &= c_1 e^{(\alpha+i\beta)t} + c_2 e^{(\alpha-i\beta)t} \\ &= c_1 e^{\alpha t}(\cos\beta t + i\sin\beta t) + c_2 e^{\alpha t}(\cos\beta t - i\sin\beta t)\ . \end{aligned}$$

Example 1 shows that in general the constants c_1 and c_2 that go into (8), for a specific initial value problem, are complex.

Example 1 Use the general solution (8) to solve the initial value problem

$$y'' + 2y' + 2y = 0\,; \qquad y(0) = 0, \qquad y'(0) = 2\ .$$

Solution The auxiliary equation is $r^2 + 2r + 2 = 0$, which has roots

$$r = \frac{-2\sqrt{4-8}}{2} = -1 \pm i\ .$$

Hence, with $\alpha = -1$, $\beta = 1$, a general solution is given by

$$y(t) = c_1 e^{-t}(\cos t + i\sin t) + c_2 e^{-t}(\cos t - i\sin t)\ .$$

For initial conditions we have

$$\begin{aligned} y(0) &= c_1 e^0(\cos 0 + i\sin 0) + c_2 e^0(\cos 0 - i\sin 0) = c_1 + c_2 = 0\,, \\ y'(0) &= -c_1 e^0(\cos 0 + i\sin 0) + c_1 e^0(-\sin 0 + i\cos 0) \\ &\quad\ -c_2 e^0(\cos 0 - i\sin 0) + c_2 e^0(-\sin 0 - i\cos 0) \\ &= (-1+i)c_1 + (-1-i)c_2 \\ &= 2\ . \end{aligned}$$

As a result, $c_1 = -i$, $c_2 = i$, and $y(t) = -ie^{-t}(\cos t + i\sin t) + ie^{-t}(\cos t - i\sin t)$, or simply $2e^{-t}\sin t$. ◆

The final form of the answer to Example 1 suggests that we should seek an alternative pair of solutions to the differential equation (2) that don't require complex arithmetic, and we now turn to that task.

In general, if $z(t)$ is a complex-valued function of the real variable t, we can write $z(t) = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real-valued functions. The derivatives of $z(t)$ are then given by

$$\frac{dz}{dt} = \frac{du}{dt} + i\frac{dv}{dt}, \qquad \frac{d^2z}{dt^2} = \frac{d^2u}{dt^2} + i\frac{d^2v}{dt^2}\ .$$

With the following lemma, we show that the complex-valued solution $e^{(\alpha+i\beta)t}$ gives rise to two linearly independent *real-valued* solutions.

Real Solutions Derived from Complex Solutions

Lemma 2. Let $z(t) = u(t) + iv(t)$ be a solution to equation (2), where a, b, and c are real numbers. Then, the real part $u(t)$ and the imaginary part $v(t)$ are real-valued solutions of (2).†

†It will be clear from the proof that this property holds for any linear homogeneous differential equation having real-valued coefficients.

Proof. By assumption, $az'' + bz' + cz = 0$, and hence

$$a(u'' + iv'') + b(u' + iv') + c(u + iv) = 0\,,$$
$$(au'' + bu' + cu) + i(av'' + bv' + cv) = 0\,.$$

But a complex number is zero if and only if its real and imaginary parts are both zero. Thus, we must have

$$au'' + bu' + cu = 0 \quad \text{and} \quad av'' + bv' + cv = 0\,,$$

which means that both $u(t)$ and $v(t)$ are real-valued solutions of (2). ◆

When we apply Lemma 2 to the solution

$$e^{(\alpha+i\beta)t} = e^{\alpha t}\cos\beta t + ie^{\alpha t}\sin\beta t\,,$$

we obtain the following.

Complex Conjugate Roots

If the auxiliary equation has complex conjugate roots $\alpha \pm i\beta$, then two linearly independent solutions to (2) are

$$e^{\alpha t}\cos\beta t \quad \textbf{and} \quad e^{\alpha t}\sin\beta t\,,$$

and a general solution is

$$\text{(9)} \qquad \boldsymbol{y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t\,,}$$

where c_1 and c_2 are arbitrary constants.

In the preceding discussion, we glossed over some important details concerning complex numbers and complex-valued functions. In particular, further analysis is required to justify the use of the law of exponents, Euler's formula, and even the fact that the derivative of e^{rt} is re^{rt} when r is a complex constant.† If you feel uneasy about our conclusions, we encourage you to substitute the expression in (9) into equation (2) to verify that it is, indeed, a solution.

You may also be wondering what would have happened if we had worked with the function $e^{(\alpha-i\beta)t}$ instead of $e^{(\alpha+i\beta)t}$. We leave it as an exercise to verify that $e^{(\alpha-i\beta)t}$ gives rise to the same general solution (9). Indeed, the sum of these two complex solutions, divided by two, gives the first real-valued solution, while their difference, divided by $2i$, gives the second.

Example 2 Find a general solution to

$$\text{(10)} \qquad y'' + 2y' + 4y = 0\,.$$

Solution The auxiliary equation is

$$r^2 + 2r + 4 = 0\,,$$

which has roots

$$r = \frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm \sqrt{-12}}{2} = -1 \pm i\sqrt{3}\,.$$

†For a detailed treatment of these topics see, for example, *Fundamentals of Complex Analysis,* 3rd ed., by E. B. Saff and A. D. Snider (Prentice Hall, Upper Saddle River, New Jersey, 2003).

Hence, with $\alpha = -1, \beta = \sqrt{3}$, a general solution for (10) is

$$y(t) = c_1 e^{-t}\cos(\sqrt{3}\,t) + c_2 e^{-t}\sin(\sqrt{3}\,t)\ . \quad \blacklozenge$$

When the auxiliary equation has complex conjugate roots, the (real) solutions oscillate between positive and negative values. This type of behavior is observed in vibrating springs.

Example 3 In Section 4.1 we discussed the mechanics of the mass–spring oscillator (Figure 4.1, page 152), and we saw how Newton's second law implies that the position $y(t)$ of the mass m is governed by the second-order differential equation

(11) $$my''(t) + by'(t) + ky(t) = 0\ ,$$

where the terms are physically identified as

$$[\text{inertia}]y'' + [\text{damping}]y' + [\text{stiffness}]y = 0\ .$$

Determine the equation of motion for a spring system when $m = 36$ kg, $b = 12$ kg/sec (which is equivalent to 12 N-sec/m), $k = 37$ kg/sec^2, $y(0) = 0.7$ m, and $y'(0) = 0.1$ m/sec. After how many seconds will the mass first cross the equilibrium point?

Solution The equation of motion is given by $y(t)$, the solution of the initial value problem for the specified values of m, b, k, $y(0)$, and $y'(0)$. That is, we seek the solution to

(12) $$36y'' + 12y' + 37y = 0\ ; \qquad y(0) = 0.7\ ,\ y'(0) = 0.1\ .$$

The auxiliary equation for (12) is

$$36r^2 + 12r + 37 = 0\ ,$$

which has roots

$$r = \frac{-12 \pm \sqrt{144 - 4(36)(37)}}{72} = \frac{-12 \pm 12\sqrt{1-37}}{72} = -\frac{1}{6} \pm i\ .$$

Hence, with $\alpha = -1/6, \quad \beta = 1$, the displacement $y(t)$ can be expressed in the form

(13) $$y(t) = c_1 e^{-t/6}\cos t + c_2 e^{-t/6}\sin t\ .$$

We can find c_1 and c_2 by substituting $y(t)$ and $y'(t)$ into the initial conditions given in (12). Differentiating (13), we get a formula for $y'(t)$:

$$y'(t) = \left(-\frac{c_1}{6} + c_2\right)e^{-t/6}\cos t + \left(-c_1 - \frac{c_2}{6}\right)e^{-t/6}\sin t\ .$$

Substituting into the initial conditions now results in the system

$$\begin{aligned} c_1 &= 0.7\ , \\ -\frac{c_1}{6} + c_2 &= 0.1\ . \end{aligned}$$

Upon solving, we find $c_1 = 0.7$ and $c_2 = 1.3/6$. With these values, the equation of motion becomes

$$y(t) = 0.7e^{-t/6}\cos t + \frac{1.3}{6}e^{-t/6}\sin t\ .$$

To determine the times when the mass will cross the equilibrium point, we set $y(t) = 0$ and solve for t:

$$0 = y(t) = 0.7e^{-t/6}\cos t + \tfrac{1.3}{6}e^{-t/6}\sin t = (\cos t)\left(0.7e^{-t/6} + \tfrac{1.3}{6}e^{-t/6}\tan t\right).$$

But $y(t)$ is not zero for $\cos t = 0$, so only the zeros of the second factor are pertinent; that is,

$$\tan t = -\tfrac{4.2}{1.3}.$$

A quick glance at the graph of tan t reveals that the first positive t for which this is true lies between $\pi/2$ and π. Since the arctangent function takes only values between $-\pi/2$ and $\pi/2$, the appropriate adjustment is

$$t = \arctan\left(\tfrac{-4.2}{1.3}\right) + \pi \approx 1.87 \text{ seconds}. \quad \blacklozenge$$

From Example 3 we see that *any* second-order constant-coefficient differential equation $ay'' + by' + cy = 0$ can be interpreted as describing a mass–spring system with mass a, damping coefficient b, spring stiffness c, and displacement y, if these constants make sense physically; that is, if a is positive and b and c are nonnegative. From the discussion in Section 4.1, then, we expect on physical grounds to see damped oscillatory solutions in such a case. This is consistent with the display in equation (9). With $a = m$ and $c = k$, the exponential decay rate α equals $-b/(2m)$, and the angular frequency β equals $\sqrt{4mk - b^2}/(2m)$, by equation (3).

It is a little surprising, then, that the solutions to the equation $y'' + 4y' + 4y = 0$ do *not* oscillate; the general solution was shown in Example 3 of Section 4.2 (page 162) to be $c_1e^{-2t} + c_2te^{-2t}$. The physical significance of this is simply that when the damping coefficient b is too high, the resulting friction prevents the mass from oscillating. Rather than overshoot the spring's equilibrium point, it merely settles in lazily. This could happen if a light mass on a weak spring were submerged in a viscous fluid.

From the above formula for the oscillation frequency β, we can see that the oscillations will not occur for $b > \sqrt{4mk}$. This *overdamping* phenomenon is discussed in more detail in Section 4.9.

It is extremely enlightening to contemplate the predictions of the mass–spring analogy when the coefficients b and c in the equation $ay'' + by' + cy = 0$ are *negative*.

Example 4 Interpret the equation

(14) $$36y'' - 12y' + 37y = 0$$

in terms of the mass–spring system.

Solution Equation (14) is a minor alteration of equation (12) in Example 3; the auxiliary equation $36r^2 - 12r + 37$ has roots $r = (+)\frac{1}{6} \pm i$. Thus, its general solution becomes

(15) $$y(t) = c_1e^{+t/6}\cos t + c_2e^{+t/6}\sin t.$$

Comparing equation (14) with the mass–spring model

(16) $$[\textbf{inertia}]y'' + [\textbf{damping}]y' + [\textbf{stiffness}]y = 0,$$

we have to envision a *negative* damping coefficient $b = -12$, giving rise to a friction force $F_{\text{friction}} = -by'$ that *imparts* energy to the system instead of draining it. The increase in energy over time must then reveal itself in oscillations of ever-greater amplitude–precisely in accordance with formula (15), for which a typical graph is drawn in Figure 4.7. $\blacklozenge$

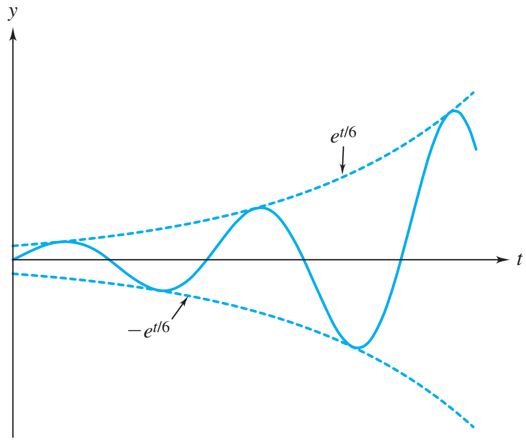

Figure 4.7 Solution graph for Example 4

Example 5 Interpret the equation

$$y'' + 5y' - 6y = 0 \tag{17}$$

in terms of the mass–spring system.

Solution Comparing the given equation with (16), we have to envision a spring with a *negative* stiffness $k = -6$. What does this mean? As the mass is moved away from the spring's equilibrium point, the spring *repels* the mass farther with a force $F_{\text{spring}} = -ky$ that intensifies as the displacement increases. Clearly the spring must "exile" the mass to (plus or minus) infinity, and we expect all solutions $y(t)$ to approach $\pm\infty$ as t increases (except for the equilibrium solution $y(t) \equiv 0$).

In fact, in Example 1 of Section 4.2, we showed the general solution to equation (17) to be

$$c_1 e^{t} + c_2 e^{-6t} . \tag{18}$$

Indeed, if we examine the solutions $y(t)$ that start with a unit displacement $y(0) = 1$ and velocity $y'(0) = v_0$, we find

$$y(t) = \frac{6 + v_0}{7} e^{t} + \frac{1 - v_0}{7} e^{-6t} , \tag{19}$$

and the plots in Figure 4.8 on page 172 confirm our prediction that all (nonequilibrium) solutions diverge—except for the one with $v_0 = -6$.

What is the physical significance of this isolated bounded solution? Evidently, if the mass is given an initial inwardly directed velocity of -6, it has barely enough energy to overcome the effect of the spring banishing it to $+\infty$ but not enough energy to cross the equilibrium point (and get pushed to $-\infty$). So it asymptotically approaches the (extremely delicate) equilibrium position $y = 0$. ◆

In Section 4.8, we will see that taking further liberties with the mass–spring interpretation enables us to predict qualitative features of more complicated equations.

Throughout this section we have assumed that the coefficients a, b, and c in the differential equation were real numbers. If we now allow them to be *complex* constants, then the roots

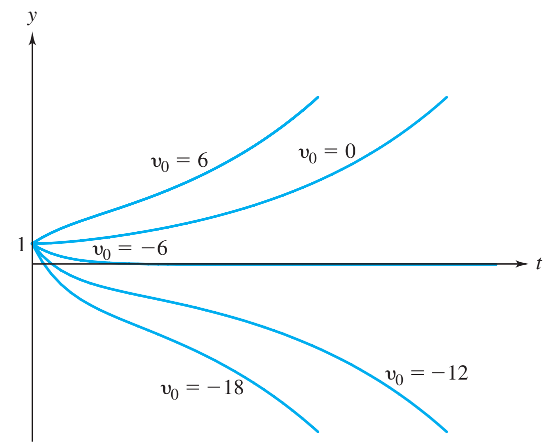

Figure 4.8 Solution graphs for Example 5

r_1, r_2 of the auxiliary equation (1) are, in general, also complex but not necessarily conjugates of each other. When $r_1 \neq r_2$, a general solution to equation (2) still has the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} ,$$

but c_1 and c_2 are now arbitrary complex-valued constants, and we have to resort to the clumsy calculations of Example 1.

We also remark that a complex differential equation can be regarded as a system of two real differential equations since we can always work separately with its real and imaginary parts. Systems are discussed in Chapters 5 and 9.

4.3 EXERCISES

In Problems 1–8, the auxiliary equation for the given differential equation has complex roots. Find a general solution.

1. $y'' + 9y = 0$ **2.** $y'' + y = 0$

3. $z'' - 6z' + 10z = 0$ **4.** $y'' - 10y' + 26y = 0$

5. $w'' + 4w' + 6w = 0$ **6.** $y'' - 4y' + 7y = 0$

7. $4y'' + 4y' + 6y = 0$ **8.** $4y'' - 4y' + 26y = 0$

In Problems 9–20, find a general solution.

9. $y'' - 8y' + 7y = 0$ **10.** $y'' + 4y' + 8y = 0$

11. $z'' + 10z' + 25z = 0$ **12.** $u'' + 7u = 0$

13. $y'' - 2y' + 26y = 0$ **14.** $y'' + 2y' + 5y = 0$

15. $y'' - 3y' - 11y = 0$ **16.** $y'' + 10y' + 41y = 0$

17. $y'' - y' + 7y = 0$ **18.** $2y'' + 13y' - 7y = 0$

19. $y''' + y'' + 3y' - 5y = 0$ **20.** $y''' - y'' + 2y = 0$

In Problems 21–27, solve the given initial value problem.

21. $y'' + 2y' + 2y = 0 ; \quad y(0) = 2 , \quad y'(0) = 1$

22. $y'' + 2y' + 17y = 0 ; \quad y(0) = 1 , \quad y'(0) = -1$

23. $w'' - 4w' + 2w = 0 ; \quad w(0) = 0 , \quad w'(0) = 1$

24. $y'' + 9y = 0 ; \quad y(0) = 1 , \quad y'(0) = 1$

25. $y'' - 2y' + 2y = 0 ; \quad y(\pi) = e^{\pi} , \quad y'(\pi) = 0$

26. $y'' - 2y' + y = 0 ; \quad y(0) = 1 , \quad y'(0) = -2$

27. $y''' - 4y'' + 7y' - 6y = 0 ; \quad y(0) = 1 , \quad y'(0) = 0, \quad y''(0) = 0$

28. To see the effect of changing the parameter b in the initial value problem

$$y'' + by' + 4y = 0 ; \quad y(0) = 1 , \quad y'(0) = 0 ,$$

solve the problem for $b = 5$, 4, and 2 and sketch the solutions.

29. Find a general solution to the following higher-order equations.
(a) $y''' - y'' + y' + 3y = 0$
(b) $y''' + 2y'' + 5y' - 26y = 0$
(c) $y^{\text{iv}} + 13y'' + 36y = 0$

30. Using the representation for $e^{(\alpha + i\beta)t}$ in (6), verify the differentiation formula (7).

31. Using the mass–spring analogy, predict the behavior as $t \to +\infty$ of the solution to the given initial value problem. Then confirm your prediction by actually solving the problem.

(a) $y'' + 16y = 0$; $y(0) = 2$, $y'(0) = 0$

(b) $y'' + 100y' + y = 0$; $y(0) = 1$, $y'(0) = 0$

(c) $y'' - 6y' + 8y = 0$; $y(0) = 1$, $y'(0) = 0$

(d) $y'' + 2y' - 3y = 0$; $y(0) = -2$, $y'(0) = 0$

(e) $y'' - y' - 6y = 0$; $y(0) = 1$, $y'(0) = 1$

32. Vibrating Spring without Damping. A vibrating spring without damping can be modeled by the initial value problem (11) in Example 3 by taking $b = 0$.

(a) If $m = 10$ kg, $k = 250$ kg/sec^2, $y(0) = 0.3$ m, and $y'(0) = -0.1$ m/sec, find the equation of motion for this undamped vibrating spring.

(b) After how many seconds will the mass in part (a) first cross the equilibrium point?

(c) When the equation of motion is of the form displayed in (9), the motion is said to be **oscillatory** with **frequency** $\beta/2\pi$. Find the frequency of oscillation for the spring system of part (a).

33. Vibrating Spring with Damping. Using the model for a vibrating spring with damping discussed in Example 3:

(a) Find the equation of motion for the vibrating spring with damping if $m = 10$ kg, $b = 60$ kg/sec, $k = 250$ kg/sec^2, $y(0) = 0.3$ m, and $y'(0) = -0.1$ m/sec.

(b) After how many seconds will the mass in part (a) first cross the equilibrium point?

(c) Find the frequency of oscillation for the spring system of part (a). [*Hint:* See the definition of frequency given in Problem 32(c).]

(d) Compare the results of Problems 32 and 33 and determine what effect the damping has on the frequency of oscillation. What other effects does it have on the solution?

34. *RLC* Series Circuit. In the study of an electrical circuit consisting of a resistor, capacitor, inductor, and an electromotive force (see Figure 4.9), we are led to an initial value problem of the form

$$L\frac{dI}{dt} + RI + \frac{q}{C} = E(t)\,; \tag{20}$$

$$q(0) = q_0\,,$$

$$I(0) = I_0\,,$$

where L is the inductance in henrys, R is the resistance in ohms, C is the capacitance in farads, $E(t)$ is the electromotive force in volts, $q(t)$ is the charge in coulombs on the capacitor at time t, and $I = dq/dt$ is the current in amperes. Find the current at time t if the charge on the capacitor is initially zero, the initial current is zero, $L = 10$ H, $R = 20\,\Omega$, $C = (6260)^{-1}$ F, and $E(t) = 100$ V. [*Hint:* Differentiate both sides of the differential equation in (20) to obtain a homogeneous linear second-order equation for $I(t)$. Then use (20) to determine dI/dt at $t = 0$.]

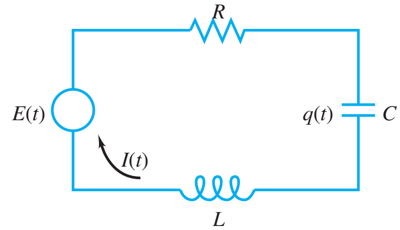

Figure 4.9 *RLC* series circuit

35. Swinging Door. The motion of a swinging door with an adjustment screw that controls the amount of friction on the hinges is governed by the initial value problem

$$I\theta'' + b\theta' + k\theta = 0\,;\quad \theta(0) = \theta_0\,,\quad \theta'(0) = v_0\,,$$

where θ is the angle that the door is open, I is the moment of inertia of the door about its hinges, $b > 0$ is a damping constant that varies with the amount of friction on the door, $k > 0$ is the spring constant associated with the swinging door, θ_0 is the initial angle that the door is opened, and v_0 is the initial angular velocity imparted to the door (see Figure 4.10). If I and k are fixed, determine for which values of b the door will *not* continually swing back and forth when closing.

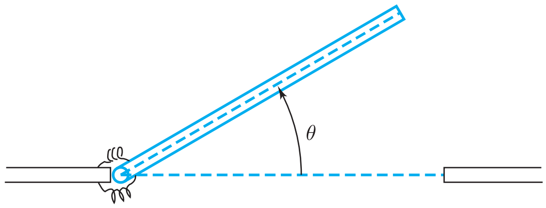

Figure 4.10 Top view of swinging door

36. Although the real general solution form (9) is convenient, it is also possible to use the form

$$d_1 e^{(\alpha+i\beta)t} + d_2 e^{(\alpha-i\beta)t} \tag{21}$$

to solve initial value problems, as illustrated in Example 1. The coefficients d_1 and d_2 are complex constants.

(a) Use the form (21) to solve Problem 21. Verify that your form is equivalent to the one derived using (9).

(b) Show that, in general, d_1 and d_2 in (21) must be complex conjugates in order that the solution be real.

37. The auxiliary equations for the following differential equations have repeated complex roots. Adapt the "repeated root" procedure of Section 4.2 to find their general solutions:

(a) $y^{\text{iv}} + 2y'' + y = 0$.

(b) $y^{\text{iv}} + 4y''' + 12y'' + 16y' + 16y = 0$. [*Hint:* The auxiliary equation is $(r^2 + 2r + 4)^2 = 0$.]

38. Prove the sum of angles formula for the sine function by following these steps. Fix x.

(a) Let $f(t) := \sin(x+t)$. Show that $f''(t) + f(t) = 0$, $f(0) = \sin x$, and $f'(0) = \cos x$.

(b) Use the auxiliary equation technique to solve the initial value problem $y'' + y = 0$, $y(0) = \sin x$, and $y'(0) = \cos x$.

(c) By uniqueness, the solution in part (b) is the same as $f(t)$ from part (a). Write this equality; this should be the standard sum of angles formula for $\sin(x+t)$.

4.4 Nonhomogeneous Equations: the Method of Undetermined Coefficients

In this section we employ "judicious guessing" to derive a simple procedure for finding a solution to a *non*homogeneous linear equation with constant coefficients

$$ay'' + by' + cy = f(t), \tag{1}$$

when the nonhomogeneity $f(t)$ is a single term of a special type. Our experience in Section 4.3 indicates that (1) will have an infinite number of solutions. For the moment we are content to find one particular solution. To motivate the procedure, let's first look at a few instructive examples.

Example 1 Find a particular solution to

$$y'' + 3y' + 2y = 3t\,. \tag{2}$$

Solution We need to find a function $y(t)$ such that the combination $y'' + 3y' + 2y$ is a linear function of t—namely, $3t$. Now what kind of function y "ends up" as a linear function after having its zeroth, first, and second derivatives combined? One immediate answer is: *another linear function*. So we might try $y_1(t) = At$ and attempt to match up $y_1'' + 3y_1' + 2y_1$ with $3t$.

Perhaps you can see that this won't work: $y_1 = At$, $y_1' = A$ and $y_1'' = 0$ gives us

$$y_1'' + 3y_1' + 2y_1 = 3A + 2At\,,$$

and for this to equal $3t$, we require both that $A = 0$ and $A = 3/2$. We'll have better luck if we append a constant term to the trial function: $y_2(t) = At + B$. Then $y_2' = A$, $y_2'' = 0$, and

$$y_2'' + 3y_2' + 2y_2 = 3A + 2(At + B) = 2At + (3A + 2B)\,,$$

which successfully matches up with $3t$ if $2A = 3$ and $3A + 2B = 0$. Solving this system gives $A = 3/2$ and $B = -9/4$. Thus, the function

$$y_2(t) = \frac{3}{2}t - \frac{9}{4}$$

is a solution to (2). ◆

Example 1 suggests the following method for finding a particular solution to the equation

$$ay'' + by' + cy = Ct^m, \quad m = 0, 1, 2, \ldots\,;$$

namely, we guess a solution of the form

$$y_p(t) = A_m t^m + \cdots + A_1 t + A_0\,,$$

with undetermined coefficients A_j, and match the corresponding powers of t in $ay'' + by' + cy$ with Ct^m.[†] This procedure involves solving $m+1$ linear equations in the $m+1$ unknowns

[†]In this case the coefficient of t^k in $ay'' + by' + cy$ will be zero for $k \neq m$ and C for $k = m$.

$A_0, A_1, \ldots, A_m$, and hopefully they have a solution. The technique is called the **method of undetermined coefficients.** Note that, as Example 1 demonstrates, we must retain *all* the powers $t^m, t^{m-1}, \ldots, t^1, t^0$ in the trial solution even though they are not present in the nonhomogeneity $f(t)$.

Example 2 Find a particular solution to

$$y'' + 3y' + 2y = 10e^{3t} . \tag{3}$$

Solution We guess $y_p(t) = Ae^{3t}$ because then y_p' and y_p'' will retain the same exponential form:

$$y_p'' + 3y_p' + 2y_p = 9Ae^{3t} + 3(3Ae^{3t}) + 2(Ae^{3t}) = 20Ae^{3t} .$$

Setting $20Ae^{3t} = 10e^{3t}$ and solving for A gives $A = 1/2$; hence,

$$y_p(t) = \frac{e^{3t}}{2}$$

is a solution to (3). ◆

Example 3 Find a particular solution to

$$y'' + 3y' + 2y = \sin t . \tag{4}$$

Solution Our initial action might be to guess $y_1(t) = A \sin t$, but this will fail because the derivatives introduce cosine terms:

$$y_1'' + 3y_1' + 2y_1 = -A \sin t + 3A \cos t + 2A \sin t = A \sin t + 3A \cos t ,$$

and matching this with $\sin t$ would require that A equal both 1 and 0. So we include the cosine term in the trial solution:

$$\begin{aligned} y_p(t) &= A \sin t + B \cos t , \\ y_p'(t) &= A \cos t - B \sin t , \\ y_p''(t) &= -A \sin t - B \cos t , \end{aligned}$$

and (4) becomes

$$\begin{aligned} y_p''(t) + 3y_p'(t) + 2y_p(t) &= -A \sin t - B \cos t + 3A \cos t - 3B \sin t \\ &\quad + 2A \sin t + 2B \cos t \\ &= (A - 3B) \sin t + (B + 3A) \cos t \\ &= \sin t . \end{aligned}$$

The equations $A - 3B = 1$, $B + 3A = 0$ have the solution $A = 0.1$, $B = -0.3$. Thus, the function

$$y_p(t) = 0.1 \sin t - 0.3 \cos t$$

is a particular solution to (4). ◆

More generally, for an equation of the form

$$ay'' + by' + cy = C \sin \beta t \quad (\text{or} \quad C \cos \beta t) , \tag{5}$$

the method of undetermined coefficients suggests that we guess

$$y_p(t) = A \cos \beta t + B \sin \beta t \tag{6}$$

and solve (5) for the unknowns A and B.

If we compare equation (5) with the mass–spring system equation

(7) $$[\text{inertia}] \times y'' + [\text{damping}] \times y' + [\text{stiffness}] \times y = F_{\text{ext}} ,$$

we can interpret (5) as describing a damped oscillator, shaken with a sinusoidal force. According to our discussion in Section 4.1, then, we would expect the mass ultimately to respond by moving in synchronization with the forcing sinusoid. In other words, the form (6) is suggested by physical, as well as mathematical, experience. A complete description of forced oscillators will be given in Section 4.10.

Example 4 Find a particular solution to

(8) $$y'' + 4y = 5t^2e^t .$$

Solution Our experience with Example 1 suggests that we take a trial solution of the form $y_p(t) = (At^2 + Bt + C)e^t$, to match the nonhomogeneity in (8). We find

$$\begin{aligned} y_p &= (At^2 + Bt + C)e^t , \\ y_p' &= (2At + B)e^t + (At^2 + Bt + C)e^t , \\ y_p'' &= 2Ae^t + 2(2At + B)e^t + (At^2 + Bt + C)e^t , \\ y_p'' + 4y_p &= e^t(2A + 2B + C + 4C) + te^t(4A + B + 4B) + t^2e^t(A + 4A) \\ &= 5t^2e^t . \end{aligned}$$

Matching like terms yields $A = 1$, $B = -4/5$, and $C = -2/25$. A solution is given by

$$y_p(t) = \left(t^2 - \frac{4t}{5} - \frac{2}{25}\right)e^t . \quad ◆$$

As our examples illustrate, when the nonhomogeneous term $f(t)$ is an exponential, a sine, a cosine function, or a nonnegative integer power of t times any of these, the function $f(t)$ itself suggests the form of a particular solution. However, certain situations thwart the straightforward application of the method of undetermined coefficients. Consider, for example, the equation

(9) $$y'' + y' = 5 .$$

Example 1 suggests that we guess $y_1(t) = A$, a zero-degree polynomial. But substitution into (9) proves futile:

$$(A)'' + (A)' = 0 \neq 5 .$$

The problem arises because any constant function, such as $y_1(t) = A$, is a solution to the corresponding homogeneous equation $y'' + y' = 0$, and the undetermined coefficient A gets lost upon substitution into the equation. We would encounter the same situation if we tried to find a solution to

(10) $$y'' - 6y' + 9y = e^{3t}$$

of the form $y_1 = Ae^{3t}$, because e^{3t} solves the associated homogeneous equation and

$$[Ae^{3t}]'' - 6[Ae^{3t}]' + 9[Ae^{3t}] = 0 \neq e^{3t} .$$

The "trick" for refining the method of undetermined coefficients in these situations smacks of the same logic as in Section 4.2, when a method was prescribed for finding second solutions to homogeneous equations with double roots. Basically, we append an extra factor of t to the trial solution suggested by the basic procedure. In other words, to solve (9) we try $y_p(t) = At$ instead of A:

(9′) $$\begin{aligned} &y_p = At, \quad y_p' = A, \quad y_p'' = 0, \\ &y_p'' + y_p' = 0 + A = 5\,, \\ &A = 5, \qquad y_p(t) = 5t\,. \end{aligned}$$

Similarly, to solve (10) we try $y_p = Ate^{3t}$ instead of Ae^{3t}. The trick won't work this time, because the characteristic equation of (10) has a double root and, consequently, Ate^{3t} also solves the homogeneous equation:

$$\left[Ate^{3t}\right]'' - 6\left[Ate^{3t}\right]' + 9\left[Ate^{3t}\right] = 0 \neq e^{3t}.$$

But if we append *another* factor of t, $y_p = At^2e^{3t}$, we succeed in finding a particular solution:†

$$\begin{aligned} y_p = At^2e^{3t},\ y_p' &= 2Ate^{3t} + 3At^2e^{3t},\ y_p'' = 2Ae^{3t} + 12Ate^{3t} + 9At^2e^{3t}\,, \\ y_p'' - 6y_p' + 9y_p &= (2Ae^{3t} + 12Ate^{3t} + 9At^2e^{3t}) - 6(2Ate^{3t} + 3At^2e^{3t}) + 9(At^2e^{3t}) \\ &= 2Ae^{3t} = e^{3t}, \end{aligned}$$

so $A = 1/2$ and $y_p(t) = t^2e^{3t}/2$.

To see why this strategy resolves the problem and to generalize it, recall the form of the original differential equation (1), $ay'' + by' + cy = f(t)$. Its associated auxiliary equation is

(11) $$\boldsymbol{ar^2 + br + c = 0\,,}$$

and if r is a double root, then

(12) $$\boldsymbol{2ar + b = 0}$$

holds also [equation (13), Section 4.2, page 163].

Now suppose the nonhomogeneity $f(t)$ has the form Ct^me^{rt}, and we seek to match this $f(t)$ by substituting $y_p(t) = (A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0)e^{rt}$ into (1), *with the power n to be determined*. For simplicity we merely list the leading terms in y_p, y_p', and y_p'':

$$\begin{aligned} y_p &= A_nt^ne^{rt} + A_{n-1}t^{n-1}e^{rt} + A_{n-2}t^{n-2}e^{rt} + \ (\textit{lower-order terms}) \\ y_p' &= A_nrt^ne^{rt} + A_nnt^{n-1}e^{rt} + A_{n-1}rt^{n-1}e^{rt} + A_{n-1}(n-1)t^{n-2}e^{rt} \\ &\quad + A_{n-2}rt^{n-2}e^{rt} + (l.o.t.)\,, \\ y_p'' &= A_nr^2t^ne^{rt} + 2A_nnrt^{n-1}e^{rt} + A_nn(n-1)t^{n-2}e^{rt} \\ &\quad + A_{n-1}r^2t^{n-1}e^{rt} + 2A_{n-1}r(n-1)t^{n-2}e^{rt} + A_{n-2}r^2t^{n-2}e^{rt} + (l.o.t)\,. \end{aligned}$$

Then the left-hand member of (1) becomes

(13) $$\begin{aligned} &ay_p'' + by_p' + cy_p \\ &\quad = A_n(\boldsymbol{ar^2 + br + c})t^ne^{rt} + [A_nn(\boldsymbol{2ar + b}) + A_{n-1}(\boldsymbol{ar^2 + br + c})]t^{n-1}e^{rt} \\ &\qquad + [A_nn(n-1)a + A_{n-1}(n-1)(\boldsymbol{2ar + b}) + A_{n-2}(\boldsymbol{ar^2 + br + c})]t^{n-2}e^{rt} \\ &\qquad + (l.o.t.)\,, \end{aligned}$$

†Indeed, the solution t^2 to the equation $y'' = 2$, computed by simple integration, can also be derived by appending two factors of t to the solution $y \equiv 1$ of the associated homogeneous equation.

and we observe the following:

Case 1. If r is *not* a root of the auxiliary equation, the leading term in (13) is $A_n(ar^2 + br + c)t^n e^{rt}$, and to match $f(t) = Ct^m e^{rt}$ we must take $n = m$:

$$y_p(t) = (A_m t^m + \cdots + A_1 t + A_0)e^{rt} .$$

Case 2. If r is a *simple* root of the auxiliary equation, (11) holds and the leading term in (13) is $A_n n(2ar + b)t^{n-1}e^{rt}$, and to match $f(t) = Ct^m e^{rt}$ we must take $n = m + 1$:

$$y_p(t) = (A_{m+1}t^{m+1} + A_m t^m + \cdots + A_1 t + A_0)e^{rt} .$$

However, now the final term $A_0 e^{rt}$ can be dropped, since it solves the associated homogeneous equation, so we can factor out t and for simplicity renumber the coefficients to write

$$y_p(t) = t(A_m t^m + \cdots + A_1 t + A_0)e^{rt} .$$

Case 3. If r is a *double* root of the auxiliary equation, (11) and (12) hold and the leading term in (13) is $A_n n(n-1)at^{n-2}e^{rt}$, and to match $f(t) = Ct^m e^{rt}$ we must take $n = m + 2$:

$$y_p(t) = (A_{m+2}t^{m+2} + A_{m+1}t^{m+1} + \cdots + A_2 t^2 + A_1 t + A_0)e^{rt},$$

but again we drop the solutions to the associated homogeneous equation and renumber to write

$$y_p(t) = t^2(A_m t^m + \cdots + A_1 t + A_0)e^{rt} .$$

We summarize with the following rule.

Method of Undetermined Coefficients

To find a particular solution to the differential equation

$$ay'' + by' + cy = Ct^m e^{rt},$$

where m is a nonnegative integer, use the form

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt}, \tag{14}$$

with

(i) $s = 0$ if r is not a root of the associated auxiliary equation;
(ii) $s = 1$ if r is a simple root of the associated auxiliary equation; and
(iii) $s = 2$ if r is a double root of the associated auxiliary equation.

To find a particular solution to the differential equation

$$ay'' + by' + cy = \begin{cases} Ct^m e^{\alpha t}\cos\beta t \\ \qquad \text{or} \\ Ct^m e^{\alpha t}\sin\beta t \end{cases}$$

for $\beta \neq 0$, use the form

$$\begin{aligned} y_p(t) &= t^s(A_m t^m + \cdots + A_1 t + A_0)e^{\alpha t}\cos\beta t \\ &\quad + t^s(B_m t^m + \cdots + B_1 t + B_0)e^{\alpha t}\sin\beta t , \end{aligned} \tag{15}$$

with

(iv) $s = 0$ if $\alpha + i\beta$ is not a root of the associated auxiliary equation; and
(v) $s = 1$ if $\alpha + i\beta$ is a root of the associated auxiliary equation.

[The (cos, sin) formulation (15) is easily derived from the exponential formulation (14) by putting $r = \alpha + i\beta$ and employing Euler's formula, as in Section 4.3.]

Remark 1. The nonhomogeneity Ct^m corresponds to the case when $r = 0$.

Remark 2. The rigorous justification of the method of undetermined coefficients [including the analysis of the terms we dropped in (13)] will be presented in a more general context in Chapter 6.

Example 5 Find the form for a particular solution to

$$y'' + 2y' - 3y = f(t)\,, \tag{16}$$

where $f(t)$ equals

(a) $7\cos 3t$ **(b)** $2te^t \sin t$ **(c)** $t^2 \cos \pi t$ **(d)** $5e^{-3t}$ **(e)** $3te^t$ **(f)** $t^2 e^t$

Solution The auxiliary equation for the homogeneous equation corresponding to (16), $r^2 + 2r - 3 = 0$, has roots $r_1 = 1$ and $r_2 = -3$. Notice that the functions in (a), (b), and (c) are associated with *complex* roots (because of the trigonometric factors). These are clearly different from r_1 and r_2, so the solution forms correspond to (15) with $s = 0$:

(a) $y_p(t) = A\cos 3t + B\sin 3t$

(b) $y_p(t) = (A_1 t + A_0)e^t \cos t + (B_1 t + B_0)e^t \sin t$

(c) $y_p(t) = (A_2 t^2 + A_1 t + A_0)\cos \pi t + (B_2 t^2 + B_1 t + B_0)\sin \pi t$

For the nonhomogeneity in (d) we appeal to (ii) and take $y_p(t) = Ate^{-3t}$ since -3 is a simple root of the auxiliary equation. Similarly, for (e) we take $y_p(t) = t(A_1 t + A_0)e^t$ and for (f) we take $y_p(t) = t(A_2 t^2 + A_1 t + A_0)e^t$. ◆

Example 6 Find the form of a particular solution to

$$y'' - 2y' + y = f(t),$$

for the same set of nonhomogeneities $f(t)$ as in Example 5.

Solution Now the auxiliary equation for the corresponding homogeneous equation is $r^2 - 2r + 1 = (r-1)^2 = 0$, with the double root $r = 1$. This root is not linked with any of the nonhomogeneities (a) through (d), so the same trial forms should be used for (a), (b), and (c) as in the previous example, and $y(t) = Ae^{-3t}$ will work for (d).

Since $r = 1$ is a double root, we have $s = 2$ in (14) and the trial forms for (e) and (f) have to be changed to

(e) $y_p(t) = t^2(A_1 t + A_0)e^t$

(f) $y_p(t) = t^2(A_2 t^2 + A_1 t + A_0)e^t$

respectively, in accordance with (iii). ◆

Example 7 Find the form of a particular solution to

$$y'' - 2y' + 2y = 5te^t \cos t\,.$$

Solution Now the auxiliary equation for the corresponding homogeneous equation is $r^2 - 2r + 2 = 0$, and it has complex roots $r_1 = 1 + i$, $r_2 = 1 - i$. Since the nonhomogeneity involves $e^{\alpha t}\cos \beta t$ with $\alpha = \beta = 1$; that is, $\alpha + i\beta = 1 + i = r_1$, the solution takes the form

$$y_p(t) = t(A_1 t + A_0)e^t \cos t + t(B_1 t + B_0)e^t \sin t\,. \quad ◆$$

The nonhomogeneity tan t in an equation like $y'' + y' + y = \tan t$ is not one of the forms for which the method of undetermined coefficients can be used; the derivatives of the "trial solution" $y(t) = A\tan t$, for example, get complicated, and it is not clear what additional terms need to be added to obtain a true solution. In Section 4.6 we discuss a different procedure that can handle such nonhomogeneous terms. **Keep in mind that the method of undetermined coefficients applies only to nonhomogeneities that are polynomials, exponentials, sines or cosines, or products of these functions.** The superposition principle in Section 4.5 shows how the method can be extended to the **sums** of such nonhomogeneities. Also, it provides the key to assembling a general solution to (1) that can accommodate initial value problems, which we have avoided so far in our examples.

4.4 EXERCISES

In Problems 1–8, decide whether or not the method of undetermined coefficients can be applied to find a particular solution of the given equation.

1. $y'' + 2y' - y = t^{-1}e^t$

2. $5y'' - 3y' + 2y = t^3 \cos 4t$

3. $2y''(x) - 6y'(x) + y(x) = (\sin x)/e^{4x}$

4. $x'' + 5x' - 3x = 3^t$

5. $y''(\theta) + 3y'(\theta) - y(\theta) = \sec\theta$

6. $2\omega''(x) - 3\omega(x) = 4x\sin^2 x + 4x\cos^2 x$

7. $8z'(x) - 2z(x) = 3x^{100}e^{4x}\cos 25x$

8. $ty'' - y' + 2y = \sin 3t$

In Problems 9–26, find a particular solution to the differential equation.

9. $y'' + 3y = -9$

10. $y'' + 2y' - y = 10$

11. $y''(x) + y(x) = 2^x$

12. $2x' + x = 3t^2$

13. $y'' - y' + 9y = 3\sin 3t$

14. $2z'' + z = 9e^{2t}$

15. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = xe^x$

16. $\theta''(t) - \theta(t) = t\sin t$

17. $y'' + 4y = 8\sin 2t$

18. $y'' - 2y' + y = 8e^t$

19. $4y'' + 11y' - 3y = -2te^{-3t}$

20. $y'' + 4y = 16t\sin 2t$

21. $x''(t) - 4x'(t) + 4x(t) = te^{2t}$

22. $x''(t) - 2x'(t) + x(t) = 24t^2e^t$

23. $y''(\theta) - 7y'(\theta) = \theta^2$

24. $y''(x) + y(x) = 4x\cos x$

25. $y'' + 2y' + 4y = 111e^{2t}\cos 3t$

26. $y'' + 2y' + 2y = 4te^{-t}\cos t$

In Problems 27–32, determine the form of a particular solution for the differential equation. (Do not evaluate coefficients.)

27. $y'' + 9y = 4t^3\sin 3t$

28. $y'' - 6y' + 9y = 5t^6e^{3t}$

29. $y'' + 3y' - 7y = t^4e^t$

30. $y'' - 2y' + y = 7e^t\cos t$

31. $y'' + 2y' + 2y = 8t^3e^{-t}\sin t$

32. $y'' - y' - 12y = 2t^6e^{-3t}$

In Problems 33–36, use the method of undetermined coefficients to find a particular solution to the given higher-order equation.

33. $y''' - y'' + y = \sin t$

34. $2y''' + 3y'' + y' - 4y = e^{-t}$

35. $y''' + y'' - 2y = te^t$

36. $y^{(4)} - 3y'' - 8y = \sin t$

4.5 The Superposition Principle and Undetermined Coefficients Revisited

The next theorem describes the superposition principle, a very simple observation which nonetheless endows the solution set for our equations with a powerful structure. It extends the applicability of the method of undetermined coefficients and enables us to solve initial value problems for nonhomogeneous differential equations.

Superposition Principle

Theorem 3. If y_1 is a solution to the differential equation

$$ay'' + by' + cy = f_1(t)\ ,$$

and y_2 is a solution to

$$ay'' + by' + cy = f_2(t)\ ,$$

then for any constants k_1 and k_2, the function $k_1y_1 + k_2y_2$ is a solution to the differential equation

$$ay'' + by' + cy = k_1f_1(t) + k_2f_2(t)\ .$$

Proof. This is straightforward; by substituting and rearranging we find

$$\begin{aligned} &a(k_1y_1 + k_2y_2)'' + b(k_1y_1 + k_2y_2)' + c(k_1y_1 + k_2y_2) \\ &\quad = k_1(ay_1'' + by_1' + cy_1) + k_2(ay_2'' + by_2' + cy_2) \\ &\quad = k_1f_1(t) + k_2f_2(t)\ . \ \blacklozenge \end{aligned}$$

Example 1 Find a particular solution to

(1) $$y'' + 3y' + 2y = 3t + 10e^{3t} \quad \text{and}$$

(2) $$y'' + 3y' + 2y = -9t + 20e^{3t}\ .$$

Solution In Example 1, Section 4.4, we found that $y_1(t) = 3t/2 - 9/4$ was a solution to $y'' + 3y' + 2y = 3t$, and in Example 2 we found that $y_2(t) = e^{3t}/2$ solved $y'' + 3y' + 2y = 10e^{3t}$. By superposition, then, $y_1 + y_2 = 3t/2 - 9/4 + e^{3t}/2$ solves equation (1).

The right-hand member of (2) equals minus three times $(3t)$ plus two times $(10e^{3t})$. Therefore, this same combination of y_1 and y_2 will solve (2):

$$y(t) = -3y_1 + 2y_2 = -3(3t/2 - 9/4) + 2(e^{3t}/2) = -9t/2 + 27/4 + e^{3t}\ . \ \blacklozenge$$

If we take a particular solution y_p to a nonhomogeneous equation like

(3) $$ay'' + by' + cy = f(t)$$

and add it to a general solution $c_1y_1 + c_2y_2$ of the homogeneous equation associated with (3),

(4) $$ay'' + by' + cy = 0\ ,$$

the sum

(5) $$y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t)$$

is again, according to the superposition principle, a solution to (3):

$$\begin{aligned} &a(y_p + c_1y_1 + c_2y_2)'' + b(y_p + c_1y_1 + c_2y_2)' + c(y_p + c_1y_1 + c_2y_2) \\ &\quad = f(t) + 0 + 0 = f(t)\ . \end{aligned}$$

Since (5) contains two parameters, one would suspect that c_1 and c_2 can be chosen to make it satisfy arbitrary initial conditions. It is easy to verify that this is indeed the case.

Existence and Uniqueness: Nonhomogeneous Case

Theorem 4. For any real numbers $a(\neq 0)$, b, c, t_0, Y_0, and Y_1, suppose $y_p(t)$ is a particular solution to (3) in an interval I containing t_0 and that $y_1(t)$ and $y_2(t)$ are linearly independent solutions to the associated homogeneous equation (4) in I. Then there exists a unique solution in I to the initial value problem

$$ay'' + by' + cy = f(t), \qquad y(t_0) = Y_0, \quad y'(t_0) = Y_1, \tag{6}$$

and it is given by (5), for the appropriate choice of the constants c_1, c_2.

Proof. We have already seen that the superposition principle implies that (5) solves the differential equation. To satisfy the initial conditions in (6) we need to choose the constants so that

$$\begin{cases} y_p(t_0) + c_1 y_1(t_0) + c_2 y_2(t_0) = Y_0, \\ y_p'(t_0) + c_1 y_1'(t_0) + c_2 y_2'(t_0) = Y_1. \end{cases} \tag{7}$$

But as in the proof of Theorem 2 in Section 4.2, simple algebra shows that the choice

$$c_1 = \frac{[Y_0 - y_p(t_0)]y_2'(t_0) - [Y_1 - y_p'(t_0)]y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}$$

and

$$c_2 = \frac{[Y_1 - y_p'(t_0)]y_1(t_0) - [Y_0 - y_p(t_0)]y_1'(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}$$

solves (7) unless the denominator is zero; Lemma 1, Section 4.2, assures us that it is not.

Why is the solution unique? If $y_{\mathrm{I}}(t)$ were another solution to (6), then the difference $y_{\mathrm{II}}(t) := y_p(t) + c_1 y_1(t) + c_2 y_2(t) - y_{\mathrm{I}}(t)$ would satisfy

$$\begin{cases} ay_{\mathrm{II}}'' + by_{\mathrm{II}}' + cy_{\mathrm{II}} = f(t) - f(t) = 0, \\ y_{\mathrm{II}}(t_0) = Y_0 - Y_0 = 0, \quad y_{\mathrm{II}}'(t_0) = Y_1 - Y_1 = 0. \end{cases} \tag{8}$$

But the initial value problem (8) admits the identically zero solution, and Theorem 1 in Section 4.2 applies since the differential equation in (8) is homogeneous. Consequently, (8) has *only* the identically zero solution. Thus, $y_{\mathrm{II}} \equiv 0$ and $y_{\mathrm{I}} = y_p + c_1 y_1 + c_2 y_2$. ◆

These deliberations entitle us to say that $y = y_p + c_1 y_1 + c_2 y_2$ is a **general solution** to the nonhomogeneous equation (3), since *any* solution $y_g(t)$ can be expressed in this form. (**Proof:** As in Section 4.2, we simply pick c_1 and c_2 so that $y_p + c_1 y_1 + c_2 y_2$ matches the value and the derivative of y_g *at any single point*; by uniqueness, $y_p + c_1 y_1 + c_2 y_2$ and y_g have to be the same function.)

Example 2 Given that $y_p(t) = t^2$ is a particular solution to

$$y'' - y = 2 - t^2,$$

find a general solution and a solution satisfying $y(0) = 1$, $y'(0) = 0$.

Solution The corresponding homogeneous equation,

$$y'' - y = 0\,,$$

has the associated auxiliary equation $r^2 - 1 = 0$. Because $r = \pm 1$ are the roots of this equation, a general solution to the homogeneous equation is $c_1 e^t + c_2 e^{-t}$. Combining this with the particular solution $y_p(t) = t^2$ of the nonhomogeneous equation, we find that a general solution is

$$y(t) = t^2 + c_1 e^t + c_2 e^{-t}\,.$$

To meet the initial conditions, set

$$\begin{aligned} y(0) &= 0^2 + c_1 e^0 + c_2 e^{-0} = 1\,, \\ y'(0) &= 2 \times 0 + c_1 e^0 - c_2 e^{-0} = 0\,, \end{aligned}$$

which yields $c_1 = c_2 = \frac{1}{2}$. The answer is

$$y(t) = t^2 + \frac{1}{2}(e^t + e^{-t}) = t^2 + \cosh t\,. \ \blacklozenge$$

Example 3 A mass–spring system is driven by a sinusoidal external force $(5 \sin t + 5 \cos t)$. The mass equals 1, the spring constant equals 2, and the damping coefficient equals 2 (in appropriate units), so the deliberations of Section 4.1 imply that the motion is governed by the differential equation

(9) $$y'' + 2y' + 2y = 5 \sin t + 5 \cos t\,.$$

If the mass is initially located at $y(0) = 1$, with a velocity $y'(0) = 2$, find its equation of motion.

Solution The associated homogeneous equation $y'' + 2y' + 2y = 0$ was studied in Example 1, Section 4.3; the roots of the auxiliary equation were found to be $-1 \pm i$, leading to a general solution $c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$.

The method of undetermined coefficients dictates that we try to find a particular solution of the form $A \sin t + B \cos t$ for the first nonhomogeneity $5 \sin t$:

(10) $$\begin{aligned} &y_p = A \sin t + B \cos t\,, \quad y_p' = A \cos t - B \sin t\,, \quad y_p'' = -A \sin t - B \cos t\,; \\ &y_p'' + 2y_p' + 2y_p = (-A - 2B + 2A) \sin t + (-B + 2A + 2B) \cos t = 5 \sin t\,. \end{aligned}$$

Matching coefficients requires $A = 1$, $B = -2$ and so $y_p = \sin t - 2 \cos t$.

The second nonhomogeneity $5 \cos t$ calls for the identical form for y_p and leads to $(-A - 2B + 2A) \sin t + (-B + 2A + 2B) \cos t = 5 \cos t$, or $A = 2, B = 1$. Hence $y_p = 2 \sin t + \cos t$.

By the superposition principle, a general solution to (9) is given by the sum

$$\begin{aligned} y &= c_1 e^{-t} \cos t + c_2 e^{-t} \sin t + \sin t - 2 \cos t + 2 \sin t + \cos t \\ &= c_1 e^{-t} \cos t + c_2 e^{-t} \sin t + 3 \sin t - \cos t\,. \end{aligned}$$

The initial conditions are

$$\begin{aligned} y(0) &= 1 = c_1 e^{-0} \cos 0 + c_2 e^{-0} \sin 0 + 3 \sin 0 - \cos 0 = c_1 - 1\,, \\ y'(0) &= 2 = c_1 \left[-e^{-t} \cos t - e^{-t} \sin t\right]_{t=0} + c_2 \left[-e^{-t} \sin t + e^{-t} \cos t\right]_{t=0} \\ &\qquad + 3 \cos 0 + \sin 0 \\ &= -c_1 + c_2 + 3\,, \end{aligned}$$

requiring $c_1 = 2$, $c_2 = 1$, and thus

(11) $$y(t) = 2e^{-t} \cos t + e^{-t} \sin t + 3 \sin t - \cos t\,. \ \blacklozenge$$

The solution (11) exemplifies the features of forced, damped oscillations that we anticipated in Section 4.1. There is a sinusoidal component $(3\sin t - \cos t)$ that is synchronous with the driving force $(5\sin t + 5\cos t)$, and a component $(2e^{-t}\cos t + e^{-t}\sin t)$ that dies out. When the system is "pumped" sinusoidally, the response is a synchronous sinusoidal oscillation, after an initial transient that depends on the initial conditions; the synchronous response is the particular solution supplied by the method of undetermined coefficients, and the transient is the solution to the associated homogeneous equation. This interpretation will be discussed in detail in Sections 4.9 and 4.10.

You may have observed that, since the two undetermined-coefficient forms in the last example were identical and were destined to be added together, we could have used the form (10) to match both nonhomogeneities at the same time, deriving the condition

$$y_p'' + 2y_p' + 2y_p = (-A - 2B + 2A)\sin t + (-B + 2A + 2B)\cos t = 5\sin t + 5\cos t\,,$$

with solution $y_p = 3\sin t - \cos t$. The next example illustrates this "streamlined" procedure.

Example 4 Find a particular solution to

$$y'' - y = 8te^t + 2e^t\,. \tag{12}$$

Solution A general solution to the associated homogeneous equation is easily seen to be $c_1e^t + c_2e^{-t}$. Thus, a particular solution for matching the nonhomogeneity $8te^t$ has the form $t(A_1t + A_0)e^t$, whereas matching $2e^t$ requires the form A_0te^t. Therefore, we can match both with the first form:

$$\begin{aligned}
y_p &= t(A_1t + A_0)e^t = (A_1t^2 + A_0t)e^t\,,\\
y_p' &= (A_1t^2 + A_0t)e^t + (2A_1t + A_0)e^t = [A_1t^2 + (2A_1 + A_0)t + A_0]e^t\,,\\
y_p'' &= [2A_1t + (2A_1 + A_0)]e^t + [A_1t^2 + (2A_1 + A_0)t + A_0]e^t\\
&= [A_1t^2 + (4A_1 + A_0)t + (2A_1 + 2A_0)]e^t\,.
\end{aligned}$$

Thus

$$\begin{aligned}
y_p'' - y_p &= [4A_1t + (2A_1 + 2A_0)]e^t\\
&= 8te^t + 2e^t\,,
\end{aligned}$$

which yields $A_1 = 2$, $A_0 = -1$, and so $y_p = (2t^2 - t)e^t$. ◆

We generalize this procedure by modifying the method of undetermined coefficients as follows.

Method of Undetermined Coefficients (Revisited)

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{rt},$$

where $P_m(t)$ is a polynomial of degree m, use the form

$$y_p(t) = t^s(A_mt^m + \cdots + A_1t + A_0)e^{rt}\,; \tag{13}$$

if r is not a root of the associated auxiliary equation, take $s = 0$; if r is a simple root of the associated auxiliary equation, take $s = 1$; and if r is a double root of the associated auxiliary equation, take $s = 2$.

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{\alpha t}\cos\beta t + Q_n(t)e^{\alpha t}\sin\beta t\,, \qquad \beta \neq 0\,,$$

where $P_m(t)$ is a polynomial of degree m and $Q_n(t)$ is a polynomial of degree n, use the form

$$\textbf{(14)} \qquad y_p(t) = t^s(A_k t^k + \cdots + A_1 t + A_0)e^{\alpha t}\cos\beta t + t^s(B_k t^k + \cdots + B_1 t + B_0)e^{\alpha t}\sin\beta t\,,$$

where k is the larger of m and n. If $\alpha + i\beta$ is not a root of the associated auxiliary equation, take $s = 0$; if $\alpha + i\beta$ is a root of the associated auxiliary equation, take $s = 1$.

Example 5 Write down the form of a particular solution to the equation

$$y'' + 2y' + 2y = 5e^{-t}\sin t + 5t^3e^{-t}\cos t\,.$$

Solution The roots of the associated homogeneous equation $y'' + 2y' + 2y = 0$ were identified in Example 3 as $-1 \pm i$. Application of (14) dictates the form

$$y_p(t) = t(A_3t^3 + A_2t^2 + A_1t + A_0)e^{-t}\cos t + t(B_3t^3 + B_2t^2 + B_1t + B_0)e^{-t}\sin t\,. \ ◆$$

The method of undetermined coefficients applies to higher-order linear differential equations with constant coefficients. Details will be provided in Chapter 6, but the following example should be clear.

Example 6 Write down the form of a particular solution to the equation

$$y''' + 2y'' + y' = 5e^{-t}\sin t + 3 + 7te^{-t}\,.$$

Solution The auxiliary equation for the associated homogeneous is $r^3 + 2r^2 + r = r(r+1)^2 = 0$, with a double root $r = -1$ and a single root $r = 0$. Term by term, the nonhomogeneities call for the forms

$$A_0e^{-t}\cos t + B_0e^{-t}\sin t \qquad (\text{for } 5e^{-t}\sin t)\,,$$
$$tA_0 \qquad (\text{for } 3)\,,$$
$$t^2(A_1t + A_0)e^{-t} \qquad (\text{for } 7te^{-t})\,.$$

(If -1 were a *triple* root, we would need $t^3(A_1t + A_0)e^{-t}$ for $7te^{-t}$.) Of course, we have to rename the coefficients, so the general form is

$$y_p(t) = Ae^{-t}\cos t + Be^{-t}\sin t + tC + t^2(Dt + E)e^{-t}\,. \ ◆$$

4.5 EXERCISES

1. Given that $y_1(t) = \cos t$ is a solution to

$$y'' - y' + y = \sin t$$

and $y_2(t) = e^{2t}/3$ is a solution to

$$y'' - y' + y = e^{2t},$$

use the superposition principle to find solutions to the following differential equations:

(a) $y'' - y' + y = 5\sin t\,.$

(b) $y'' - y' + y = \sin t - 3e^{2t}\,.$

(c) $y'' - y' + y = 4\sin t + 18e^{2t}\,.$

2. Given that $y_1(t) = (1/4)\sin 2t$ is a solution to $y'' + 2y' + 4y = \cos 2t$ and that $y_2(t) = t/4 - 1/8$ is a solution to $y'' + 2y' + 4y = t$, use the superposition principle to find solutions to the following:

(a) $y'' + 2y' + 4y = t + \cos 2t$.

(b) $y'' + 2y' + 4y = 2t - 3\cos 2t$.

(c) $y'' + 2y' + 4y = 11t - 12\cos 2t$.

In Problems 3–8, a nonhomogeneous equation and a particular solution are given. Find a general solution for the equation.

3. $y'' - y = t\,, \quad y_p(t) = -t$

4. $y'' + y' = 1\,, \quad y_p(t) = t$

5. $\theta'' - \theta' - 2\theta = 1 - 2t\,, \quad \theta_p(t) = t - 1$

6. $y'' + 5y' + 6y = 6x^2 + 10x + 2 + 12e^x$, $y_p(x) = e^x + x^2$

7. $y'' = 2y + 2\tan^3 x\,, \quad y_p(x) = \tan x$

8. $y'' = 2y' - y + 2e^x\,, \quad y_p(x) = x^2 e^x$

In Problems 9–16 decide whether the method of undetermined coefficients together with superposition can be applied to find a particular solution of the given equation. Do not solve the equation.

9. $3y'' + 2y' + 8y = t^2 + 4t - t^2 e^t \sin t$

10. $y'' - y' + y = (e^t + t)^2$

11. $y'' - 6y' - 4y = 4\sin 3t - e^{3t}t^2 + 1/t$

12. $y'' + y' + ty = e^t + 7$

13. $y'' - 2y' + 3y = \cosh t + \sin^3 t$

14. $2y'' + 3y' - 4y = 2t + \sin^2 t + 3$

15. $y'' + e^t y' + y = 7 + 3t$

16. $2y'' - y' + 6y = t^2 e^{-t}\sin t - 8t\cos 3t + 10^t$

In Problems 17–22, find a general solution to the differential equation.

17. $y'' - 2y' - 3y = 3t^2 - 5$

18. $y'' - y = -11t + 1$

19. $y''(x) - 3y'(x) + 2y(x) = e^x \sin x$

20. $y''(\theta) + 4y(\theta) = \sin\theta - \cos\theta$

21. $y''(\theta) + 2y'(\theta) + 2y(\theta) = e^{-\theta}\cos\theta$

22. $y''(x) + 6y'(x) + 10y(x) = 10x^4 + 24x^3 + 2x^2 - 12x + 18$

In Problems 23–30, find the solution to the initial value problem.

23. $y' - y = 1\,, \quad y(0) = 0$

24. $y'' = 6t\,; \quad y(0) = 3\,, \quad y'(0) = -1$

25. $z''(x) + z(x) = 2e^{-x}\,; \quad z(0) = 0\,, \quad z'(0) = 0$

26. $y'' + 9y = 27\,; \quad y(0) = 4\,, \quad y'(0) = 6$

27. $y''(x) - y'(x) - 2y(x) = \cos x - \sin 2x \;;$ $y(0) = -7/20\,, \quad y'(0) = 1/5$

28. $y'' + y' - 12y = e^t + e^{2t} - 1\,; y(0) = 1\,, \quad y'(0) = 3$

29. $y''(\theta) - y(\theta) = \sin\theta - e^{2\theta}\,;$ $y(0) = 1\,, \quad y'(0) = -1$

30. $y'' + 2y' + y = t^2 + 1 - e^t\,; \quad y(0) = 0\,, \quad y'(0) = 2$

In Problems 31–36, determine the form of a particular solution for the differential equation. Do not solve.

31. $y'' + y = \sin t + t\cos t + 10^t$

32. $y'' - y = e^{2t} + te^{2t} + t^2 e^{2t}$

33. $x'' - x' - 2x = e^t\cos t - t^2 + \cos^3 t$

34. $y'' + 5y' + 6y = \sin t - \cos 2t$

35. $y'' - 4y' + 5y = e^{5t} + t\sin 3t - \cos 3t$

36. $y'' - 4y' + 4y = t^2 e^{2t} - e^{2t}$

In Problems 37–40, find a particular solution to the given higher-order equation.

37. $y''' - 2y'' - y' + 2y = 2t^2 + 4t - 9$

38. $y^{(4)} - 5y'' + 4y = 10\cos t - 20\sin t$

39. $y''' + y'' - 2y = te^t + 1$

40. $y^{(4)} - 3y''' + 3y'' - y' = 6t - 20$

41. Discontinuous Forcing Term. In certain physical models, the nonhomogeneous term, or **forcing term,** $g(t)$ in the equation

$$ay'' + by' + cy = g(t)$$

may not be continuous but may have a jump discontinuity. If this occurs, we can still obtain a reasonable solution using the following procedure. Consider the initial value problem

$$y'' + 2y' + 5y = g(t)\,; \quad y(0) = 0\,, \quad y'(0) = 0,$$

where

$$g(t) = \begin{cases} 10 & \text{if } 0 \le t \le 3\pi/2 \\ 0 & \text{if } t > 3\pi/2 \end{cases}.$$

(a) Find a solution to the initial value problem for $0 \le t \le 3\pi/2$.

(b) Find a general solution for $t > 3\pi/2$.

(c) Now choose the constants in the general solution from part (b) so that the solution from part (a) and the solution from part (b) agree, together with their first derivatives, at $t = 3\pi/2$. This gives us a continuously differentiable function that satisfies the differential equation except at $t = 3\pi/2$.

42. Forced Vibrations. As discussed in Section 4.1, a vibrating spring with damping that is under external force can be modeled by

$$my'' + by' + ky = g(t)\,, \tag{15}$$

where $m > 0$ is the mass of the spring system, $b > 0$ is the damping constant, $k > 0$ is the spring constant, $g(t)$ is the force on the system at time t, and $y(t)$ is the displacement from the equilibrium of the spring system at time t. Assume $b^2 < 4mk$.

(a) Determine the form of the equation of motion for the spring system when $g(t) = \sin\beta t$ by finding a general solution to equation (15).

(b) Discuss the long-term behavior of this system. [*Hint:* Consider what happens to the general solution obtained in part (a) as $t \to +\infty$.]

43. A mass–spring system is driven by a sinusoidal external force $g(t) = 5\sin t$. The mass equals 1, the spring constant equals 3, and the damping coefficient equals 4. If the mass is initially located at $y(0) = 1/2$ and at rest, i.e., $y'(0) = 0$, find its equation of motion.

44. A mass–spring system is driven by the external force $g(t) = 2\sin 3t + 10\cos 3t$. The mass equals 1, the spring constant equals 5, and the damping coefficient equals 2. If the mass is initially located at $y(0) = -1$, with initial velocity $y'(0) = 5$, find its equation of motion.

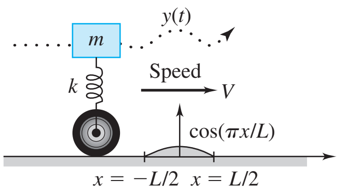

Figure 4.11 Speed bump

45. Speed Bumps. Often bumps like the one depicted in Figure 4.11 are built into roads to discourage speeding. The figure suggests that a crude model of the vertical motion $y(t)$ of a car encountering the speed bump with the speed V is given by

$$y(t) = 0 \qquad \text{for } t \le -L/(2V)\,,$$

$$my'' + ky = \begin{cases} F_0\cos(\pi Vt/L) & \text{for } |t| < L/(2V) \\ 0 & \text{for } t \ge L/(2V)\,. \end{cases}$$

(The absence of a damping term indicates that the car's shock absorbers are not functioning.)

(a) Taking $m = k = 1$, $L = \pi$, and $F_0 = 1$ in appropriate units, solve this initial value problem. Thereby show that the formula for the oscillatory motion *after the car has traversed the speed bump* is $y(t) = A\sin t$, where the constant A depends on the speed V.

(b) Plot the amplitude $|A|$ of the solution $y(t)$ found in part (a) versus the car's speed V. From the graph, estimate the speed that produces the most violent shaking of the vehicle.

46. Show that the *boundary value problem*

$$y'' + \lambda^2 y = \sin t\,; \quad y(0) = 0\,, \quad y(\pi) = 1\,,$$

has a solution if and only if $\lambda \ne \pm 1, \pm 2, \pm 3, \ldots$.

47. Find the solution(s) to

$$y'' + 9y = 27\cos 6t$$

(if it exists) satisfying the *boundary conditions*

(a) $y(0) = -1\,, \quad y(\pi/6) = 3\,.$

(b) $y(0) = -1\,, \quad y(\pi/3) = 5\,.$

(c) $y(0) = -1\,, \quad y(\pi/3) = -1\,.$

48. All that is known concerning a mysterious second-order constant-coefficient differential equation $y'' + py' + qy = g(t)$ is that $t^2 + 1 + e^t\cos t$, $t^2 + 1 + e^t\sin t$, and $t^2 + 1 + e^t\cos t + e^t\sin t$ are solutions.

(a) Determine two linearly independent solutions to the corresponding homogeneous equation.

(b) Find a suitable choice of p, q, and $g(t)$ that enables these solutions.

4.6 Variation of Parameters

We have seen that the method of undetermined coefficients is a simple procedure for determining a particular solution when the equation has constant coefficients and the nonhomogeneous term is of a special type. Here we present a more general method, called **variation of parameters,**† for finding a particular solution.

Consider the nonhomogeneous linear second-order equation

$$ay'' + by' + cy = f(t) \tag{1}$$

and let $\{y_1(t), y_2(t)\}$ be two linearly independent solutions for the corresponding homogeneous equation

$$ay'' + by' + cy = 0\,.$$

†*Historical Footnote:* The method of variation of parameters was invented by Joseph Lagrange in 1774.

Then we know that a general solution to this homogeneous equation is given by

(2) $$y_h(t) = c_1y_1(t) + c_2y_2(t) ,$$

where c_1 and c_2 are constants. To find a particular solution to the nonhomogeneous equation, the strategy of variation of parameters is to replace the constants in (2) by functions of t. That is, we seek a solution of (1) of the form[†]

(3) $$\boldsymbol{y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) .}$$

Because we have introduced two unknown functions, $v_1(t)$ and $v_2(t)$, it is reasonable to expect that we can impose two equations (requirements) on these functions. Naturally, one of these equations should come from (1). Let's therefore plug $y_p(t)$ given by (3) into (1). To accomplish this, we must first compute $y_p'(t)$ and $y_p''(t)$. From (3) we obtain

$$y_p' = (v_1'y_1 + v_2'y_2) + (v_1y_1' + v_2y_2') .$$

To simplify the computation and to avoid second-order derivatives for the unknowns v_1, v_2 in the expression for y_p'', we impose the requirement

(4) $$v_1'y_1 + v_2'y_2 = 0 .$$

Thus, the formula for y_p' becomes

(5) $$y_p' = v_1y_1' + v_2y_2' ,$$

and so

(6) $$y_p'' = v_1'y_1' + v_1y''{}_1 + v_2'y_2' + v_2y_2'' .$$

Now, substituting y_p, y_p', and y_p'', as given in (3), (5), and (6), into (1), we find

(7) $$\begin{aligned} f &= ay_p'' + by_p' + cy_p \\ &= a(v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'') + b(v_1y_1' + v_2y_2') + c(v_1y_1 + v_2y_2) \\ &= a(v_1'y_1' + v_2'y_2') + v_1(ay_1'' + by_1' + cy_1) + v_2(ay_2'' + by_2' + cy_2) \\ &= a(v_1'y_1' + v_2'y_2') + 0 + 0 \end{aligned}$$

since y_1 and y_2 are solutions to the homogeneous equation. Thus, (7) reduces to

(8) $$v_1'y_1' + v_2'y_2' = \frac{f}{a} .$$

To summarize, if we can find v_1 and v_2 that satisfy both (4) and (8), that is,

(9) $$\begin{aligned} &\boldsymbol{y_1v_1' + y_2v_2' = 0 ,} \\ &\boldsymbol{y_1'v_1' + y_2'v_2' = \frac{f}{a} ,} \end{aligned}$$

then y_p given by (3) will be a particular solution to (1). To determine v_1 and v_2, we first solve the linear system (9) for v_1' and v_2'. Algebraic manipulation or Cramer's rule (see Appendix D) immediately gives

$$v_1'(t) = \frac{-f(t)y_2(t)}{a\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]} \quad \text{and} \quad v_2'(t) = \frac{f(t)y_1(t)}{a\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]} ,$$

[†]In Exercises 2.3, Problem 36, we developed this approach for first-order linear equations. Because of the similarity of equations (2) and (3), this technique is sometimes known as "variation of constants."

where the bracketed expression in the denominator (the Wronskian) is never zero because of Lemma 1, Section 4.2. Upon integrating these equations, we finally obtain

$$v_1(t) = \int \frac{-f(t)y_2(t)}{a\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]}\,dt \quad \text{and} \quad v_2(t) = \int \frac{f(t)y_1(t)}{a\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]}\,dt\,. \tag{10}$$

Let's review this procedure.

Method of Variation of Parameters

To determine a particular solution to $ay'' + by' + cy = f$:

(a) Find two linearly independent solutions $\{y_1(t), y_2(t)\}$ to the corresponding homogeneous equation and take

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)\,.$$

(b) Determine $v_1(t)$ and $v_2(t)$ by solving the system in (9) for $v_1'(t)$ and $v_2'(t)$ and integrating.

(c) Substitute $v_1(t)$ and $v_2(t)$ into the expression for $y_p(t)$ to obtain a particular solution.

Of course, in step (b) one could use the formulas in (10), but $v_1(t)$ and $v_2(t)$ are so easy to derive that you are advised not to memorize them.

Example 1 Find a general solution on $(-\pi/2, \pi/2)$ to

$$\frac{d^2y}{dt^2} + y = \tan t\,. \tag{11}$$

Solution Observe that two independent solutions to the homogeneous equation $y'' + y = 0$ are $\cos t$ and $\sin t$. We now set

$$y_p(t) = v_1(t)\cos t + v_2(t)\sin t \tag{12}$$

and, referring to (9), solve the system

$$\begin{aligned} (\cos t)v_1'(t) + (\sin t)v_2'(t) &= 0\,, \\ (-\sin t)v_1'(t) + (\cos t)v_2'(t) &= \tan t\,, \end{aligned}$$

for $v_1'(t)$ and $v_2'(t)$. This gives

$$\begin{aligned} v_1'(t) &= -\tan t\ \sin t\,, \\ v_2'(t) &= \tan t\ \cos t = \sin t\,. \end{aligned}$$

Integrating, we obtain

$$\begin{aligned} v_1(t) &= -\int \tan t \sin t\,dt = -\int \frac{\sin^2 t}{\cos t}\,dt \\ &= -\int \frac{1-\cos^2 t}{\cos t}\,dt = \int (\cos t - \sec t)\,dt \\ &= \sin t - \ln|\sec t + \tan t| + C_1\,, \end{aligned} \tag{13}$$

$$v_2(t) = \int \sin t\,dt = -\cos t + C_2\,. \tag{14}$$

We need only one particular solution, so we take both C_1 and C_2 to be zero for simplicity. Then, substituting $v_1(t)$ and $v_2(t)$ in (12), we obtain

$$y_p(t) = (\sin t - \ln|\sec t + \tan t|)\cos t - \cos t \sin t ,$$

which simplifies to

$$y_p(t) = -(\cos t) \ln|\sec t + \tan t| .$$

We may drop the absolute value symbols because $\sec t + \tan t = (1 + \sin t)/\cos t > 0$ for $-\pi/2 < t < \pi/2$.

Recall that a general solution to a nonhomogeneous equation is given by the sum of a general solution to the homogeneous equation and a particular solution. Consequently, a general solution to equation (11) on the interval $(-\pi/2, \pi/2)$ is

$$y(t) = c_1 \cos t + c_2 \sin t - (\cos t) \ln(\sec t + \tan t) . \quad (15)$$ ◆

Note that in the above example the constants C_1 and C_2 appearing in (13) and (14) were chosen to be zero. If we had retained these arbitrary constants, the ultimate effect would be just to add $C_1 \cos t + C_2 \sin t$ to (15), which is clearly redundant.

Example 2 Find a particular solution on $(-\pi/2, \pi/2)$ to

$$\frac{d^2y}{dt^2} + y = \tan t + 3t - 1 . \quad (16)$$

Solution With $f(t) = \tan t + 3t - 1$, the variation of parameters procedure will lead to a solution. But it is simpler in this case to consider separately the equations

$$\frac{d^2y}{dt^2} + y = \tan t , \quad (17)$$

$$\frac{d^2y}{dt^2} + y = 3t - 1 \quad (18)$$

and then use the superposition principle (Theorem 3, page 181).

In Example 1 we found that

$$y_q(t) = -(\cos t) \ln(\sec t + \tan t)$$

is a particular solution for equation (17). For equation (18) the method of undetermined coefficients can be applied. On seeking a solution to (18) of the form $y_r(t) = At + B$, we quickly obtain

$$y_r(t) = 3t - 1 .$$

Finally, we apply the superposition principle to get

$$\begin{aligned} y_p(t) &= y_q(t) + y_r(t) \\ &= -(\cos t) \ln(\sec t + \tan t) + 3t - 1 \end{aligned}$$

as a particular solution for equation (16). ◆

Note that we could not have solved Example 1 by the method of undetermined coefficients; the nonhomogeneity $\tan t$ is unsuitable. Another important advantage of the method of variation of parameters is its applicability to linear equations whose coefficients a, b, c are functions of t. Indeed, on reviewing the derivation of the system (9) and the formulas (10), one can check that we did not make any use of the constant coefficient property; i.e., the method works provided we know a pair of linearly independent solutions to the corresponding homogeneous equation. We illustrate the method in the next example.

Example 3 Find a particular solution of the variable coefficient linear equation

$$t^2y'' - 4ty' + 6y = 4t^3\,, \quad t > 0\,, \tag{19}$$

given that $y_1(t) = t^2$ and $y_2(t) = t^3$ are solutions to the corresponding homogeneous equation.

Solution The functions t^2 and t^3 are linearly independent solutions to the corresponding homogeneous equation on $(0, \infty)$ (verify this!) and so (19) has a particular solution of the form

$$y_p(t) = v_1(t)t^2 + v_2(t)t^3\,.$$

To determine the unknown functions v_1 and v_2, we solve the system (9) with $f(t) = 4t^3$ and $a = a(t) = t^2$:

$$t^2v_1'(t) + t^3v_2'(t) = 0$$
$$2tv_1'(t) + 3t^2v_2'(t) = f/a = 4t\,.$$

The solutions are readily found to be $v_1'(t) = -4$ and $v_2'(t) = 4/t$, which gives $v_1(t) = -4t$ and $v_2(t) = 4 \ln t$. Consequently,

$$y_p(t) = (-4t)t^2 + (4\ln t)t^3 = 4t^3(-1 + \ln t)$$

is a solution to (19). ◆

Variable coefficient linear equations will be discussed in more detail in the next section.

4.6 EXERCISES

In Problems 1–8, find a general solution to the differential equation using the method of variation of parameters.

1. $y'' + 4y = \tan 2t$
2. $y'' + y = \sec t$
3. $y'' - 2y' + y = t^{-1}e^t$
4. $y'' + 2y' + y = e^{-t}$
5. $y''(\theta) + 16y(\theta) = \sec 4\theta$
6. $y'' + 9y = \sec^2(3t)$
7. $y'' + 4y' + 4y = e^{-2t}\ln t$
8. $y'' + 4y = \csc^2(2t)$

In Problems 9 and 10, find a particular solution first by undetermined coefficients, and then by variation of parameters. Which method was quicker?

9. $y'' - y = 2t + 4$
10. $2x''(t) - 2x'(t) - 4x(t) = 2e^{2t}$

In Problems 11–18, find a general solution to the differential equation.

11. $y'' + y = \tan t + e^{3t} - 1$
12. $y'' + y = \tan^2 t$
13. $v'' + 4v = \sec^4(2t)$
14. $y''(\theta) + y(\theta) = \sec^3\theta$
15. $y'' + y = 3\sec t - t^2 + 1$
16. $y'' + 5y' + 6y = 18t^2$
17. $\frac{1}{2}y'' + 2y = \tan 2t - \frac{1}{2}e^t$
18. $y'' - 6y' + 9y = t^{-3}e^{3t}$
19. Express the solution to the initial value problem
$$y'' - y = \frac{1}{t}\,, \quad y(1) = 0\,, \quad y'(1) = -2\,,$$
using definite integrals. Using numerical integration (Appendix C) to approximate the integrals, find an approximation for $y(2)$ to two decimal places.
20. Use the method of variation of parameters to show that
$$y(t) = c_1\cos t + c_2\sin t + \int_0^t f(s)\sin(t - s)\,ds$$
is a general solution to the differential equation
$$y'' + y = f(t)\,,$$
where $f(t)$ is a continuous function on $(-\infty, \infty)$. [*Hint:* Use the trigonometric identity $\sin(t - s) = \sin t\cos s - \sin s\cos t$.]

21. Suppose y satisfies the equation $y'' + 10y' + 25y = e^{t^3}$ subject to $y(0) = 1$ and $y'(0) = -5$. Estimate $y(0.2)$ to within ± 0.0001 by numerically approximating the integrals in the variation of parameters formula.

In Problems 22 through 25, use variation of parameters to find a general solution to the differential equation given that the functions y_1 and y_2 are linearly independent solutions to the corresponding homogeneous equation for $t > 0$.

22. $t^2y'' - 4ty' + 6y = t^3 + 1$;
$y_1 = t^2$, $y_2 = t^3$

23. $ty'' - (t+1)y' + y = t^2$;
$y_1 = e^t$, $y_2 = t + 1$

24. $ty'' + (1-2t)y' + (t-1)y = te^t$;
$y_1 = e^t$, $y_2 = e^t \ln t$

25. $ty'' + (5t-1)y' - 5y = t^2e^{-5t}$;
$y_1 = 5t - 1$, $y_2 = e^{-5t}$

4.7 Variable-Coefficient Equations

The techniques of Sections 4.2 and 4.3 have explicitly demonstrated that solutions to a linear homogeneous constant-coefficient differential equation,

$$ay'' + by' + cy = 0\,, \tag{1}$$

are defined and satisfy the equation over the whole interval $(-\infty, +\infty)$. After all, such solutions are combinations of exponentials, sinusoids, and polynomials.

The variation of parameters formula of Section 4.6 extended this to nonhomogeneous constant-coefficient problems,

$$ay'' + by' + cy = f(t)\,, \tag{2}$$

yielding solutions valid *over all intervals where $f(t)$ is continuous* (ensuring that the integrals in (10) of Section 4.6 containing $f(t)$ exist and are differentiable). We could hardly hope for more; indeed, it is debatable what *meaning* the differential equation (2) would have at a point where $f(t)$ is undefined, or discontinuous.

Therefore, when we move to the realm of equations with *variable* coefficients of the form

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)\,, \tag{3}$$

the most we can expect is that there are solutions that are valid over intervals where all four "governing" functions—$a_2(t)$, $a_1(t)$, $a_0(t)$, and $f(t)$—are continuous. Fortunately, this expectation is fulfilled except for an important technical requirement—namely, that the coefficient function $a_2(t)$ must be nonzero over the interval.†

Typically, one divides by the nonzero coefficient $a_2(t)$ and expresses the theorem for the equation in **standard form** [see (4), below] as follows.

Existence and Uniqueness of Solutions

Theorem 5. If $p(t)$, $q(t)$, and $g(t)$ are continuous on an interval (a, b) that contains the point t_0, then for any choice of the initial values Y_0 and Y_1, there exists a unique solution $y(t)$ on the same interval (a, b) to the initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)\,; \quad y(t_0) = Y_0\,, \quad y'(t_0) = Y_1\,. \tag{4}$$

†Indeed, the whole nature of the equation—reduction from *second*-order to *first*-order—changes at points where $a_2(t)$ is zero.

Example 1 Determine the largest interval for which Theorem 5 ensures the existence and uniqueness of a solution to the initial value problem

$$(5)\qquad (t-3)\frac{d^2y}{dt^2}+\frac{dy}{dt}+\sqrt{t}\,y=\ln t\,;\quad y(1)=3\,,\quad y'(1)=-5\,.$$

Solution The data $p(t)$, $q(t)$, and $g(t)$ in the standard form of the equation,

$$y''+py'+qy=\frac{d^2y}{dt^2}+\frac{1}{(t-3)}\frac{dy}{dt}+\frac{\sqrt{t}}{(t-3)}y=\frac{\ln t}{(t-3)}=g\,,$$

are simultaneously continuous in the intervals $0<t<3$ and $3<t<\infty$. The former contains the point $t_0=1$, where the initial conditions are specified, so Theorem 5 guarantees (5) has a unique solution in $0<t<3$. ◆

Theorem 5, embracing existence and uniqueness for the variable-coefficient case, is difficult to prove because we can't construct explicit solutions in the general case. So the proof is deferred to Chapter 13.† However, it is instructive to examine a special case that we can solve explicitly.

Cauchy–Euler, or Equidimensional, Equations

Definition 2. A linear second-order equation that can be expressed in the form

$$(6)\qquad at^2y''(t)+bty'(t)+cy=f(t)\,,$$

where a, b, and c are constants, is called a **Cauchy–Euler,** or **equidimensional, equation.**

For example, the differential equation

$$3t^2y''+11ty'-3y=\sin t$$

is a Cauchy–Euler equation, whereas

$$2y''-3ty'+11y=3t-1$$

is *not* because the coefficient of y'' is 2, which is not a constant times t^2.

The nomenclature *equidimensional* comes about because if y has the dimensions of, say, meters and t has dimensions of time, then each term t^2y'', ty', and y has the same dimensions (meters). The coefficient of $y''(t)$ in (6) is at^2, and it is zero at $t=0$; equivalently, the standard form

$$y''+\frac{b}{at}y'+\frac{c}{at^2}y=\frac{f(t)}{at^2}$$

has discontinuous coefficients at $t=0$. Therefore, we can expect the solutions to be valid only for $t>0$ or $t<0$. Discontinuities in f, of course, will impose further restrictions.

†All references to Chapters 11–13 refer to the expanded text, *Fundamentals of Differential Equations and Boundary Value Problems*, 7th ed.

To solve a *homogeneous* Cauchy–Euler equation, for $t > 0$, we exploit the equidimensional feature by looking for solutions of the form $y = t^r$, because then t^2y'', ty', and y each have the form *(constant)* $\times\, t^r$:

$$y = t^r\,, \quad ty' = trt^{r-1} = rt^r\,, \quad t^2y'' = t^2r(r-1)t^{r-2} = r(r-1)t^r\,,$$

and substitution into the homogeneous form of (6) (that is, with $g = 0$) yields a simple quadratic equation for r:

$$ar(r-1)t^r + brt^r + ct^r = [ar^2 + (b-a)r + c]t^r = 0\,, \quad \text{or}$$

(7) $$\boldsymbol{ar^2 + (b-a)r + c = 0\,,}$$

which we call the associated *characteristic equation.*

Example 2 Find two linearly independent solutions to the equation

$$3t^2y'' + 11ty' - 3y = 0\,, \quad t > 0\,.$$

Solution Inserting $y = t^r$ yields, according to (7),

$$3r^2 + (11-3)r - 3 = 3r^2 + 8r - 3 = 0\,,$$

whose roots $r = 1/3$ and $r = -3$ produce the independent solutions

$$y_1(t) = t^{1/3}\,, \quad y_2(t) = t^{-3} \quad (\text{for } t > 0)\,. \ \blacklozenge$$

Clearly, the substitution $y = t^r$ into a homogeneous *equidimensional* equation has the same simplifying effect as the insertion of $y = e^{rt}$ into the homogeneous *constant-coefficient* equation in Section 4.2. That means we will have to deal with the same encumbrances:

1. What to do when the roots of (7) are complex
2. What to do when the roots of (7) are equal

If r is complex, $r = \alpha + i\beta$, we can interpret $t^{\alpha+i\beta}$ by using the identity $t = e^{\ln t}$ and invoking Euler's formula [equation (5), Section 4.3]:

$$t^{\alpha+i\beta} = t^\alpha t^{i\beta} = t^\alpha e^{i\beta\ln t} = t^\alpha[\cos(\beta\ln t) + i\sin(\beta\ln t)]\,.$$

Then we simplify as in Section 4.3 by taking the real and imaginary parts to form independent solutions:

(8) $$\boldsymbol{y_1 = t^\alpha\cos(\beta\ln t)\,, \quad y_2 = t^\alpha\sin(\beta\ln t)\,.}$$

If r is a double root of the characteristic equation (7), then independent solutions of the Cauchy–Euler equation on $(0, \infty)$ are given by

(9) $$\boldsymbol{y_1 = t^r\,, \quad y_2 = t^r\ln t\,.}$$

This can be verified by direct substitution into the differential equation. Alternatively, the second, linearly independent, solution can be obtained by *reduction of order*, a procedure to be discussed shortly in Theorem 8. Furthermore, Problem 23 demonstrates that the substitution $t = e^x$ changes the homogeneous Cauchy–Euler equation into a homogeneous constant-coefficient equation, and the formats (8) and (9) then follow from our earlier deliberations.

We remark that if a homogeneous Cauchy–Euler equation is to be solved for $t < 0$, then one simply introduces the change of variable $t = -\tau$, where $\tau > 0$. The reader should verify via the chain rule that the identical characteristic equation (7) arises when $\tau^r = (-t)^r$ is substituted in the equation. Thus the solutions take the same form as (8), (9), but with t replaced by $-t$; for example, if r is a double root of (7), we get $(-t)^r$ and $(-t)^r \ln(-t)$ as two linearly independent solutions on $(-\infty, 0)$.

Example 3 Find a pair of linearly independent solutions to the following Cauchy–Euler equations for $t > 0$.

(a) $t^2y'' + 5ty' + 5y = 0$ **(b)** $t^2y'' + ty' = 0$

Solution For part (a), the characteristic equation becomes $r^2 + 4r + 5 = 0$, with the roots $r = -2 \pm i$, and (8) produces the real solutions $t^{-2}\cos(\ln t)$ and $t^{-2}\sin(\ln t)$.

For part (b), the characteristic equation becomes simply $r^2 = 0$ with the double root $r = 0$, and (9) yields the solutions $t^0 = 1$ and $\ln t$. ◆

In Chapter 8 we will see how one can obtain power series expansions for solutions to variable-coefficient equations when the coefficients are *analytic* functions. But, as we said, there is no procedure for explicitly solving the general case. Nonetheless, thanks to the existence/uniqueness result of Theorem 5, most of the other theorems and concepts of the preceding sections are easily extended to the variable-coefficient case, with the proviso that they apply only over intervals in which the governing functions $p(t)$, $q(t)$, $g(t)$ are continuous. Thus we have the following analog of Lemma 1, page 160.

A Condition for Linear Dependence of Solutions

Lemma 3. If $y_1(t)$ and $y_2(t)$ are any two solutions to the homogeneous differential equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0 \tag{10}$$

on an interval *I* where the functions $p(t)$ and $q(t)$ are continuous and if the Wronskian†

$$W[y_1, y_2](t) := y_1(t)y_2'(t) - y_1'(t)y_2(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

is zero at *any* point t **of *I***, then y_1 and y_2 are linearly dependent **on *I***.

As in the constant-coefficient case, the Wronskian of two solutions is either identically zero or never zero on I, with the latter implying linear independence on I.

Precisely as in the proof for the constant-coefficient case, it can be verified that any linear combination $c_1y_1 + c_2y_2$ of solutions y_1 and y_2 to (10) is also a solution. In fact, these are the only solutions to (10) as stated in the following result.

Representation of Solutions to Initial Value Problems

Theorem 6. If $y_1(t)$ and $y_2(t)$ are any two solutions to the homogeneous differential equation (10) that are linearly independent on an interval I, then every solution to (10) on I is expressible as a linear combination of y_1 and y_2. Moreover, the initial value problem consisting of equation (10) and the initial conditions $y(t_0) = Y_0$, $y'(t_0) = Y_1$ has a unique solution on I for any point t_0 in I and any constants Y_0, Y_1.

†The determinant representation of the Wronskian was introduced in Problem 34, Section 4.2.

As in the constant-coefficient case, the linear combination $y_h = c_1y_1 + c_2y_2$ is called a **general solution** to (10) on I if y_1, y_2 are linearly independent solutions on I.

For the nonhomogeneous equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)\ , \tag{11}$$

a general solution on I is given by $y = y_p + y_h$, where $y_h = c_1y_1 + c_2y_2$ is a general solution to the corresponding homogeneous equation (10) **on I and y_p is a particular solution to** (11) **on I.** In other words, the solution to the initial value problem stated in Theorem 5 must be of this form for a suitable choice of the constants c_1, c_2. This follows, just as before, from a straightforward extension of the superposition principle for variable-coefficient equations described in Problem 30.

As illustrated at the end of the Section 4.6, if linearly independent solutions to the homogeneous equation (10) are known, then y_p can be determined for (11) by the variation of parameters method.

Variation of Parameters

Theorem 7. If y_1 and y_2 are two linearly independent solutions to the homogeneous equation (10) on an interval I where $p(t)$, $q(t)$, and $g(t)$ are continuous, then a particular solution to (11) is given by $y_p = v_1y_1 + v_2y_2$, where v_1 and v_2 are determined up to a constant by the pair of equations

$$\mathbf{y_1v_1' + y_2v_2' = 0\ ,}$$
$$\mathbf{y_1'v_1' + y_2'v_2' = g\ ,}$$

which have the solution

$$v_1(t) = \int \frac{-g(t)\,y_2(t)}{W[y_1, y_2](t)}\,dt\ , \quad v_2(t) = \int \frac{g(t)y_1(t)}{W[y_1, y_2](t)}\,dt\ . \tag{12}$$

Note the formulation (12) presumes that the differential equation has been put into standard form [that is, divided by $a_2(t)$].

The proofs of the constant-coefficient versions of these theorems in Sections 4.2 and 4.5 did not make use of the constant-coefficient property, so one can prove them in the general case by literally copying those proofs but interpreting the coefficients as variables. Unfortunately, however, there is no construction analogous to the method of undetermined coefficients for the variable-coefficient case.

What does all this mean? The only stumbling block for our completely solving nonhomogeneous initial value problems for equations with variable coefficients,

$$y'' + p(t)y' + q(t)y = g(t)\ ; \quad y(t_0) = Y_0\ , \quad y'(t_0) = Y_1\ ,$$

is the lack of an explicit procedure for constructing independent solutions to the associated homogeneous equation (10). If we *had* y_1 and y_2 as described in the variation of parameters formula, we could implement (12) to find y_p, formulate the general solution of (11) as $y_p + c_1y_1 + c_2y_2$, and (with the assurance that the Wronskian is nonzero) fit the constants to the initial conditions. But with the exception of the Cauchy–Euler equation and the ponderous power series machinery of Chapter 8, we are stymied at the outset; there is no general procedure for finding y_1 and y_2.

Ironically, we only need *one* nontrivial solution to the associated homogeneous equation, thanks to a procedure known as *reduction of order* that constructs a second, linearly independent solution y_2 from a known one y_1. So one might well feel that the following theorem rubs salt into the wound.

Reduction of Order

Theorem 8. If $y_1(t)$ is a solution, not identically zero, to the homogeneous differential equation (10) in an interval I (see page 195), then

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t)dt}}{y_1(t)^2}\, dt \tag{13}$$

is a second, linearly independent solution.

This remarkable formula can be confirmed directly, but the following derivation shows how the procedure got its name.

Proof of Theorem 8. Our strategy is similar to that used in the derivation of the variation of parameters formula, Section 4.6. Bearing in mind that cy_1 is a solution of (10) for any constant c, we replace c by a *function* $v(t)$ and propose the trial solution $y_2(t) = v(t)y_1(t)$, spawning the formulas

$$y_2' = vy_1' + v'y_1 \,, \quad y_2'' = vy_1'' + 2v'y_1' + v''y_1 \,.$$

Substituting these expressions into the differential equation (10) yields

$$(vy_1'' + 2v'y_1' + v''y_1) + p(vy_1' + v'y_1) + qvy_1 = 0 \,,$$

or, on regrouping,

$$(y_1'' + py_1' + qy_1)v + y_1v'' + (2y_1' + py_1)v' = 0 \,. \tag{14}$$

The group in front of the undifferentiated $v(t)$ is simply a copy of the left-hand member of the original differential equation (10), so it is zero.[†] Thus (14) reduces to

$$y_1v'' + (2y_1' + py_1)v' = 0 \,, \tag{15}$$

which is actually a *first*-order equation in the variable $w \equiv v'$:

$$y_1w' + (2y_1' + py_1)w = 0 \,. \tag{16}$$

Indeed, (16) is separable and can be solved immediately using the procedure of Section 2.2. Problem 48 on page 201 requests the reader to carry out the details of this procedure to complete the derivation of (13). ◆

Example 4 Given that $y_1(t) = t$ is a solution to

$$y'' - \frac{1}{t}y' + \frac{1}{t^2}y = 0 \,, \tag{17}$$

use the reduction of order procedure to determine a second linearly independent solution for $t > 0$.

[†]This is hardly a surprise; if v were constant, vy would be a solution with $v' = v'' = 0$ in (14).

Solution Rather than implementing the formula (13), let's apply the strategy used to derive it. We set $y_2(t) = v(t)y_1(t) = v(t)t$ and substitute $y_2' = v't + v$, $y_2'' = v''t + 2v'$ into (17) to find

$$v''t + 2v' - \frac{1}{t}(v't + v) + \frac{1}{t^2}vt = v''t + (2v' - v') = v''t + v' = 0 . \tag{18}$$

As promised, (18) is a *separable first*-order equation in v', simplifying to $(v')'/(v') = -1/t$ with a solution $v' = 1/t$, or $v = \ln t$ (taking integration constants to be zero). Therefore, a second solution to (17) is $y_2 = vt = t \ln t$.

Of course (17) is a Cauchy–Euler equation for which (7) has equal roots:

$$ar^2 + (b - a)r + c = r^2 - 2r + 1 = (r - 1)^2 = 0 ,$$

and y_2 is precisely the form for the independent solution predicted by (9). ◆

Example 5 The following equation arises in the mathematical modeling of reverse osmosis.†

$$(\sin t)y'' - 2(\cos t)y' - (\sin t)y = 0 , \quad 0 < t < \pi . \tag{19}$$

Find a general solution.

Solution As we indicated above, the tricky part is to find a single nontrivial solution. Inspection of (19) suggests that $y = \sin t$ or $y = \cos t$, combined with a little luck with trigonometric identities, might be solutions. In fact, trial and error shows that the cosine function works:

$$y_1 = \cos t , \quad y_1' = -\sin t , \quad y_1'' = -\cos t ,$$
$$(\sin t)y_1'' - 2(\cos t)y_1' - (\sin t)y_1 = (\sin t)(-\cos t) - 2(\cos t)(-\sin t) - (\sin t)(\cos t) = 0 .$$

Unfortunately, the sine function fails (try it).

So we use reduction of order to construct a second, independent solution. Setting $y_2(t) = v(t)y_1(t) = v(t)\cos t$ and computing $y_2' = v'\cos t - v\sin t$, $y_2'' = v''\cos t - 2v'\sin t - v\cos t$, we substitute into (19) to derive

$$\begin{aligned}(\sin t)[v''\cos t - 2v'\sin t - v\cos t] - 2(\cos t)[v'\cos t - v\sin t] - (\sin t)[v\cos t] \\ = v''(\sin t)(\cos t) - 2v'(\sin^2 t + \cos^2 t) = 0,\end{aligned}$$

which is equivalent to the separated first-order equation

$$\frac{(v')'}{(v')} = \frac{2}{(\sin t)(\cos t)} = 2\frac{\sec^2 t}{\tan t} .$$

Taking integration constants to be zero yields $\ln v' = 2\ln(\tan t)$ or $v' = \tan^2 t$, and $v = \tan t - t$. Therefore, a second solution to (19) is $y_2 = (\tan t - t)\cos t = \sin t - t\cos t$. We conclude that a general solution is $c_1 \cos t + c_2 (\sin t - t \cos t)$. ◆

In this section we have seen that the *theory* for variable-coefficient equations differs only slightly from the constant-coefficient case (in that solution domains are restricted to intervals), but explicit solutions can be hard to come by. In the next section, we will supplement our exposition by describing some nonrigorous procedures that sometimes can be used to predict qualitative features of the solutions.

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†Reverse osmosis is a process used to fortify the alcoholic content of wine, among other applications.

4.7 EXERCISES

In Problems 1 through 4, use Theorem 5 to discuss the existence and uniqueness of a solution to the differential equation that satisfies the initial conditions $y(1) = Y_0$, $y'(1) = Y_1$, where Y_0 and Y_1 are real constants.

1. $(1 + t^2)y'' + ty' - y = \tan t$

2. $t(t-3)y'' + 2ty' - y = t^2$

3. $t^2y'' + y = \cos t$

4. $e^t y'' - \dfrac{y'}{t-3} + y = \ln t$

In Problems 5 through 8, determine whether Theorem 5 applies. If it does, then discuss what conclusions can be drawn. If it does not, explain why.

5. $t^2z'' + tz' + z = \cos t\,; \quad z(0) = 1\,, \quad z'(0) = 0$

6. $y'' + yy' = t^2 - 1\,; \quad y(0) = 1\,, \quad y'(0) = -1$

7. $y'' + ty' - t^2y = 0\,; \quad y(0) = 0\,, \quad y(1) = 0$

8. $(1-t)y'' + ty' - 2y = \sin t\,; \quad y(0) = 1\,, \quad y'(0) = 1$

In Problems 9 through 14, find a general solution to the given Cauchy–Euler equation for $t > 0$.

9. $t^2y''(t) + 7ty'(t) - 7y(t) = 0$

10. $t^2\dfrac{d^2y}{dt^2} + 2t\dfrac{dy}{dt} - 6y = 0$

11. $t^2\dfrac{d^2z}{dt^2} + 5t\dfrac{dz}{dt} + 4z = 0$

12. $\dfrac{d^2w}{dt^2} + \dfrac{6}{t}\dfrac{dw}{dt} + \dfrac{4}{t^2}w = 0$

13. $9t^2y''(t) + 15ty'(t) + y(t) = 0$

14. $t^2y''(t) - 3ty'(t) + 4y(t) = 0$

In Problems 15 through 18, find a general solution for $t < 0$.

15. $y''(t) - \dfrac{1}{t}y'(t) + \dfrac{5}{t^2}y(t) = 0$

16. $t^2y''(t) - 3ty'(t) + 6y(t) = 0$

17. $t^2y''(t) + 9ty'(t) + 17y(t) = 0$

18. $t^2y''(t) + 3ty'(t) + 5y(t) = 0$

In Problems 19 and 20, solve the given initial value problem for the Cauchy–Euler equation.

19. $t^2y''(t) - 4ty'(t) + 4y(t) = 0\,;$
$y(1) = -2\,, \quad y'(1) = -11$

20. $t^2y''(t) + 7ty'(t) + 5y(t) = 0\,;$
$y(1) = -1\,, \quad y'(1) = 13$

In Problems 21 and 22, devise a modification of the method for Cauchy–Euler equations to find a general solution to the given equation.

21. $(t-2)^2y''(t) - 7(t-2)y'(t) + 7y(t) = 0\,, \quad t > 2$

22. $(t+1)^2y''(t) + 10(t+1)y'(t) + 14y(t) = 0\,,$
$t > -1$

23. To justify the solution formulas (8) and (9), perform the following analysis.

(a) Show that if the substitution $t = e^x$ is made in the function $y(t)$ and x is regarded as the new independent variable in $Y(x) := y(e^x)$, the chain rule implies the following relationships:

$$t\frac{dy}{dt} = \frac{dY}{dx}\,, \qquad t^2\frac{d^2y}{dt^2} = \frac{d^2Y}{dx^2} - \frac{dY}{dx}\,.$$

(b) Using part (a), show that if the substitution $t = e^x$ is made in the Cauchy–Euler differential equation (6), the result is a constant-coefficient equation for $Y(x) = y(e^x)$, namely,

$$a\frac{d^2Y}{dx^2} + (b-a)\frac{dY}{dx} + cY = f(e^x)\,. \tag{20}$$

(c) Observe that the auxiliary equation (recall Section 4.2) for the homogeneous form of (20) is the same as (7) in this section. If the roots of the former are complex, linearly independent solutions of (20) have the form $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$; if they are equal, linearly independent solutions of (20) have the form e^{rx} and xe^{rx}. Express x in terms of t to derive the corresponding solution forms (8) and (9).

24. Solve the following Cauchy–Euler equations by using the substitution described in Problem 23 to change them to constant coefficient equations, finding their general solutions by the methods of the preceding sections, and restoring the original independent variable t.

(a) $t^2y'' + ty' - 9y = 0$

(b) $t^2y'' + 3ty' + 10y = 0$

(c) $t^2y'' + 3ty' + y = t + t^{-1}$

(d) $t^2y'' + ty' + 9y = -\tan(3\ln t)$

25. Let y_1 and y_2 be two functions defined on $(-\infty, \infty)$.

(a) True or False: If y_1 and y_2 are linearly dependent on the interval $[a, b]$, then y_1 and y_2 are linearly dependent on the smaller interval $[c, d] \subset [a, b]$.

(b) True or False: If y_1 and y_2 are linearly dependent on the interval $[a, b]$, then y_1 and y_2 are linearly dependent on the larger interval $[C, D] \supset [a, b]$.

26. Let $y_1(t) = t^3$ and $y_2(t) = |t^3|$. Are y_1 and y_2 linearly independent on the following intervals?

(a) $[0, \infty)$ **(b)** $(-\infty, 0]$ **(c)** $(-\infty, \infty)$

(d) Compute the Wronskian $W[y_1, y_2](t)$ on the interval $(-\infty, \infty)$.

27. Consider the linear equation

$$t^2y'' - 3ty' + 3y = 0\,, \quad \text{for } -\infty < t < \infty\,. \tag{21}$$

(a) Verify that $y_1(t) := t$ and $y_2(t) := t^3$ are two solutions to (21) on $(-\infty, \infty)$. Furthermore, show that $y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$ for $t_0 = 1$.

(b) Prove that $y_1(t)$ and $y_2(t)$ are linearly independent on $(-\infty, \infty)$.

(c) Verify that the function $y_3(t) := |t|^3$ is also a solution to (21) on $(-\infty, \infty)$.

(d) Prove that there is *no* choice of constants c_1, c_2 such that $y_3(t) = c_1y_1(t) + c_2y_2(t)$ for *all* t in $(-\infty, \infty)$. [*Hint:* Argue that the contrary assumption leads to a contradiction.]

(e) From parts (c) and (d), we see that there is at least one solution to (21) on $(-\infty, \infty)$ that is not expressible as a linear combination of the solutions $y_1(t)$, $y_2(t)$. Does this provide a counterexample to the theory in this section? Explain.

28. Let $y_1(t) = t^2$ and $y_2(t) = 2t|t|$. Are y_1 and y_2 linearly independent on the interval:

(a) $[0, \infty)$? **(b)** $(-\infty, 0]$? **(c)** $(-\infty, \infty)$?

(d) Compute the Wronskian $W[y_1, y_2](t)$ on the interval $(-\infty, \infty)$.

29. Prove that if y_1 and y_2 are linearly independent solutions of $y'' + py' + qy = 0$ on (a, b), then they cannot both be zero at the same point t_0 in (a, b).

30. Superposition Principle. Let y_1 be a solution to

$$y''(t) + p(t)y'(t) + q(t)y(t) = g_1(t)$$

on the interval I and let y_2 be a solution to

$$y''(t) + p(t)y'(t) + q(t)y(t) = g_2(t)$$

on the same interval. Show that for any constants k_1 and k_2, the function $k_1y_1 + k_2y_2$ is a solution on I to

$$y''(t) + p(t)y'(t) + q(t)y(t) = k_1g_1(t) + k_2g_2(t) .$$

31. Determine whether the following functions can be Wronskians on $-1 < t < 1$ for a pair of solutions to some equation $y'' + py' + qy = 0$ (with p and q continuous).

(a) $w(t) = 6e^{4t}$ **(b)** $w(t) = t^3$

(c) $w(t) = (t + 1)^{-1}$ **(d)** $w(t) \equiv 0$

32. By completing the following steps, prove that the Wronskian of any two solutions y_1, y_2 to the equation $y'' + py' + qy = 0$ on (a, b) is given by **Abel's formula**†

$$W[y_1, y_2](t) = C \exp\left\{-\int_{t_0}^{t} p(\tau)\, d\tau\right\} ,$$
t_0 and t in (a, b) ,

where the constant C depends on y_1 and y_2.

(a) Show that the Wronskian W satisfies the equation $W' + pW = 0$.

(b) Solve the separable equation in part (a).

(c) How does Abel's formula clarify the fact that the Wronskian is either identically zero or never zero on (a, b)?

†*Historical Footnote:* Niels Abel derived this identity in 1827.

33. Use Abel's formula (Problem 32) to determine (up to a constant multiple) the Wronskian of two solutions on $(0, \infty)$ to

$$ty'' + (t - 1)y' + 3y = 0 .$$

34. All that is known concerning a mysterious differential equation $y'' + p(t)y' + q(t)y = g(t)$ is that the functions t, t^2, and t^3 are solutions.

(a) Determine two linearly independent solutions to the corresponding homogeneous differential equation.

(b) Find the solution to the original equation satisfying the initial conditions $y(2) = 2$, $y'(2) = 5$.

(c) What is $p(t)$? [*Hint:* Use Abel's formula for the Wronskian, Problem 32.]

35. Given that $1 + t$, $1 + 2t$, and $1 + 3t^2$ are solutions to the differential equation $y'' + p(t)y' + q(t)y = g(t)$, find the solution to this equation that satisfies $y(1) = 2$, $y'(1) = 0$.

36. Verify that the given functions y_1 and y_2 are linearly independent solutions of the following differential equation and find the solution that satisfies the given initial conditions.

$$ty'' - (t + 2)y' + 2y = 0 ;$$
$$y_1(t) = e^t , \quad y_2(t) = t^2 + 2t + 2 ;$$
$$y(1) = 0 , \quad y'(1) = 1$$

In Problems 37 through 39, find general solutions to the nonhomogeneous Cauchy–Euler equations using variation of parameters.

37. $t^2z'' + tz' + 9z = -\tan(3 \ln t)$

38. $t^2y'' + 3ty' + y = t^{-1}$

39. $t^2z'' - tz' + z = t\left(1 + \dfrac{3}{\ln t}\right)$

40. The **Bessel equation** of order one-half

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = 0 , \quad t > 0$$

has two linearly independent solutions,

$$y_1(t) = t^{-1/2} \cos t, \quad y_2(t) = t^{-1/2} \sin t .$$

Find a general solution to the nonhomogeneous equation

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = t^{5/2}, \quad t > 0 .$$

In Problems 41 through 44, a differential equation and a nontrivial solution f are given. Find a second linearly independent solution using reduction of order.

41. $t^2y'' - 2ty' - 4y = 0 , \quad t > 0 ; \quad f(t) = t^{-1}$

42. $t^2y'' + 6ty' + 6y = 0 , \quad t > 0 ; \quad f(t) = t^{-2}$

43. $tx'' - (t + 1)x' + x = 0 , \quad t > 0 ; \quad f(t) = e^t$

44. $ty'' + (1 - 2t)y' + (t - 1)y = 0 , \quad t > 0 ; \quad f(t) = e^t$

45. Find a particular solution to the nonhomogeneous equation

$$ty'' - (t+1)y' + y = t^2e^{2t},$$

given that $f(t) = e^t$ is a solution to the corresponding homogeneous equation.

46. Find a particular solution to the nonhomogeneous equation

$$(1-t)y'' + ty' - y = (1-t)^2,$$

given that $f(t) = t$ is a solution to the corresponding homogeneous equation.

47. In quantum mechanics, the study of the Schrödinger equation for the case of a harmonic oscillator leads to a consideration of **Hermite's equation,**

$$y'' - 2ty' + \lambda y = 0,$$

where λ is a parameter. Use the reduction of order formula to obtain an integral representation of a second linearly independent solution to Hermite's equation for the given value of λ and corresponding solution $f(t)$.

(a) $\lambda = 4, \quad f(t) = 1 - 2t^2$

(b) $\lambda = 6, \quad f(t) = 3t - 2t^3$

48. Complete the proof of Theorem 8 by solving equation (16).

49. The reduction of order procedure can be used more generally to reduce a homogeneous linear nth-order equation to a homogeneous linear $(n-1)$th-order equation. For the equation

$$ty''' - ty'' + y' - y = 0,$$

which has $f(t) = e^t$ as a solution, use the substitution $y(t) = v(t)f(t)$ to reduce this third-order equation to a homogeneous linear second-order equation in the variable $w = v'$.

50. The equation

$$ty''' + (1-t)y'' + ty' - y = 0$$

has $f(t) = t$ as a solution. Use the substitution $y(t) = v(t)f(t)$ to reduce this third-order equation to a homogeneous linear second-order equation in the variable $w = v'$.

51. **Isolated Zeros.** Let $\phi(t)$ be a solution to $y'' + py' + qy = 0$ on (a, b), where p, q are continuous on (a, b). By completing the following steps, prove that if ϕ is not identically zero, then its zeros in (a, b) are *isolated*, i.e., if $\phi(t_0) = 0$, then there exists a $\delta > 0$ such that $\phi(t) \neq 0$ for $0 < |t - t_0| < \delta$.

(a) Suppose $\phi(t_0) = 0$ and assume to the contrary that for each $n = 1, 2, \ldots$, the function ϕ has a zero at t_n, where $0 < |t_0 - t_n| < 1/n$. Show that this implies $\phi'(t_0) = 0$. [*Hint:* Consider the difference quotient for ϕ at t_0.]

(b) With the assumptions of part (a), we have $\phi(t_0) = \phi'(t_0) = 0$. Conclude from this that ϕ must be identically zero, which is a contradiction. Hence, there is some integer n_0 such that $\phi(t)$ is not zero for $0 < |t - t_0| < 1/n_0$.

52. The reduction of order formula (13) can also be derived from Abels' identity (Problem 32). Let $f(t)$ be a nontrivial solution to (10) and $y(t)$ a second linearly independent solution. Show that

$$\left(\frac{y}{f}\right)' = \frac{W[f, y]}{f^2}$$

and then use Abel's identity for the Wronskian $W[f, y]$ to obtain the reduction of order formula.

4.8 Qualitative Considerations for Variable-Coefficient and Nonlinear Equations

There are no techniques for obtaining explicit, closed-form solutions to second-order linear differential equations with variable coefficients (with certain exceptions) or for nonlinear equations. In general, we will have to settle for numerical solutions or power series expansions. So it would be helpful to be able to derive, with simple calculations, some nonrigorous, qualitative conclusions about the behavior of the solutions before we launch the heavy computational machinery. In this section we first display a few examples that illustrate the profound differences that can occur when the equations have variable coefficients or are nonlinear. Then we show how the mass–spring analogy, discussed in Section 4.1, can be exploited to predict some of the attributes of solutions of these more complicated equations.

To begin our discussion we display a linear constant-coefficient, a linear variable-coefficient, and two nonlinear equations.

or

$$c_1 = \frac{y_0 - 2x_0}{2}, \quad c_2 = \frac{y_0 + 2x_0}{2}.$$

Thus, the mass of salt in tanks A and B at time t are, respectively,

$$\begin{aligned} x(t) &= -\left(\frac{y_0 - 2x_0}{4}\right)e^{-t/2} + \left(\frac{y_0 + 2x_0}{4}\right)e^{-t/6}, \\ y(t) &= \left(\frac{y_0 - 2x_0}{2}\right)e^{-t/2} + \left(\frac{y_0 + 2x_0}{2}\right)e^{-t/6}. \end{aligned} \tag{5}$$

The ad hoc elimination procedure that we used to solve this example will be generalized and formalized in the next section, to find solutions of all *linear systems with constant coefficients*. Furthermore, in later sections we will show how to extend our numerical algorithms for first-order equations to *general* systems and will consider applications to coupled oscillators and electrical systems.

It is interesting to note from (5) that all solutions of the interconnected-tanks problem tend to the constant solution $x(t) \equiv 0, y(t) \equiv 0$ as $t \to +\infty$. (This is of course consistent with our physical expectations.) This constant solution will be identified as a *stable equilibrium solution* in Section 5.4, in which we introduce phase plane analysis. It turns out that, for a general class of systems, equilibria can be identified and classified so as to give qualitative information about the other solutions even when we cannot solve the system explicitly.

5.2 Differential Operators and the Elimination Method* for Systems

The notation $y'(t) = \dfrac{dy}{dt} = \dfrac{d}{dt}y$ was devised to suggest that the derivative of a function y is the result of *operating* on the function y with the differentiation operator $\dfrac{d}{dt}$. Indeed, second derivatives are formed by iterating the operation: $y''(t) = \dfrac{d^2y}{dt^2} = \dfrac{d}{dt}\dfrac{d}{dt}y$. Commonly, the symbol D is used instead of $\dfrac{d}{dt}$, and the second-order differential equation

$$y'' + 4y' + 3y = 0$$

is represented[†] by

$$D^2y + 4Dy + 3y = (D^2 + 4D + 3)[y] = 0.$$

So, we have implicitly adopted the convention that the operator "product," D times D, is interpreted as the *composition* of D with itself when it operates on functions: D^2y means $D(D[y])$; i.e., the second derivative. Similarly, the product $(D + 3)(D + 1)$ operates on a function via

$$\begin{aligned} (D+3)(D+1)[y] &= (D+3)[(D+1)[y]] = (D+3)[y' + y] \\ &= D[y' + y] + 3[y' + y] \\ &= (y'' + y') + (3y' + 3y) = y'' + 4y' + 3y = (D^2 + 4D + 3)[y]. \end{aligned}$$

*An alternative procedure to the methodology of this section will be described in Chapter 9. Although it involves the machinery of matrix analysis, it is preferable for large systems.

[†]Some authors utilize the identity operator I, defined by $I[y] = y$, and write more formally $D^2 + 4D + 3I$ instead of $D^2 + 4D + 3$.

Thus, $(D+3)(D+1)$ is the same operator as D^2+4D+3; when they are applied to twice-differentiable functions, the results are identical.

Example 1 Show that the operator $(D+1)(D+3)$ is also the same as D^2+4D+3.

Solution For any twice-differentiable function $y(t)$, we have

$$\begin{aligned}(D+1)(D+3)[y] &= (D+1)[(D+3)[y]] = (D+1)[y'+3y]\\ &= D[y'+3y]+1[y'+3y] = (y''+3y')+(y'+3y)\\ &= y''+4y'+3y = (D^2+4D+3)[y]\,.\end{aligned}$$

Hence, $(D+1)(D+3) = D^2+4D+3$. ◆

Since $(D+1)(D+3) = (D+3)(D+1) = D^2+4D+3$, it is tempting to generalize and propose that one can treat expressions like aD^2+bD+c as if they were ordinary polynomials in D. This is true, as long as we restrict the coefficients a, b, c to be *constants*. The following example, which has *variable* coefficients, is instructive.

Example 2 Show that $(D+3t)D$ is *not* the same as $D(D+3t)$.

Solution With $y(t)$ as before,

$$\begin{aligned}(D+3t)D[y] &= (D+3t)[y'] = y''+3ty'\,;\\ D(D+3t)[y] &= D[y'+3ty] = y''+3y+3ty'\,.\end{aligned}$$

They are not the same! ◆

Because the coefficient $3t$ is not a constant, it "interrupts" the interaction of the differentiation operator D with the function $y(t)$. As long as we only deal with expressions like aD^2+bD+c with *constant* coefficients a, b, and c, the "algebra" of differential operators follows the same rules as the algebra of polynomials. (See Problem 39 for elaboration on this point.)

This means that the familiar elimination method, used for solving *algebraic* systems like

$$\begin{aligned}3x-2y+z &= 4\,,\\ x+y-z &= 0\,,\\ 2x-y+3z &= 6\,,\end{aligned}$$

can be adapted to solve any system of *linear differential equations with constant coefficients.* In fact, we used this approach in solving the system that arose in the interconnected tanks problem of Section 5.1. Our goal in this section is to formalize this **elimination method** so that we can tackle more general linear constant coefficient systems.

We first demonstrate how the method applies to a linear system of two first-order differential equations of the form

$$\begin{aligned}a_1x'(t)+a_2x(t)+a_3y'(t)+a_4y(t) &= f_1(t)\,,\\ a_5x'(t)+a_6x(t)+a_7y'(t)+a_8y(t) &= f_2(t)\,,\end{aligned}$$

where $a_1, a_2, \ldots, a_8$ are constants and $x(t)$, $y(t)$ is the function pair to be determined. In operator notation this becomes

$$\begin{aligned}(a_1D+a_2)[x]+(a_3D+a_4)[y] &= f_1\,,\\ (a_5D+a_6)[x]+(a_7D+a_8)[y] &= f_2\,.\end{aligned}$$

Example 3 Solve the system

$$\text{(1)}\qquad \begin{aligned} x'(t) &= 3x(t) - 4y(t) + 1\,, \\ y'(t) &= 4x(t) - 7y(t) + 10t\,. \end{aligned}$$

Solution The alert reader may observe that since y' is absent from the first equation, we could use the latter to express y in terms of x and x' and substitute into the second equation to derive an "uncoupled" equation containing only x and its derivatives. However, this simple trick will not work on more general systems (Problem 18 is an example).

To utilize the elimination method, we first write the system using the operator notation:

$$\text{(2)}\qquad \begin{aligned} (D-3)[x] + 4y &= 1\,, \\ -4x + (D+7)[y] &= 10t\,. \end{aligned}$$

Imitating the elimination procedure for algebraic systems, we can eliminate x from this system by adding 4 times the first equation to $(D-3)$ applied to the second equation. This gives

$$\big(16 + (D-3)(D+7)\big)[y] = 4 \cdot 1 + (D-3)[10t] = 4 + 10 - 30t\,,$$

which simplifies to

$$\text{(3)}\qquad (D^2 + 4D - 5)[y] = 14 - 30t\,.$$

Now equation (3) is just a second-order linear equation in y with constant coefficients that has the general solution

$$\text{(4)}\qquad y(t) = C_1e^{-5t} + C_2e^{t} + 6t + 2\,,$$

which can be found using undetermined coefficients.

To find $x(t)$, we have two options.

Method 1. We return to system (2) and eliminate y. This is accomplished by "multiplying" the first equation in (2) by $(D+7)$ and the second equation by -4 and then adding to obtain

$$(D^2 + 4D - 5)[x] = 7 - 40t\,.$$

This equation can likewise be solved using undetermined coefficients to yield

$$\text{(5)}\qquad x(t) = K_1e^{-5t} + K_2e^{t} + 8t + 5\,,$$

where we have taken K_1 and K_2 to be the arbitrary constants, which are not necessarily the same as C_1 and C_2 used in formula (4).

It is reasonable to expect that system (1) will involve only *two* arbitrary constants, since it consists of two first-order equations. Thus, the four constants C_1, C_2, K_1, and K_2 are not independent. To determine the relationships, we substitute the expressions for $x(t)$ and $y(t)$ given in (4) and (5) into one of the equations in (1)—say, the first one. This yields

$$\begin{aligned} -5K_1e^{-5t} + K_2e^{t} + 8 = \\ 3K_1e^{-5t} + 3K_2e^{t} + 24t + 15 - 4C_1e^{-5t} - 4C_2e^{t} - 24t - 8 + 1\,, \end{aligned}$$

which simplifies to

$$(4C_1 - 8K_1)e^{-5t} + (4C_2 - 2K_2)e^{t} = 0\,.$$

Because e^t and e^{-5t} are linearly independent functions on any interval, this last equation holds for all t only if

$$4C_1 - 8K_1 = 0 \qquad \text{and} \qquad 4C_2 - 2K_2 = 0\,.$$

Therefore, $K_1 = C_1/2$ and $K_2 = 2C_2$.

A solution to system (1) is then given by the pair

$$x(t) = \frac{1}{2}C_1e^{-5t} + 2C_2e^t + 8t + 5\,, \quad y(t) = C_1e^{-5t} + C_2e^t + 6t + 2\,. \tag{6}$$

As you might expect, this pair is a **general solution** to (1) in the sense that *any* solution to (1) can be expressed in this fashion.

Method 2. A simpler method for determining $x(t)$ once $y(t)$ is known is to use the system to obtain an equation for $x(t)$ in terms of $y(t)$ and $y'(t)$. In this example we can directly solve the second equation in (1) for $x(t)$:

$$x(t) = \frac{1}{4}y'(t) + \frac{7}{4}y(t) - \frac{5}{2}t\,.$$

Substituting $y(t)$ as given in (4) yields

$$\begin{aligned} x(t) &= \frac{1}{4}[-5C_1e^{-5t} + C_2e^t + 6] + \frac{7}{4}[C_1e^{-5t} + C_2e^t + 6t + 2] - \frac{5}{2}t \\ &= \frac{1}{2}C_1e^{-5t} + 2C_2e^t + 8t + 5\,, \end{aligned}$$

which agrees with (6). ◆

The above procedure works, more generally, for any linear system of two equations and two unknowns with *constant coefficients* regardless of the order of the equations. For example, if we let L_1, L_2, L_3, and L_4 denote linear differential operators with constant coefficients (i.e., polynomials in D), then the method can be applied to the linear system

$$\begin{aligned} L_1[x] + L_2[y] &= f_1\,, \\ L_3[x] + L_4[y] &= f_2\,. \end{aligned}$$

Because the system has constant coefficients, the operators commute (e.g., $L_2L_4 = L_4L_2$) and we can eliminate variables in the usual algebraic fashion. Eliminating the variable y gives

$$(L_1L_4 - L_2L_3)[x] = g_1\,, \tag{7}$$

where $g_1 := L_4[f_1] - L_2[f_2]$. Similarly, eliminating the variable x yields

$$(L_1L_4 - L_2L_3)[y] = g_2\,, \tag{8}$$

where $g_2 := L_1[f_2] - L_3[f_1]$. Now if $L_1L_4 - L_2L_3$ is a differential operator of order n, then a general solution for (7) contains n arbitrary constants, and a general solution for (8) also contains n arbitrary constants. Thus, a total of $2n$ constants arise. However, as we saw in Example 3, there are only n of these that are independent for the system; the remaining constants can be expressed in terms of these.[†] The pair of general solutions to (7) and (8) written in terms of the n independent constants is called a **general solution for the system.**

[†]For a proof of this fact, see *Ordinary Differential Equations*, by M. Tenenbaum and H. Pollard (Dover, New York, 1985), Chapter 7.

If it turns out that $L_1L_4 - L_2L_3$ is the zero operator, the system is said to be **degenerate.** As with the anomalous problem of solving for the points of intersection of two parallel or coincident lines, a degenerate system may have no solutions, or if it does possess solutions, they may involve any number of arbitrary constants (see Problems 23 and 24).

Elimination Procedure for 2 × 2 Systems

To find a general solution for the system

$$\begin{aligned} L_1[x] + L_2[y] &= f_1 \,, \\ L_3[x] + L_4[y] &= f_2 \,, \end{aligned}$$

where L_1, L_2, L_3, and L_4 are polynomials in $D = d/dt$:

(a) Make sure that the system is written in operator form.
(b) Eliminate one of the variables, say, y, and solve the resulting equation for $x(t)$. If the system is degenerate, stop! A separate analysis is required to determine whether or not there are solutions.
(c) (*Shortcut*) If possible, use the system to derive an equation that involves $y(t)$ but not its derivatives. [Otherwise, go to step (d).] Substitute the found expression for $x(t)$ into this equation to get a formula for $y(t)$. The expressions for $x(t)$, $y(t)$ give the desired general solution. ◆
(d) Eliminate x from the system and solve for $y(t)$. [Solving for $y(t)$ gives more constants—in fact, twice as many as needed.]
(e) Remove the extra constants by substituting the expressions for $x(t)$ and $y(t)$ into one or both of the equations in the system. Write the expressions for $x(t)$ and $y(t)$ in terms of the remaining constants. ◆

Example 4 Find a general solution for

$$\begin{aligned} x''(t) + y'(t) - x(t) + y(t) &= -1 \,, \\ x'(t) + y'(t) - x(t) &= t^2 \,. \end{aligned} \tag{9}$$

Solution We begin by expressing the system in operator notation:

$$\begin{aligned} (D^2 - 1)[x] + (D+1)[y] &= -1 \,, \\ (D-1)[x] + D[y] &= t^2 \,. \end{aligned} \tag{10}$$

Here $L_1 := D^2 - 1$, $L_2 := D + 1$, $L_3 := D - 1$, and $L_4 := D$.

Eliminating y gives [see (7)]:

$$\big((D^2-1)D - (D+1)(D-1)\big)[x] = D[-1] - (D+1)[t^2] \,,$$

which reduces to

$$(D^2-1)(D-1)[x] = -2t - t^2 \,, \quad \text{or}$$

$$(D-1)^2(D+1)[x] = -2t - t^2 \,. \tag{11}$$

Since $(D-1)^2(D+1)$ is third order, we should expect three arbitrary constants in a general solution to system (9).

Although the methods of Chapter 4 focused on solving second-order equations, we have seen several examples of how they extend in a natural way to higher-order equations.[†] Applying this strategy to the third-order equation (11), we observe that the corresponding homogeneous equation has the auxiliary equation $(r-1)^2(r+1) = 0$ with roots $r = 1, 1, -1$. Hence, a general solution for the homogeneous equation is

$$x_h(t) = C_1e^t + C_2te^t + C_3e^{-t} .$$

To find a particular solution to (11), we use the method of undetermined coefficients with $x_p(t) = At^2 + Bt + C$. Substituting into (11) and solving for A, B, and C yields (after a little algebra)

$$x_p(t) = -t^2 - 4t - 6 .$$

Thus, a general solution to equation (11) is

$$x(t) = x_h(t) + x_p(t) = C_1e^t + C_2te^t + C_3e^{-t} - t^2 - 4t - 6 . \tag{12}$$

To find $y(t)$, we take the shortcut described in step (c) of the elimination procedure box. Subtracting the second equation in (10) from the first, we find

$$(D^2 - D)[x] + y = -1 - t^2 ,$$

so that

$$y = (D - D^2)[x] - 1 - t^2 .$$

Inserting the expression for $x(t)$, given in (12), we obtain

$$\begin{aligned} y(t) &= C_1e^t + C_2(te^t + e^t) - C_3e^{-t} - 2t - 4 \\ &\quad -[C_1e^t + C_2(te^t + 2e^t) + C_3e^{-t} - 2] - 1 - t^2 , \quad \text{or} \end{aligned}$$

$$y(t) = -C_2e^t - 2C_3e^{-t} - t^2 - 2t - 3 . \tag{13}$$

The formulas for $x(t)$ in (12) and $y(t)$ in (13) give the desired general solution to (9). ◆

The elimination method also applies to linear systems with three or more equations and unknowns; however, the process becomes more cumbersome as the number of equations and unknowns increases. The matrix methods presented in Chapter 9 are better suited for handling larger systems. Here we illustrate the elimination technique for a 3×3 system.

Example 5 Find a general solution to

$$\begin{aligned} x'(t) &= x(t) + 2y(t) - z(t) , \\ y'(t) &= x(t) + z(t) , \\ z'(t) &= 4x(t) - 4y(t) + 5z(t) . \end{aligned} \tag{14}$$

Solution We begin by expressing the system in operator notation:

$$\begin{aligned} (D-1)[x] - 2y + z &= 0 , \\ -x + D[y] - z &= 0 , \\ -4x + 4y + (D-5)[z] &= 0 . \end{aligned} \tag{15}$$

Eliminating z from the first two equations (by adding them) and then from the last two equations yields (after some algebra, which we omit)

$$\begin{aligned} (D-2)[x] + (D-2)[y] &= 0 , \\ -(D-1)[x] + (D-1)(D-4)[y] &= 0 . \end{aligned} \tag{16}$$

[†] More detailed treatment of higher-order equations is given in Chapter 6.

On eliminating x from this 2×2 system, we eventually obtain

$$(D-1)(D-2)(D-3)[y] = 0 ,$$

which has the general solution

$$y(t) = C_1e^t + C_2e^{2t} + C_3e^{3t} . \tag{17}$$

Taking the shortcut approach, we add the two equations in (16) to get an expression for x in terms of y and its derivatives, which simplifies to

$$x = (D^2 - 4D + 2)[y] = y'' - 4y' + 2y .$$

When we substitute the expression (17) for $y(t)$ into this equation, we find

$$x(t) = -C_1e^t - 2C_2e^{2t} - C_3e^{3t} . \tag{18}$$

Finally, using the second equation in (14) to solve for $z(t)$, we get

$$z(t) = y'(t) - x(t) ,$$

and substituting in for $y(t)$ and $x(t)$ yields

$$z(t) = 2C_1e^t + 4C_2e^{2t} + 4C_3e^{3t} . \tag{19}$$

The expressions for $x(t)$ in (18), $y(t)$ in (17), and $z(t)$ in (19) give a general solution with C_1, C_2, and C_3 as arbitrary constants. ◆

5.2 EXERCISES

1. Let $A = D - 1$, $B = D + 2$, $C = D^2 + D - 2$, where $D = d/dt$. For $y = t^3 - 8$, compute
(a) $A[y]$ **(b)** $B[A[y]]$ **(c)** $B[y]$
(d) $A[B[y]]$ **(e)** $C[y]$

2. Show that the operator $(D-1)(D+2)$ is the same as the operator $D^2 + D - 2$.

In Problems 3–18, use the elimination method to find a general solution for the given linear system, where differentiation is with respect to t.

3. $x' + 2y = 0 ,$
$x' - y' = 0$

4. $x' = x - y ,$
$y' = y - 4x$

5. $x' + y' - x = 5 ,$
$x' + y' + y = 1$

6. $x' = 3x - 2y + \sin t ,$
$y' = 4x - y - \cos t$

7. $(D+1)[u] - (D+1)[v] = e^t ,$
$(D-1)[u] + (2D+1)[v] = 5$

8. $(D-3)[x] + (D-1)[y] = t ,$
$(D+1)[x] + (D+4)[y] = 1$

9. $x' + y' + 2x = 0 ,$
$x' + y' - x - y = \sin t$

10. $2x' + y' - x - y = e^{-t} ,$
$x' + y' + 2x + y = e^t$

11. $(D^2 - 1)[u] + 5v = e^t ,$
$2u + (D^2 + 2)[v] = 0$

12. $D^2[u] + D[v] = 2 ,$
$4u + D[v] = 6$

13. $\dfrac{dx}{dt} = x - 4y ,$
$\dfrac{dy}{dt} = x + y$

14. $\dfrac{dx}{dt} + y = t^2 ,$
$-x + \dfrac{dy}{dt} = 1$

15. $\dfrac{dw}{dt} = 5w + 2z + 5t ,$
$\dfrac{dz}{dt} = 3w + 4z + 17t$

16. $\dfrac{dx}{dt} + x + \dfrac{dy}{dt} = e^{4t} ,$
$2x + \dfrac{d^2y}{dt^2} = 0$

17. $x'' + 5x - 4y = 0 ,$
$-x + y'' + 2y = 0$

18. $x'' + y'' - x' = 2t ,$
$x'' + y' - x + y = -1$

In Problems 19–21, solve the given initial value problem.

19. $\dfrac{dx}{dt} = 4x + y ; \quad x(0) = 1 ,$
$\dfrac{dy}{dt} = -2x + y ; \quad y(0) = 0$

20. $\dfrac{dx}{dt} = 2x + y - e^{2t} ; \quad x(0) = 1 ,$
$\dfrac{dy}{dt} = x + 2y ; \quad y(0) = -1$

21. $\dfrac{d^2x}{dt^2} = y\,; \quad x(0) = 3\,, \qquad x'(0) = 1\,,$

$\dfrac{d^2y}{dt^2} = x\,; \quad y(0) = 1\,, \qquad y'(0) = -1$

22. Verify that the solution to the initial value problem

$$x' = 5x - 3y - 2\,; \quad x(0) = 2\,,$$
$$y' = 4x - 3y - 1\,; \quad y(0) = 0$$

satisfies $|x(t)| + |y(t)| \to +\infty$ as $t \to +\infty$.

In Problems 23 and 24, show that the given linear system is degenerate. In attempting to solve the system, determine whether it has no solutions or infinitely many solutions.

23. $(D-1)[x] + (D-1)[y] = -3e^{-2t}\,,$
$(D+2)[x] + (D+2)[y] = 3e^{t}$

24. $D[x] + (D+1)[y] = e^{t}\,,$
$D^2[x] + (D^2+D)[y] = 0$

In Problems 25–28, use the elimination method to find a general solution for the given system of three equations in the three unknown functions $x(t)$, $y(t)$, $z(t)$.

25. $x' = x + 2y - z\,,$
$y' = x + z\,,$
$z' = 4x - 4y + 5z$

26. $x' = 3x + y - z\,,$
$y' = x + 2y - z\,,$
$z' = 3x + 3y - z$

27. $x' = 4x - 4z\,,$
$y' = 4y - 2z\,,$
$z' = -2x - 4y + 4z$

28. $x' = x + 2y + z\,,$
$y' = 6x - y\,,$
$z' = -x - 2y - z$

*In Problems 29 and 30, determine the range of values (if any) of the parameter λ that will ensure **all** solutions $x(t)$, $y(t)$ of the given system remain bounded as $t \to +\infty$.*

29. $\dfrac{dx}{dt} = \lambda x - y\,,$
$\dfrac{dy}{dt} = 3x + y$

30. $\dfrac{dx}{dt} = -x + \lambda y\,,$
$\dfrac{dy}{dt} = x - y$

31. Two large tanks, each holding 100 L of liquid, are interconnected by pipes, with the liquid flowing from tank A into tank B at a rate of 3 L/min and from B into A at a rate of 1 L/min (see Figure 5.2). The liquid inside each tank is kept well stirred. A brine solution with a concentration of 0.2 kg/L of salt flows into tank A at a rate of 6 L/min. The (diluted) solution flows out of the system from tank A at 4 L/min and from tank B at 2 L/min. If, initially, tank A contains pure water and tank B contains 20 kg of salt, determine the mass of salt in each tank at time $t \geq 0$.

32. In Problem 31, 3 L/min of liquid flowed from tank A into tank B and 1 L/min from B into A. Determine the mass of salt in each tank at time $t \geq 0$ if, instead, 5 L/min flows from A into B and 3 L/min flows from B into A, with all other data the same.

33. In Problem 31, assume that no solution flows out of the system from tank B, only 1 L/min flows from A into B, and only 4 L/min of brine flows into the system at tank A, other data being the same. Determine the mass of salt in each tank at time $t \geq 0$.

34. Feedback System with Pooling Delay. Many physical and biological systems involve time delays. A pure time delay has its output the same as its input but shifted in time. A more common type of delay is *pooling delay*. An example of such a feedback system is shown in Figure 5.3 on page 251. Here the level of fluid in tank B determines the rate at which fluid enters tank A. Suppose this rate is given by $R_1(t) = \alpha[V - V_2(t)]$, where α and V are positive constants and $V_2(t)$ is the volume of fluid in tank B at time t.

(a) If the outflow rate R_3 from tank B is constant and the flow rate R_2 from tank A into B is $R_2(t) = KV_1(t)$, where K is a positive constant and $V_1(t)$ is the volume of fluid in tank A at time t, then show that this feedback system is governed by the system

$$\frac{dV_1}{dt} = \alpha\big(V - V_2(t)\big) - KV_1(t)\,,$$
$$\frac{dV_2}{dt} = KV_1(t) - R_3\,.$$

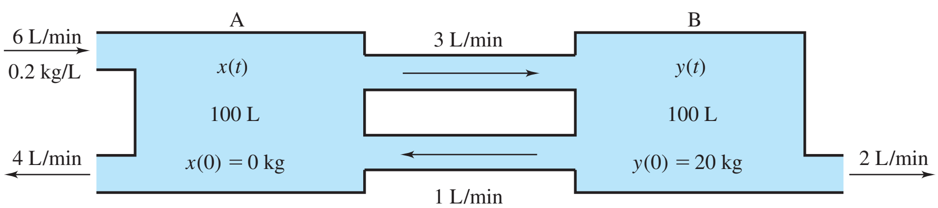

Figure 5.2 Mixing problem for interconnected tanks

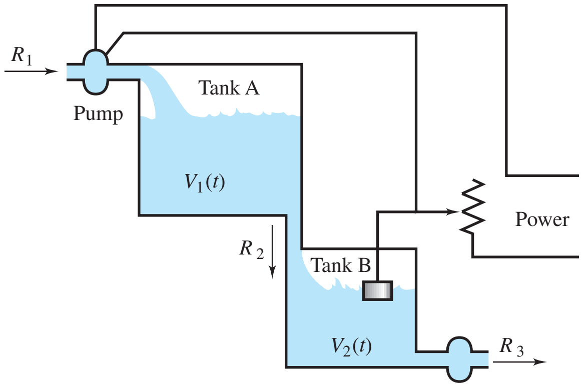

Figure 5.3 Feedback system with pooling delay

(b) Find a general solution for the system in part (a) when $\alpha = 5\,(\text{min})^{-1}$, $V = 20$ L, $K = 2\,(\text{min})^{-1}$, and $R_3 = 10$ L/min.

(c) Using the general solution obtained in part (b), what can be said about the volume of fluid in each of the tanks as $t \to +\infty$?

35. A house, for cooling purposes, consists of two zones: the attic area zone A and the living area zone B (see Figure 5.4). The living area is cooled by a 2-ton air conditioning unit that removes 24,000 Btu/hr. The heat capacity of zone B is $1/2$°F per thousand Btu. The time constant for heat transfer between zone A and the outside is 2 hr, between zone B and the outside is 4 hr, and between the two zones is 4 hr. If the outside temperature stays at 100°F, how warm does it eventually get in the attic zone A? (Heating and cooling of buildings was treated in Section 3.3 on page 102.)

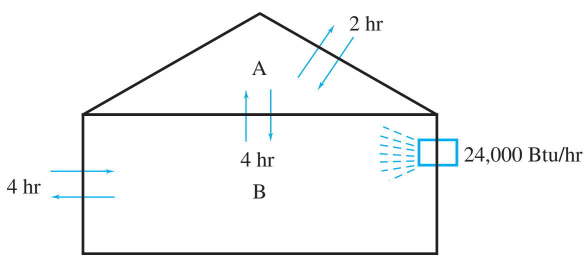

Figure 5.4 Air-conditioned house with attic

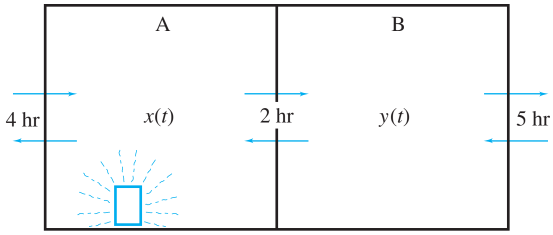

Figure 5.5 Two-zone building with one zone heated

36. A building consists of two zones A and B (see Figure 5.5). Only zone A is heated by a furnace, which generates 80,000 Btu/hr. The heat capacity of zone A is $1/4$°F per thousand Btu. The time constant for heat transfer between zone A and the outside is 4 hr, between the unheated zone B and the outside is 5 hr, and between the two zones is 2 hr. If the outside temperature stays at 0°F, how cold does it eventually get in the unheated zone B?

37. In Problem 36, if a small furnace that generates 1000 Btu/hr is placed in zone B, determine the coldest it would eventually get in zone B if zone B has a heat capacity of 2°F per thousand Btu.

38. **Arms Race.** A simplified mathematical model for an arms race between two countries whose expenditures for defense are expressed by the variables $x(t)$ and $y(t)$ is given by the linear system

$$\frac{dx}{dt} = 2y - x + a\,; \qquad x(0) = 1\,,$$
$$\frac{dy}{dt} = 4x - 3y + b\,; \qquad y(0) = 4\,,$$

where a and b are constants that measure the trust (or distrust) each country has for the other. Determine whether there is going to be disarmament (x and y approach 0 as t increases), a stabilized arms race (x and y approach a constant as $t \to +\infty$), or a runaway arms race (x and y approach $+\infty$ as $t \to +\infty$).

39. Let A, B, and C represent three linear differential operators with constant coefficients; for example,

$$A := a_2D^2 + a_1D + a_0\,,\ B := b_2D^2 + b_1D + b_0\,,$$
$$C := c_2D^2 + c_1D + c_0\,,$$

where the a's, b's, and c's are constants. Verify the following properties:[†]

(a) Commutative laws:

$$A + B = B + A\,,$$
$$AB = BA\,.$$

(b) Associative laws:

$$(A + B) + C = A + (B + C)\,,$$
$$(AB)C = A(BC)\,.$$

(c) Distributive law: $A(B + C) = AB + AC$.

[†]We say that two operators A and B are *equal* if $A[y] = B[y]$ for all functions y with the necessary derivatives.

5.3 Solving Systems and Higher-Order Equations Numerically

Although we studied a half-dozen analytic methods for obtaining solutions to first-order ordinary differential equations in Chapter 2, the techniques for higher-order equations, or systems of equations, are much more limited. Chapter 4 focused on solving the linear constant-coefficient second-order equation. The elimination method of the previous section is also restricted to constant-coefficient systems. And, indeed, higher-order linear constant-coefficient equations and systems can be solved analytically by extensions of these methods, as we will see in Chapters 6, 7, and 9.

However, if the equations—even a single second-order linear equation—have variable coefficients, the solution process is much less satisfactory. As will be seen in Chapter 8, the solutions are expressed as infinite series, and their computation can be very laborious (with the notable exception of the Cauchy–Euler, or equidimensional, equation). And we know virtually nothing about how to obtain exact solutions to nonlinear second-order equations.

Fortunately, all the cases that arise (constant or variable coefficients, nonlinear, higher-order equations or systems) can be addressed by a single formulation that lends itself to a multitude of *numerical* approaches. In this section we'll see how to express differential equations as a *system in normal form* and then show how the basic Euler method for computer solution can be easily "vectorized" to apply to such systems. Although subsequent chapters will return to analytic solution methods, the vectorized version of the Euler technique or the more efficient Runge–Kutta technique will hereafter be available as fallback methods for numerical exploration of intractable problems.

Normal Form

A system of m differential equations in the m unknown functions $x_1(t), x_2(t), \ldots, x_m(t)$ expressed as

$$\begin{aligned} x_1'(t) &= f_1(t, x_1, x_2, \ldots, x_m) \ , \\ x_2'(t) &= f_2(t, x_1, x_2, \ldots, x_m) \ , \\ &\vdots \\ x_m'(t) &= f_m(t, x_1, x_2, \ldots, x_m) \end{aligned} \tag{1}$$

is said to be in **normal form.** Notice that (1) consists of m first-order equations that collectively look like a *vectorized* version of the single generic first-order equation

$$x' = f(t, x) \ , \tag{2}$$

and that the system expressed in equation (1) of Section 5.1 takes this form, as do equations (1) and (14) in Section 5.2. An initial value problem for (1) entails finding a solution to this system that satisfies the initial conditions

$$x_1(t_0) = a_1, \quad x_2(t_0) = a_2, \quad \ldots, \quad x_m(t_0) = a_m$$

for prescribed values $t_0, a_1, a_2, \ldots, a_m$.

The importance of the normal form is underscored by the fact that most professional codes for initial value problems presume that the system is written in this form. Furthermore, for a *linear* system in normal form, the powerful machinery of linear algebra can be readily applied. [Indeed, in Chapter 9 we will show how the solutions $x(t) = ce^{at}$ of the simple equation $x' = ax$ can be generalized to constant-coefficient systems in normal form.]

For these reasons it is gratifying to note that a (single) higher-order equation can always be converted to an equivalent system of first-order equations.

To convert an mth-order differential equation

$$y^{(m)}(t) = f\big(t, y, y', \ldots, y^{(m-1)}\big) \tag{3}$$

into a first-order system, we introduce, as additional unknowns, the sequence of derivatives of y:

$$x_1(t) := y(t), \quad x_2(t) := y'(t), \quad \ldots, \quad x_m(t) := y^{(m-1)}(t) .$$

With this scheme, we obtain $m-1$ first-order equations quite trivially:

$$\begin{aligned} x_1'(t) &= y'(t) = x_2(t) , \\ x_2'(t) &= y''(t) = x_3(t) , \\ &\vdots \\ x_{m-1}'(t) &= y^{(m-1)}(t) = x_m(t) . \end{aligned} \tag{4}$$

The mth and final equation then constitutes a restatement of the original equation (3) in terms of the new unknowns:

$$x_m'(t) = y^{(m)}(t) = f(t, x_1, x_2, \ldots, x_m) . \tag{5}$$

If equation (3) has initial conditions $y(t_0) = a_1, y'(t_0) = a_2, \ldots, y^{(m-1)}(t_0) = a_m$, then the system (4)–(5) has initial conditions $x_1(t_0) = a_1, x_2(t_0) = a_2, \ldots, x_m(t_0) = a_m$.

Example 1 Convert the initial value problem

$$y''(t) + 3ty'(t) + y(t)^2 = \sin t ; \quad y(0) = 1 , \quad y'(0) = 5 \tag{6}$$

into an initial value problem for a system in normal form.

Solution We first express the differential equation in (6) as

$$y''(t) = -3ty'(t) - y(t)^2 + \sin t .$$

Setting $x_1(t) := y(t)$ and $x_2(t) := y'(t)$, we obtain

$$\begin{aligned} x_1'(t) &= x_2(t) , \\ x_2'(t) &= -3tx_2(t) - x_1(t)^2 + \sin t . \end{aligned}$$

The initial conditions transform to $x_1(0) = 1, x_2(0) = 5$. ◆

Euler's Method for Systems in Normal Form

Recall from Section 1.4 that Euler's method for solving a single first-order equation (2) is based on estimating the solution x at time $(t_0 + h)$ using the approximation

$$x(t_0 + h) \approx x(t_0) + hx'(t_0) = x(t_0) + hf\big(t_0, x(t_0)\big) , \tag{7}$$

and that as a consequence the algorithm can be summarized by the recursive formulas

$$t_{n+1} = t_n + h , \tag{8}$$

$$x_{n+1} = x_n + hf(t_n, x_n), \qquad n = 0, 1, 2, \ldots \tag{9}$$

[compare equations (2) and (3), Section 1.4]. Now we can apply the approximation (7) to each of the equations in the system (1):

$$x_k(t_0 + h) \approx x_k(t_0) + hx_k'(t_0) = x_k(t_0) + hf_k\big(t_0, x_1(t_0), x_2(t_0), \ldots, x_m(t_0)\big) , \tag{10}$$

and for $k = 1, 2, \ldots m$, we are led to the recursive formulas

$$t_{n+1} = t_n + h\,, \tag{11}$$

$$\begin{aligned} x_{1;n+1} &= x_{1;n} + hf_1(t_n, x_{1;n}, x_{2;n}, \ldots, x_{m;n})\,, \\ x_{2;n+1} &= x_{2;n} + hf_2(t_n, x_{1;n}, x_{2;n}, \ldots, x_{m;n})\,, \\ &\vdots \\ x_{m;n+1} &= x_{m;n} + hf_m(t_n, x_{1;n}, x_{2;n}, \ldots, x_{m;n}) \qquad (n = 0, 1, 2, \ldots)\,. \end{aligned} \tag{12}$$

Here we are burdened with the ungainly notation $x_{p;n}$ for the approximation to the value of the pth-function x_p at time $t = t_0 + nh$; i.e., $x_{p;n} \approx x_p(t_0 + nh)$. However, if we treat the unknowns and right-hand members of (1) as components of vectors

$$\begin{aligned} \mathbf{x}(t) &\coloneqq [x_1(t), x_2(t), \ldots, x_m(t)]\,, \\ \mathbf{f}(t, \mathbf{x}) &\coloneqq [f_1(t, x_1, x_2, \ldots, x_m), f_2(t, x_1, x_2, \ldots, x_m), \ldots, f_m(t, x_1, x_2, \ldots, x_m)]\,, \end{aligned}$$

then (12) can be expressed in the much neater form

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{f}(t_n, \mathbf{x}_n)\,. \tag{13}$$

Example 2 Use the vectorized Euler method with step size $h = 0.1$ to find an approximation for the solution to the initial value problem

$$y''(t) + 4y'(t) + 3y(t) = 0; \quad y(0) = 1.5\,, \quad y'(0) = -2.5\,, \tag{14}$$

on the interval $[0, 1]$.

Solution For the given step size, the method will yield approximations for $y(0.1), y(0.2), \ldots, y(1.0)$. To apply the vectorized Euler method to (14), we first convert it to normal form. Setting $x_1 = y$ and $x_2 = y'$, we obtain the system

$$\begin{aligned} x_1' &= x_2; & x_1(0) &= 1.5\,, \\ x_2' &= -4x_2 - 3x_1; & x_2(0) &= -2.5\,. \end{aligned} \tag{15}$$

Comparing (15) with (1) we see that $f_1(t, x_1, x_2) = x_2$ and $f_2(t, x_1, x_2) = -4x_2 - 3x_1$. With the starting values of $t_0 = 0$, $x_{1;0} = 1.5$, and $x_{2;0} = -2.5$, we compute

$$\begin{cases} x_1(0.1) \approx x_{1;1} = x_{1;0} + hx_{2;0} = 1.5 + 0.1(-2.5) = 1.25\,, \\ x_2(0.1) \approx x_{2;1} = x_{2;0} + h(-4x_{2;0} - 3x_{1;0}) = -2.5 + 0.1[-4(-2.5) - 3 \cdot 1.5] = -1.95\,; \end{cases}$$

$$\begin{cases} x_1(0.2) \approx x_{1;2} = x_{1;1} + hx_{2;1} = 1.25 + 0.1(-1.95) = 1.055\,, \\ x_2(0.2) \approx x_{2;2} = x_{2;1} + h(-4x_{2;1} - 3x_{1;1}) = -1.95 + 0.1[-4(-1.95) - 3 \cdot 1.25] = -1.545\,. \end{cases}$$

Continuing the algorithm we compute the remaining values. These are listed in Table 5.1 on page 255, along with the exact values calculated via the methods of Chapter 4. Note that the $x_{2;n}$ column gives approximations to $y'(t)$, since $x_2(t) \equiv y'(t)$. ◆

TABLE 5.1 Approximations of the Solution to (14) in Example 2

$t = n(0.1)$	$x_{1;n}$	y Exact	$x_{2;n}$	y' Exact
0	1.5	1.5	−2.5	−2.5
0.1	1.25	1.275246528	−1.95	−2.016064749
0.2	1.055	1.093136571	−1.545	−1.641948207
0.3	0.9005	0.944103051	−1.2435	−1.35067271
0.4	0.77615	0.820917152	−1.01625	−1.122111364
0.5	0.674525	0.71809574	−0.842595	−0.9412259
0.6	0.5902655	0.63146108	−0.7079145	−0.796759968
0.7	0.51947405	0.557813518	−0.60182835	−0.680269946
0.8	0.459291215	0.494687941	−0.516939225	−0.585405894
0.9	0.407597293	0.440172416	−0.4479509	−0.507377929
1	0.362802203	0.392772975	−0.391049727	−0.442560044

Euler's method is modestly accurate for this problem with a step size of $h = 0.1$. The next example demonstrates the effects of using a sequence of smaller values of h to improve the accuracy.

Example 3 For the initial value problem of Example 2, use Euler's method to estimate $y(1)$ for successively halved step sizes $h = 0.1, 0.05, 0.025, 0.0125, 0.00625$.

Solution Using the same scheme as in Example 2, we find the following approximations, denoted by $y(1;h)$ (obtained with step size h):

h	0.1	0.05	0.025	0.0125	0.00625
$y(1;h)$	0.36280	0.37787	0.38535	0.38907	0.39092

[Recall that the exact value, rounded to 5 decimal places, is $y(1) = 0.39277$.] ◆

The Runge–Kutta scheme described in Section 3.7 is easy to vectorize also; details are given on the following page. As would be expected, its performance is considerably more accurate, yielding five-decimal agreement with the exact solution for a step size of 0.05:

h	0.1	0.05	0.025	0.0125	0.00625
$y(1;h)$	0.39278	0.39277	0.39277	0.39277	0.39277

As in Section 3.7, both algorithms can be coded so as to repeat the calculation of $y(1)$ with a sequence of smaller step sizes until two consecutive estimates agree to within some prespecified tolerance ε. Here one should interpret "two estimates agree to within ε" to mean that *each component* of the successive vector approximants [i.e., approximants to $y(1)$ and $y'(1)$] should agree to within ε.

An Application to Population Dynamics

A mathematical model for the population dynamics of competing species, one a predator with population $x_2(t)$ and the other its prey with population $x_1(t)$, was developed independently in the early 1900s by A. J. Lotka and V. Volterra. It assumes that there is plenty of food available for the prey to eat, so the birthrate of the prey should follow the Malthusian or exponential law (see Section 3.2); that is, the birthrate of the prey is Ax_1, where A is a positive constant. The death rate of the prey depends on the number of interactions between the predators and the prey. This is modeled by the expression Bx_1x_2, where B is a positive constant. Therefore, the rate of change in the population of the prey per unit time is $dx_1/dt = Ax_1 - Bx_1x_2$. Assuming that the predators depend entirely on the prey for their food, it is argued that the birthrate of the predators depends on the number of interactions with the prey; that is, the birthrate of predators is Dx_1x_2, where D is a positive constant. The death rate of the predators is assumed to be Cx_2 because without food the population would die off at a rate proportional to the population present. Hence, the rate of change in the population of predators per unit time is $dx_2/dt = -Cx_2 + Dx_1x_2$. Combining these two equations, we obtain the Volterra–Lotka system for the population dynamics of two competing species:

$$\begin{aligned} x_1' &= Ax_1 - Bx_1x_2 , \\ x_2' &= -Cx_2 + Dx_1x_2 . \end{aligned} \tag{16}$$

Such systems are in general not explicitly solvable. In the following example, we obtain an approximate solution for such a system by utilizing the vectorized form of the Runge–Kutta algorithm.

For the system of two equations

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2) , \\ x_2' &= f_2(t, x_1, x_2) , \end{aligned}$$

with initial conditions $x_1(t_0) = x_{1;0}$, $x_2(t_0) = x_{2;0}$, the vectorized form of the Runge–Kutta recursive equations (cf. (14), page 254) becomes

$$\begin{cases} t_{n+1} := t_n + h \qquad (n = 0, 1, 2, \dots) , \\ x_{1;n+1} := x_{1;n} + \frac{1}{6}(k_{1,1} + 2k_{1,2} + 2k_{1,3} + k_{1,4}) , \\ x_{2;n+1} := x_{2;n} + \frac{1}{6}(k_{2,1} + 2k_{2,2} + 2k_{2,3} + k_{2,4}) , \end{cases} \tag{17}$$

where h is the step size and, for $i = 1$ and 2,

$$\begin{cases} k_{i,1} := hf_i(t_n, x_{1;n}, x_{2;n}) , \\ k_{i,2} := hf_i\left(t_n + \frac{h}{2},\ x_{1;n} + \frac{1}{2}k_{1,1},\ x_{2;n} + \frac{1}{2}k_{2,1}\right) , \\ k_{i,3} := hf_i\left(t_n + \frac{h}{2},\ x_{1;n} + \frac{1}{2}k_{1,2},\ x_{2;n} + \frac{1}{2}k_{2,2}\right) , \\ k_{i,4} := hf_i(t_n + h,\ x_{1;n} + k_{1,3},\ x_{2;n} + k_{2,3}) . \end{cases} \tag{18}$$

It is important to note that both $k_{1,1}$ and $k_{2,1}$ must be computed before either $k_{1,2}$ or $k_{2,2}$. Similarly, both $k_{1,2}$ and $k_{2,2}$ are needed to compute $k_{1,3}$ and $k_{2,3}$, etc. In Appendix F, program outlines are given for applying the method to graph approximate solutions over a specified interval $[t_0, t_1]$ or to obtain approximations of the solutions at a specified point to within a desired tolerance.

Example 4 Use the classical fourth-order Runge–Kutta algorithm for systems to approximate the solution of the initial value problem

$$\textbf{(19)}\qquad \begin{aligned} x_1' &= 2x_1 - 2x_1x_2\,; \qquad & x_1(0) &= 1\,, \\ x_2' &= x_1x_2 - x_2\,; & x_2(0) &= 3 \end{aligned}$$

at $t = 1$. Starting with $h = 1$, continue halving the step size until two successive approximations of $x_1(1)$ and of $x_2(1)$ differ by at most 0.0001.

Solution Here $f_1(t, x_1, x_2) = 2x_1 - 2x_1x_2$ and $f_2(t, x_1, x_2) = x_1x_2 - x_2$. With the inputs $t_0 = 0$, $x_{1;0} = 1$, $x_{2;0} = 3$, we proceed with the algorithm to compute $x_1(1; 1)$ and $x_2(1; 1)$, the approximations to $x_1(1)$, $x_2(1)$ using $h = 1$. We find from the formulas in (18) that

$$\begin{aligned}
k_{1,1} &= h(2x_{1;0} - 2x_{1;0}x_{2;0}) = 2(1) - 2(1)(3) = -4\,, \\
k_{2,1} &= h(x_{1;0}x_{2;0} - x_{2;0}) = (1)(3) - 3 = 0\,, \\
k_{1,2} &= h\left[2\left(x_{1;0} + \tfrac{1}{2}k_{1,1}\right) - 2\left(x_{1;0} + \tfrac{1}{2}k_{1,1}\right)\left(x_{2;0} + \tfrac{1}{2}k_{2,1}\right)\right] \\
&= 2\left[1 + \tfrac{1}{2}(-4)\right] - 2\left[1 + \tfrac{1}{2}(-4)\right]\left[3 + \tfrac{1}{2}(0)\right] \\
&= -2 + 2(3) = 4\,, \\
k_{2,2} &= h\left[\left(x_{1;0} + \tfrac{1}{2}k_{1,1}\right)\left(x_{2;0} + \tfrac{1}{2}k_{2,1}\right) - \left(x_{2;0} + \tfrac{1}{2}k_{2,1}\right)\right] \\
&= \left[1 + \tfrac{1}{2}(-4)\right]\left[3 + \tfrac{1}{2}(0)\right] - \left[3 + \tfrac{1}{2}(0)\right] \\
&= (-1)(3) - 3 = -6\,,
\end{aligned}$$

and similarly we compute

$$\begin{aligned}
k_{1,3} &= h\left[2\left(x_{1;0} + \tfrac{1}{2}k_{1,2}\right) - 2\left(x_{1;0} + \tfrac{1}{2}k_{1,2}\right)\left(x_{2;0} + \tfrac{1}{2}k_{2,2}\right)\right] = 6\,, \\
k_{2,3} &= h\left[\left(x_{1;0} + \tfrac{1}{2}k_{1,2}\right)\left(x_{2;0} + \tfrac{1}{2}k_{2,2}\right) - \left(x_{2;0} + \tfrac{1}{2}k_{2,2}\right)\right] = 0\,, \\
k_{1,4} &= h[2(x_{1;0} + k_{1,3}) - 2(x_{1;0} + k_{1,3})(x_{2;0} + k_{2,3})] = -28\,, \\
k_{2,4} &= h[(x_{1;0} + k_{1,3})(x_{2;0} + k_{2,3}) - (x_{2;0} + k_{2,3})] = -18\,.
\end{aligned}$$

Inserting these values into formula (17), we get

$$\begin{aligned}
x_{1;1} &= x_{1;0} + \frac{1}{6}(k_{1,1} + 2k_{1,2} + 2k_{1,3} + k_{1,4}) \\
&= 1 + \frac{1}{6}(-4 + 8 + 12 - 28) = -1\,, \\
x_{2;1} &= x_{2;0} + \frac{1}{6}(k_{2,1} + 2k_{2,2} + 2k_{2,3} + k_{2,4})\,, \\
&= 3 + \frac{1}{6}(0 - 12 + 0 + 18) = 4\,,
\end{aligned}$$

as the respective approximations to $x_1(1)$ and $x_2(1)$.

Repeating the algorithm with $h = 1/2$ $(N = 2)$ we obtain the approximations $x_1(1; 2^{-1})$ and $x_2(1; 2^{-1})$ for $x_1(1)$ and $x_2(1)$. In Table 5.2, we list the approximations $x_1(1; 2^{-m})$ and $x_2(1; 2^{-m})$ for $x_1(1)$ and $x_2(1)$ using step size $h = 2^{-m}$ for $m = 0, 1, 2, 3,$ and 4. We stopped at $m = 4$, since both

$$|x_1(1; 2^{-3}) - x_1(1; 2^{-4})| = 0.00006 < 0.0001$$

and

$$|x_2(1; 2^{-3}) - x_2(1; 2^{-4})| = 0.00001 < 0.0001 \, .$$

Hence, $x_1(1) \approx 0.07735$ and $x_2(1) \approx 1.46445$, with tolerance 0.0001. ◆

TABLE 5.2 Approximations of the Solution to System (19) in Example 4

m	h	$x_1(1; h)$	$x_2(1; h)$
0	1.0	−1.0	4.0
1	0.5	0.14662	1.47356
2	0.25	0.07885	1.46469
3	0.125	0.07741	1.46446
4	0.0625	0.07735	1.46445

To get a better feel for the solution to system (19), we have graphed in Figure 5.6 an approximation of the solution for $0 \le t \le 12$, using linear interpolation to connect the vectorized Runge–Kutta approximants for the points $t = 0, 0.125, 0.25, \ldots, 12.0$ (i.e., with $h = 0.125$). From the graph it appears that the components x_1 and x_2 are periodic in the variable t. Phase plane analysis is used in Section 5.5 to show that, indeed, Volterra–Lotka equations have periodic solutions.

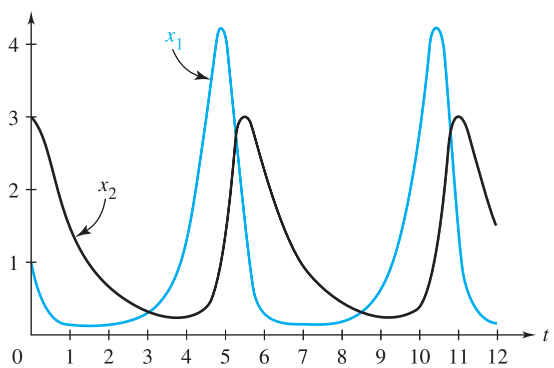

Figure 5.6 Graphs of the components of an approximate solution to the Volterra–Lotka system (17)

5.3 EXERCISES

In Problems 1–7, convert the given initial value problem into an initial value problem for a system in normal form.

1. $y''(t) + ty'(t) - 3y(t) = t^2$;
$y(0) = 3, \quad y'(0) = -6$

2. $y''(t) = \cos(t - y) + y^2(t)$;
$y(0) = 1, \quad y'(0) = 0$

3. $y^{(4)}(t) - y^{(3)}(t) + 7y(t) = \cos t$;
$y(0) = y'(0) = 1, \quad y''(0) = 0, \quad y^{(3)}(0) = 2$

4. $y^{(6)}(t) = [y'(t)]^3 - \sin\big(y(t)\big) + e^{2t}$;
$y(0) = y'(0) = \cdots = y^{(5)}(0) = 0$

5. $x'' + y - x' = 2t; \quad x(3) = 5, \quad x'(3) = 2,$
$y'' - x + y = -1; \quad y(3) = 1, \quad y'(3) = -1$
[*Hint:* Set $x_1 = x, \quad x_2 = x', \quad x_3 = y, \quad x_4 = y'$.]

6. $3x'' + 5x - 2y = 0; \quad x(0) = -1, \quad x'(0) = 0,$
$4y'' + 2y - 6x = 0; \quad y(0) = 1, \quad y'(0) = 2$

7. $x''' - y = t; \quad x(0) = x'(0) = x''(0) = 4,$
$2x'' + 5y'' - 2y = 1; \quad y(0) = y'(0) = 1$

8. Sturm–Liouville Form. A second-order equation is said to be in **Sturm–Liouville form** if it is expressed as

$$[p(t)y'(t)]' + q(t)y(t) = 0 .$$

Show that the substitutions $x_1 = y, x_2 = py'$ result in the normal form

$$x_1' = x_2/p ,$$
$$x_2' = -qx_1 .$$

If $y(0) = a$ and $y'(0) = b$ are the initial values for the Sturm–Liouville problem, what are $x_1(0)$ and $x_2(0)$?

9. In Section 3.6, we discussed the improved Euler's method for approximating the solution to a first-order equation. Extend this method to normal systems and give the recursive formulas for solving the initial value problem.

In Problems 10–13, use the vectorized Euler method with $h = 0.25$ to find an approximation for the solution to the given initial value problem on the specified interval.

10. $y'' + ty' + y = 0$;
$y(0) = 1, \quad y'(0) = 0 \quad$ on $[0, 1]$

11. $(1 + t^2)y'' + y' - y = 0$;
$y(0) = 1, \quad y'(0) = -1 \quad$ on $[0, 1]$

12. $t^2y'' + y = t + 2$;
$y(1) = 1, \quad y'(1) = -1 \quad$ on $[1, 2]$

13. $y'' = t^2 - y^2$;
$y(0) = 0, \quad y'(0) = 1 \quad$ on $[0, 1]$
(Can you guess the solution?)

In Problems 14–24, you will need a computer and a programmed version of the vectorized classical fourth-order Runge–Kutta algorithm. (At the instructor's discretion, other algorithms may be used.)†

14. Using the vectorized Runge–Kutta algorithm with $h = 0.5$, approximate the solution to the initial value problem

$$3t^2y'' - 5ty' + 5y = 0;$$
$$y(1) = 0, \quad y'(1) = \frac{2}{3}$$

at $t = 8$. Compare this approximation to the actual solution $y(t) = t^{5/3} - t$.

15. Using the vectorized Runge–Kutta algorithm, approximate the solution to the initial value problem

$$y'' = t^2 + y^2; \quad y(0) = 1, \quad y'(0) = 0$$

at $t = 1$. Starting with $h = 1$, continue halving the step size until two successive approximations [of both $y(1)$ and $y'(1)$] differ by at most 0.01.

16. Using the vectorized Runge–Kutta algorithm for systems with $h = 0.125$, approximate the solution to the initial value problem

$$x' = 2x - y; \quad x(0) = 0,$$
$$y' = 3x + 6y; \quad y(0) = -2$$

at $t = 1$. Compare this approximation to the actual solution

$$x(t) = e^{5t} - e^{3t}, \quad y(t) = e^{3t} - 3e^{5t} .$$

17. Using the vectorized Runge–Kutta algorithm, approximate the solution to the initial value problem

$$\frac{du}{dx} = 3u - 4v; \quad u(0) = 1,$$
$$\frac{dv}{dx} = 2u - 3v; \quad v(0) = 1$$

at $x = 1$. Starting with $h = 1$, continue halving the step size until two successive approximations of $u(1)$ and $v(1)$ differ by at most 0.001.

†Appendix G describes various websites and commercial software that sketch direction fields and automate most of the differential equation algorithms discussed in this book.

18. Combat Model. A simplified mathematical model for conventional versus guerrilla combat is given by the system

$$x_1' = -(0.1)x_1x_2\,; \qquad x_1(0) = 10\,,$$
$$x_2' = -x_1\,; \qquad x_2(0) = 15\,,$$

where x_1 and x_2 are the strengths of guerrilla and conventional troops, respectively, and 0.1 and 1 are the *combat effectiveness coefficients*. Who will win the conflict: the conventional troops or the guerrillas? [*Hint:* Use the vectorized Runge–Kutta algorithm for systems with $h = 0.1$ to approximate the solutions.]

19. Predator–Prey Model. The Volterra–Lotka predator–prey model predicts some rather interesting behavior that is evident in certain biological systems. For example, suppose you fix the initial population of prey but increase the initial population of predators. Then the population cycle for the prey becomes more severe in the sense that there is a long period of time with a reduced population of prey followed by a short period when the population of prey is very large. To demonstrate this behavior, use the vectorized Runge–Kutta algorithm for systems with $h = 0.5$ to approximate the populations of prey x and of predators y over the period $[0, 5]$ that satisfy the Volterra–Lotka system

$$x' = x(3 - y)\,,$$
$$y' = y(x - 3)$$

under each of the following initial conditions:

(a) $x(0) = 2\,, \quad y(0) = 4\,.$
(b) $x(0) = 2\,, \quad y(0) = 5\,.$
(c) $x(0) = 2\,, \quad y(0) = 7\,.$

20. In Project C of Chapter 4, it was shown that the simple pendulum equation

$$\theta''(t) + \sin\theta(t) = 0$$

has periodic solutions when the initial displacement and velocity are small. Show that the period of the solution may depend on the initial conditions by using the vectorized Runge–Kutta algorithm with $h = 0.02$ to approximate the solutions to the simple pendulum problem on $[0, 4]$ for the initial conditions:

(a) $\theta(0) = 0.1\,, \quad \theta'(0) = 0\,.$
(b) $\theta(0) = 0.5\,, \quad \theta'(0) = 0\,.$
(c) $\theta(0) = 1.0\,, \quad \theta'(0) = 0\,.$

[*Hint:* Approximate the length of time it takes to reach $-\theta(0)$.]

21. Fluid Ejection. In the design of a sewage treatment plant, the following equation arises:†

$$60 - H = (77.7)H'' + (19.42)(H')^2\,;$$
$$H(0) = H'(0) = 0\,,$$

where H is the level of the fluid in an ejection chamber and t is the time in seconds. Use the vectorized Runge–Kutta algorithm with $h = 0.5$ to approximate $H(t)$ over the interval $[0, 5]$.

22. Oscillations and Nonlinear Equations. For the initial value problem

$$x'' + (0.1)(1 - x^2)x' + x = 0\,;$$
$$x(0) = x_0\,, \qquad x'(0) = 0\,,$$

use the vectorized Runge–Kutta algorithm with $h = 0.02$ to illustrate that as t increases from 0 to 20, the solution x exhibits damped oscillations when $x_0 = 1$, whereas x exhibits expanding oscillations when $x_0 = 2.1$.

23. Nonlinear Spring. The **Duffing equation**

$$y'' + y + ry^3 = 0\,,$$

where r is a constant, is a model for the vibrations of a mass attached to a *nonlinear* spring. For this model, does the period of vibration vary as the parameter r is varied? Does the period vary as the initial conditions are varied? [*Hint:* Use the vectorized Runge–Kutta algorithm with $h = 0.1$ to approximate the solutions for $r = 1$ and 2, with initial conditions $y(0) = a$, $y'(0) = 0$ for $a = 1$, 2, and 3.]

24. Pendulum with Varying Length. A pendulum is formed by a mass m attached to the end of a wire that is attached to the ceiling. Assume that the length $l(t)$ of the wire varies with time in some predetermined fashion. If $\theta(t)$ is the angle in radians between the pendulum and the vertical, then the motion of the pendulum is governed for small angles by the initial value problem

$$l^2(t)\theta''(t) + 2l(t)l'(t)\theta'(t) + gl(t)\sin\big(\theta(t)\big) = 0\,;$$
$$\theta(0) = \theta_0\,, \qquad \theta'(0) = \theta_1\,,$$

where g is the acceleration due to gravity. Assume that

$$l(t) = l_0 + l_1\cos(\omega t - \phi)\,,$$

where l_1 is much smaller than l_0. (This might be a model for a person on a swing, where the *pumping* action changes the distance from the center of mass of the swing to the point where the swing is attached.) To simplify the computations, take $g = 1$. Using the Runge–Kutta algorithm with $h = 0.1$, study the motion of the pendulum when $\theta_0 = 0.05$, $\theta_1 = 0$, $l_0 = 1$, $l_1 = 0.1$, $\omega = 1$, and $\phi = 0.02$. In particular, does the pendulum ever attain an angle greater in absolute value than the initial angle θ_0?

†See *Numerical Solution of Differential Equations*, by William Milne (Dover, New York, 1970), p. 82.

In Problems 25–30, use a software package or the SUBROUTINE in Appendix F.

25. Using the Runge–Kutta algorithm for systems with $h = 0.05$, approximate the solution to the initial value problem

$$y''' + y'' + y^2 = t ;$$
$$y(0) = 1 , \qquad y'(0) = 0 , \qquad y''(0) = 1$$

at $t = 1$.

26. Use the Runge–Kutta algorithm for systems with $h = 0.1$ to approximate the solution to the initial value problem

$$\begin{aligned} x' &= yz ; & x(0) &= 0 , \\ y' &= -xz ; & y(0) &= 1 , \\ z' &= -xy/2 ; & z(0) &= 1 , \end{aligned}$$

at $t = 1$.

27. Generalized Blasius Equation. H. Blasius, in his study of laminar flow of a fluid, encountered an equation of the form

$$y''' + yy'' = (y')^2 - 1 .$$

Use the Runge–Kutta algorithm for systems with $h = 0.1$ to approximate the solution that satisfies the initial conditions $y(0) = 0, y'(0) = 0$, and $y''(0) = 1.32824$. Sketch this solution on the interval $[0, 2]$.

28. Lunar Orbit. The motion of a moon moving in a planar orbit about a planet is governed by the equations

$$\frac{d^2x}{dt^2} = -G\frac{mx}{r^3} , \qquad \frac{d^2y}{dt^2} = -G\frac{my}{r^3} ,$$

where $r := (x^2 + y^2)^{1/2}$, G is the gravitational constant, and m is the mass of the planet. Assume $Gm = 1$. When $x(0) = 1, x'(0) = y(0) = 0$, and $y'(0) = 1$, the motion is a circular orbit of radius 1 and period 2π.

(a) Setting $x_1 = x$, $x_2 = x'$, $x_3 = y$, $x_4 = y'$, express the governing equations as a first-order system in normal form.

(b) Using $h = 2\pi/100 \approx 0.0628318$, compute one orbit of this moon (i.e., do $N = 100$ steps.). Do your approximations agree with the fact that the orbit is a circle of radius 1?

29. Competing Species. Let $p_i(t)$ denote, respectively, the populations of three competing species S_{i}, $i = 1, 2, 3$. Suppose these species have the same growth rates, and the maximum population that the habitat can support is the same for each species. (We assume it to be one unit.) Also suppose the competitive advantage that S_1 has over S_2 is the same as that of S_2 over S_3 and S_3 over S_1. This situation is modeled by the system

$$\begin{aligned} p_1' &= p_1(1 - p_1 - ap_2 - bp_3) , \\ p_2' &= p_2(1 - bp_1 - p_2 - ap_3) , \\ p_3' &= p_3(1 - ap_1 - bp_2 - p_3) , \end{aligned}$$

where a and b are positive constants. To demonstrate the population dynamics of this system when $a = b = 0.5$, use the Runge–Kutta algorithm for systems with $h = 0.1$ to approximate the populations p_{i} over the time interval $[0, 10]$ under each of the following initial conditions:

(a) $p_1(0) = 1.0 , \quad p_2(0) = 0.1 , \quad p_3(0) = 0.1 .$
(b) $p_1(0) = 0.1 , \quad p_2(0) = 1.0 , \quad p_3(0) = 0.1 .$
(c) $p_1(0) = 0.1 , \quad p_2(0) = 0.1 , \quad p_3(0) = 1.0 .$

On the basis of the results of parts (a)–(c), decide what you think will happen to these populations as $t \to +\infty$.

30. Spring Pendulum. Let a mass be attached to one end of a spring with spring constant k and the other end attached to the ceiling. Let l_0 be the natural length of the spring and let $l(t)$ be its length at time t. If $\theta(t)$ is the angle between the pendulum and the vertical, then the motion of the spring pendulum is governed by the system

$$l''(t) - l(t)\theta'(t) - g\cos\theta(t) + \frac{k}{m}(l - l_0) = 0 ,$$
$$l^2(t)\theta''(t) + 2l(t)l'(t)\theta'(t) + gl(t)\sin\theta(t) = 0 .$$

Assume $g = 1$, $k = m = 1$, and $l_0 = 4$. When the system is at rest, $l = l_0 + mg/k = 5$.

(a) Describe the motion of the pendulum when $l(0) = 5.5, l'(0) = 0, \theta(0) = 0$, and $\theta'(0) = 0$.

(b) When the pendulum is both stretched and given an angular displacement, the motion of the pendulum is more complicated. Using the Runge–Kutta algorithm for systems with $h = 0.1$ to approximate the solution, sketch the graphs of the length l and the angular displacement θ on the interval $[0, 10]$ if $l(0) = 5.5$, $l'(0) = 0, \theta(0) = 0.5$, and $\theta'(0) = 0$.

CHAPTER 6

Theory of Higher-Order Linear Differential Equations

In this chapter we discuss the basic theory of linear higher-order differential equations. The material is a generalization of the results we obtained in Chapter 4 for second-order constant-coefficient equations. In the statements and proofs of these results, we use concepts usually covered in an elementary linear algebra course—namely, linear dependence, determinants, and methods for solving systems of linear equations. These concepts also arise in the matrix approach for solving systems of differential equations and are discussed in Chapter 9, which includes a brief review of linear algebraic equations and determinants.

Since this chapter is more mathematically oriented—that is, not tied to any particular physical application—we revert to the customary practice of calling the independent variable "x" and the dependent variable "y."

6.1 Basic Theory of Linear Differential Equations

A *linear* differential equation of order n is an equation that can be written in the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_0(x)y(x) = b(x)\,, \tag{1}$$

where $a_0(x), a_1(x), \ldots, a_n(x)$ and $b(x)$ depend only on x, not y. When $a_0, a_1, \ldots, a_n$ are all constants, we say equation (1) has **constant coefficients;** otherwise it has **variable coefficients**. If $b(x) \equiv 0$, equation (1) is called **homogeneous;** otherwise it is **nonhomogeneous**.

In developing a basic theory, we assume that $a_0(x), a_1(x), \ldots, a_n(x)$ and $b(x)$ are all continuous on an interval I and $a_n(x) \neq 0$ on I. Then, on dividing by $a_n(x)$, we can rewrite (1) in the **standard form**

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x)\,, \tag{2}$$

where the functions $p_1(x), \ldots, p_n(x)$, and $g(x)$ are continuous on I.

For a linear higher-order differential equation, the initial value problem always has a unique solution.

Existence and Uniqueness

Theorem 1. Suppose $p_1(x), \ldots, p_n(x)$ and $g(x)$ are each continuous on an interval (a, b) that contains the point x_0. Then, for any choice of the initial values $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$, there exists a unique solution $y(x)$ on the whole interval (a, b) to the initial value problem

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x)\,, \tag{3}$$

$$y(x_0) = \gamma_0, y'(x_0) = \gamma_1, \ldots, y^{(n-1)}(x_0) = \gamma_{n-1}\,. \tag{4}$$

The proof of Theorem 1 can be found in Chapter 13.†

Example 1 For the initial value problem

(5) $$x(x-1)y''' - 3xy'' + 6x^2y' - (\cos x)y = \sqrt{x+5}\,;$$

(6) $$y(x_0) = 1\,, \qquad y'(x_0) = 0\,, \qquad y''(x_0) = 7\,,$$

determine the values of x_0 and the intervals (a, b) containing x_0 for which Theorem 1 guarantees the existence of a unique solution on (a, b).

Solution Putting equation (5) in standard form, we find that $p_1(x) = -3/(x-1)$, $p_2(x) = 6x/(x-1)$, $p_3(x) = -(\cos x)/[x(x-1)]$, and $g(x) = \sqrt{x+5}/[x(x-1)]$. Now $p_1(x)$ and $p_2(x)$ are continuous on every interval not containing $x = 1$, while $p_3(x)$ is continuous on every interval not containing $x = 0$ or $x = 1$. The function $g(x)$ is not defined for $x < -5$, $x = 0$, and $x = 1$, but is continuous on $(-5, 0)$, $(0, 1)$, and $(1, \infty)$. Hence, the functions p_1, p_2, p_3, and g are *simultaneously* continuous on the intervals $(-5, 0)$, $(0, 1)$, and $(1, \infty)$. From Theorem 1 it follows that if we choose $x_0 \in (-5, 0)$, then there exists a unique solution to the initial value problem (5)–(6) on the whole interval $(-5, 0)$. Similarly, for $x_0 \in (0, 1)$, there is a unique solution on $(0, 1)$ and, for $x_0 \in (1, \infty)$, a unique solution on $(1, \infty)$. ◆

If we let the left-hand side of equation (3) define the *differential operator L*,

(7) $$L[y] := \frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_ny = (D^n + p_1D^{n-1} + \cdots + p_n)[y]\,,$$

then we can express equation (3) in the operator form

(8) $$L[y](x) = g(x)\,.$$

It is essential to keep in mind that L is a *linear* operator—that is, it satisfies

(9) $$L[y_1 + y_2 + \cdots + y_m] = L[y_1] + L[y_2] + \cdots + L[y_m]\,,$$

(10) $$L[cy] = cL[y] \qquad (c \text{ any constant})\,.$$

These are familiar properties for the differentiation operator D, from which (9) and (10) follow (see Problem 25).

As a consequence of this linearity, if $y_1, \ldots, y_m$ are solutions to the *homogeneous* equation

(11) $$L[y](x) = 0\,,$$

then any linear combination of these functions, $C_1y_1 + \cdots + C_my_m$, is also a solution, because

$$L[C_1y_1 + C_2y_2 + \cdots + C_my_m] = C_1 \cdot 0 + C_2 \cdot 0 + \cdots + C_m \cdot 0 = 0\,.$$

Imagine now that we have found n solutions $y_1, \ldots, y_n$ to the nth-order linear equation (11). Is it true that *every* solution to (11) can be represented by

(12) $$C_1y_1 + C_2y_2 + \cdots + C_ny_n$$

†All references to Chapters 11–13 refer to the expanded text, *Fundamentals of Differential Equations and Boundary Value Problems*, 7th ed.

for appropriate choices of the constants $C_1, \ldots, C_n$? The answer is yes, provided the solutions $y_1, \ldots, y_n$ satisfy a certain property that we now derive.

Let $\phi(x)$ be a solution to (11) on the interval (a, b) and let x_0 be a fixed number in (a, b). If it is possible to choose the constants $C_1, \ldots, C_n$ so that

$$\begin{aligned} C_1 y_1(x_0) &+ \cdots + C_n y_n(x_0) &&= \phi(x_0)\,, \\ C_1 y_1'(x_0) &+ \cdots + C_n y_n'(x_0) &&= \phi'(x_0)\,, \\ &\vdots \\ C_1 y_1^{(n-1)}(x_0) &+ \cdots + C_n y_n^{(n-1)}(x_0) &&= \phi^{(n-1)}(x_0)\,, \end{aligned} \tag{13}$$

then, since $\phi(x)$ and $C_1 y_1(x) + \cdots + C_n y_n(x)$ are two solutions satisfying the same initial conditions at x_0, the uniqueness conclusion of Theorem 1 gives

$$\phi(x) = C_1 y_1(x) + \cdots + C_n y_n(x) \tag{14}$$

for all x in (a, b).

The system (13) consists of n linear equations in the n unknowns $C_1, \ldots, C_n$. It has a unique solution for all possible values of $\phi(x_0), \phi'(x_0), \ldots, \phi^{(n-1)}(x_0)$ if and only if the determinant† of the coefficients is different from zero; that is, if and only if

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{vmatrix} \neq 0\,. \tag{15}$$

Hence, if $y_1, \ldots, y_n$ are solutions to equation (11) and there is some point x_0 in (a, b) such that (15) holds, then every solution $\phi(x)$ to (11) is a linear combination of $y_1, \ldots, y_n$. Before formulating this fact as a theorem, it is convenient to identify the determinant by name.

Wronskian

Definition 1. Let $f_1, \ldots, f_n$ be any n functions that are $(n-1)$ times differentiable. The function

$$W[f_1, \ldots, f_n](x) := \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix} \tag{16}$$

is called the **Wronskian** of $f_1, \ldots, f_n$.

We now state the representation theorem that we proved above for solutions to homogeneous linear differential equations.

†Readers unfamiliar with determinants and cofactor expansions can find these topics discussed in any linear algebra book (such as *Fundamentals of Matrix Analysis with Applications*, by Edward Barry Saff and Arthur David Snider, John Wiley & Sons, Hoboken, New Jersey, 2016.)

Representation of Solutions (Homogeneous Case)

Theorem 2. Let $y_1, \ldots, y_n$ be n solutions on (a, b) of

(17) $$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0\,,$$

where $p_1, \ldots, p_n$ are continuous on (a, b). If at some point x_0 in (a, b) these solutions satisfy

(18) $$W[y_1, \ldots, y_n](x_0) \neq 0\,,$$

then every solution of (17) on (a, b) can be expressed in the form

(19) $$y(x) = C_1y_1(x) + \cdots + C_ny_n(x)\,,$$

where $C_1, \ldots, C_n$ are constants.

The linear combination of $y_1, \ldots, y_n$ in (19), written with arbitrary constants $C_1, \ldots, C_n$, is referred to as a **general solution** to (17).

In linear algebra a set of m column vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$, each having m components, is said to be linearly dependent if and only if at least one of them can be expressed as a linear combination of the others.† A basic theorem then states that if a determinant is zero, its column vectors are linearly dependent, and conversely. So if a Wronskian of solutions to (17) is zero at a point x_0, one of its columns (the final column, say; we can always renumber!) equals a linear combination of the others:

(20) $$\begin{bmatrix} y_n(x_0) \\ y_n'(x_0) \\ \vdots \\ y_n^{(n-1)}(x_0) \end{bmatrix} = d_1\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \\ \vdots \\ y_1^{(n-1)}(x_0) \end{bmatrix} + d_2\begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \\ \vdots \\ y_2^{(n-1)}(x_0) \end{bmatrix} + \cdots + d_{n-1}\begin{bmatrix} y_{n-1}(x_0) \\ y_{n-1}'(x_0) \\ \vdots \\ y_{n-1}^{(n-1)}(x_0) \end{bmatrix}.$$

Now consider the two functions $y_n(x)$ and $[d_1y_1(x) + d_2y_2(x) + \cdots + d_{n-1}y_{n-1}(x)]$. They are both solutions to (17), and we can interpret (20) as stating that they satisfy the same initial conditions at $x = x_0$. By the uniqueness theorem, then, they are one and the same function:

(21) $$y_n(x) = d_1y_1(x) + d_2y_2(x) + \cdots + d_{n-1}y_{n-1}(x)$$

for all x in the interval I. Consequently, their derivatives are the same also, and so

(22) $$\begin{bmatrix} y_n(x) \\ y_n'(x) \\ \vdots \\ y_n^{(n-1)}(x) \end{bmatrix} = d_1\begin{bmatrix} y_1(x) \\ y_1'(x) \\ \vdots \\ y_1^{(n-1)}(x) \end{bmatrix} + d_2\begin{bmatrix} y_2(x) \\ y_2'(x) \\ \vdots \\ y_2^{(n-1)}(x) \end{bmatrix} + \cdots + d_{n-1}\begin{bmatrix} y_{n-1}(x) \\ y_{n-1}'(x) \\ \vdots \\ y_{n-1}^{(n-1)}(x) \end{bmatrix}$$

for all x in I. Hence, the final column of the Wronskian $W[y_1, y_2, \ldots, y_n]$ is *always* a linear combination of the other columns, and consequently the Wronskian is always zero.

In summary, the Wronskian of n solutions to the homogeneous equation (17) is either identically zero, or never zero, on the interval (a, b). We have also shown that, in the former

†This is equivalent to saying there exist constants $c_1, c_2, \ldots, c_m$ **not all zero**, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$ equals the zero vector.

case, (21) holds throughout (a, b). Such a relationship among functions is an extension of the notion of linear dependence introduced in Section 4.2. We employ the same nomenclature for the general case.

Linear Dependence of Functions

Definition 2. The m functions $f_1, f_2, \ldots, f_m$ are said to be **linearly dependent on an interval *I*** if at least one of them can be expressed as a linear combination of the others on I; equivalently, they are linearly dependent if there exist constants $c_1, c_2, \ldots, c_m$, *not all zero,* such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_m f_m(x) = 0 \tag{23}$$

for all x in I. Otherwise, they are said to be **linearly independent on *I***.

Example 2 Show that the functions $f_1(x) = e^x, f_2(x) = e^{-2x}$, and $f_3(x) = 3e^x - 2e^{-2x}$ are linearly dependent on $(-\infty, \infty)$.

Solution Obviously, f_3 is a linear combination of f_1 and f_2:

$$f_3(x) = 3e^x - 2e^{-2x} = 3f_1(x) - 2f_2(x) .$$

Note further that the corresponding identity $3f_1(x) - 2f_2(x) - f_3(x) = 0$ matches the pattern (23). Moreover, observe that f_1, f_2, and f_3 are *pairwise* linearly independent on $(-\infty, \infty)$, but this does not suffice to make the triplet independent. ◆

To prove that functions $f_1, f_2, \ldots, f_m$ are linearly *independent* on the interval (a, b), a convenient approach is the following: *Assume* that equation (23) holds on (a, b) and show that this forces $c_1 = c_2 = \cdots = c_m = 0$.

Example 3 Show that the functions $f_1(x) = x, f_2(x) = x^2$, and $f_3(x) = 1 - 2x^2$ are linearly independent on $(-\infty, \infty)$.

Solution Assume c_1, c_2, and c_3 are constants for which

$$c_1 x + c_2 x^2 + c_3(1 - 2x^2) = 0 \tag{24}$$

holds at every x. If we can prove that (24) implies $c_1 = c_2 = c_3 = 0$, then linear independence follows. Let's set $x = 0$, 1, and -1 in equation (24). These x values are, essentially, "picked out of a hat," but will get the job done. Substituting in (24) gives

$$\begin{aligned} c_3 &= 0 \quad (x = 0)\,, \\ c_1 + c_2 - c_3 &= 0 \quad (x = 1)\,, \\ -c_1 + c_2 - c_3 &= 0 \quad (x = -1)\,. \end{aligned} \tag{25}$$

When we solve this system (or compute the determinant of the coefficients), we find that the only possible solution is $c_1 = c_2 = c_3 = 0$. Consequently, the functions f_1, f_2, and f_3 are linearly independent on $(-\infty, \infty)$.

A neater solution is to note that if (24) holds for all x, so do its first and second derivatives. At $x = 0$ these conditions are $c_3 = 0$, $c_1 = 0$, and $2c_2 - 4c_3 = 0$. Obviously, each coefficient must be zero. ◆

Linear dependence of functions is, *prima facie*, different from linear dependence of vectors in the Euclidean space $\mathbf{R}^n$, because (23) is a functional equation that imposes a condition at every point of an interval. However, we have seen in (21) that when the functions are all solutions to the same homogeneous differential equation, linear dependence of the column vectors of the Wronskian (at *any* point x_0) implies linear dependence of the functions. The converse is also true, as demonstrated by (21) and (22). Theorem 3 summarizes our deliberations.

Linear Dependence and the Wronskian

Theorem 3. If $y_1, y_2, \ldots, y_n$ are n solutions to $y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0$ on the interval (a, b), with $p_1, p_2, \ldots, p_n$ continuous on (a, b), then the following statements are equivalent:

(i) $y_1, y_2, \ldots, y_n$ are linearly dependent on (a, b).
(ii) The Wronskian $W[y_1, y_2, \ldots, y_n](x_0)$ is zero at some point x_0 in (a, b).
(iii) The Wronskian $W[y_1, y_2, \ldots, y_n](x)$ is identically zero on (a, b).

The contrapositives of these statements are also equivalent:

(iv) $y_1, y_2, \ldots, y_n$ are linearly independent on (a, b).
(v) The Wronskian $W[y_1, y_2, \ldots, y_n](x_0)$ is nonzero at some point x_0 in (a, b).
(vi) The Wronskian $W[y_1, y_2, \ldots, y_n](x)$ is never zero on (a, b).

Whenever (iv), (v), or (vi) is met, $\{y_1, y_2, \ldots, y_n\}$ is called a **fundamental solution set** for (17) on (a, b).

The Wronskian is a curious function. If we take $W[f_1, f_2, \ldots, f_n](x)$ for n *arbitrary* functions, we simply get a function of x with no particularly interesting properties. But if the n functions are all solutions to the same homogeneous differential equation, then either it is identically zero or never zero. In fact, one can prove **Abel's identity** when the functions are all solutions to (17):

$$W[y_1, y_2, \ldots, y_n](x) = W[y_1, y_2, \ldots, y_n](x_0) \exp\left(\int_{x_0}^{x} p_1(t)\,dt \right), \tag{26}$$

which clearly exhibits this property. Problem 30 on page 327 outlines a proof of (26) for $n = 3$.[†]

It is useful to keep in mind that the following sets consist of functions that are linearly independent on every open interval (a, b):

$$\{1, x, x^2, \ldots, x^n\}$$
$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx\},$$
$$\{e^{\alpha_1 x}, e^{\alpha_2 x}, \ldots, e^{\alpha_n x}\} \quad (\alpha_i\text{'s distinct constants}).$$

[See Problems 27 and 28 on page 326, and Section 6.2 (page 327).]

If we combine the linearity (superposition) properties (9) and (10) with the representation theorem for solutions of the homogeneous equation, we obtain the following representation theorem for nonhomogeneous equations.

[†]See Problem 32, Exercises 4.7, for the case $n = 2$.

Representation of Solutions (Nonhomogeneous Case)

Theorem 4. Let $y_p(x)$ be a particular solution to the nonhomogeneous equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x) \tag{27}$$

on the interval (a, b) with $p_1, p_2, \ldots, p_n$ continuous on (a, b), and let $\{y_1, \ldots, y_n\}$ be a fundamental solution set for the corresponding homogeneous equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0 . \tag{28}$$

Then every solution of (27) on the interval (a, b) can be expressed in the form

$$\mathbf{y(x) = y_p(x) + C_1 y_1(x) + \cdots + C_n y_n(x)} . \tag{29}$$

Proof. Let $\phi(x)$ be any solution to (27). Because both $\phi(x)$ and $y_p(x)$ are solutions to (27), by linearity the difference $\phi(x) - y_p(x)$ is a solution to the homogeneous equation (28). It then follows from Theorem 2 that

$$\phi(x) - y_p(x) = C_1 y_1(x) + \cdots + C_n y_n(x)$$

for suitable constants $C_1, \ldots, C_n$. The last equation is equivalent to (29) $[$with $\phi(x)$ in place of $y(x)]$, so the theorem is proved. ◆

The linear combination of $y_p, y_1, \ldots, y_n$ in (29) written with arbitrary constants $C_1, \ldots, C_n$ is, for obvious reasons, referred to as a **general solution** to (27). Theorem 4 can be easily generalized. For example, if L denotes the operator appearing as the left-hand side in equation (27) and if $L[y_{p1}] = g_1$ and $L[y_{p2}] = g_2$, then any solution of $L[y] = c_1 g_1 + c_2 g_2$ can be expressed as

$$y(x) = c_1 y_{p1}(x) + c_2 y_{p2}(x) + C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) ,$$

for a suitable choice of the constants $C_1, C_2, \ldots, C_n$.

Example 4 Find a general solution on the interval $(-\infty, \infty)$ to

$$L[y] := y''' - 2y'' - y' + 2y = 2x^2 - 2x - 4 - 24e^{-2x} , \tag{30}$$

given that $y_{p1}(x) = x^2$ is a particular solution to $L[y] = 2x^2 - 2x - 4$, that $y_{p2}(x) = e^{-2x}$ is a particular solution to $L[y] = -12e^{-2x}$, and that $y_1(x) = e^{-x}$, $y_2(x) = e^x$, and $y_3(x) = e^{2x}$ are solutions to the corresponding homogeneous equation.

Solution We previously remarked that the functions e^{-x}, e^x, e^{2x} are linearly independent because the exponents -1, 1, and 2 are distinct. Since each of these functions is a solution to the corresponding homogeneous equation, then $\{e^{-x}, e^x, e^{2x}\}$ is a fundamental solution set. It now follows from the remarks above for nonhomogeneous equations that a general solution to (30) is

$$\begin{aligned} y(x) &= y_{p1} + 2y_{p2} + C_1 y_1 + C_2 y_2 + C_3 y_3 \\ &= x^2 + 2e^{-2x} + C_1 e^{-x} + C_2 e^x + C_3 e^{2x} . \end{aligned} \tag{31}$$ ◆

6.1 EXERCISES

In Problems 1–6, determine the largest interval (a, b) for which Theorem 1 guarantees the existence of a unique solution on (a, b) to the given initial value problem.

1. $xy''' - 3y' + e^x y = x^2 - 1\,;$
 $y(-2) = 1\,, \quad y'(-2) = 0\,, \quad y''(-2) = 2$
2. $y''' - \sqrt{x}y = \sin x\,;$
 $y(\pi) = 0\,, \quad y'(\pi) = 11\,, \quad y''(\pi) = 3$
3. $y''' - y'' + \sqrt{x-1}\,y = \tan x\,;$
 $y(5) = y'(5) = y''(5) = 1$
4. $x(x+1)y''' - 3xy' + y = 0\,;$
 $y(-1/2) = 1\,, \quad y'(-1/2) = y''(-1/2) = 0$
5. $x\sqrt{x+1}\,y''' - y' + xy = 0\,;$
 $y(1/2) = y'(1/2) = -1\,, \quad y''(1/2) = 1$
6. $(x^2 - 1)y''' + e^x y = \ln x\,;$
 $y(3/4) = 1\,, \quad y'(3/4) = y''(3/4) = 0$

In Problems 7–14, determine whether the given functions are linearly dependent or linearly independent on the specified interval. Justify your decisions.

7. $\{e^{3x}, e^{5x}, e^{-x}\}$ on $(-\infty, \infty)$
8. $\{x^2, x^2 - 1, 5\}$ on $(-\infty, \infty)$
9. $\{\sin^2 x, \cos^2 x, 1\}$ on $(-\infty, \infty)$
10. $\{\sin x, \cos x, \tan x\}$ on $(-\pi/2, \pi/2)$
11. $\{x^{-1}, x^{1/2}, x\}$ on $(0, \infty)$
12. $\{\cos 2x, \cos^2 x, \sin^2 x\}$ on $(-\infty, \infty)$
13. $\{x, x^2, x^3, x^4\}$ on $(-\infty, \infty)$
14. $\{x, xe^x, 1\}$ on $(-\infty, \infty)$

Using the Wronskian in Problems 15–18, verify that the given functions form a fundamental solution set for the given differential equation and find a general solution.

15. $y''' + 2y'' - 11y' - 12y = 0\,;$
 $\{e^{3x}, e^{-x}, e^{-4x}\}$
16. $y''' - y'' + 4y' - 4y = 0\,;$
 $\{e^x, \cos 2x, \sin 2x\}$
17. $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0\,, \quad x > 0\,;$
 $\{x, x^2, x^3\}$
18. $y^{(4)} - y = 0\,; \quad \{e^x, e^{-x}, \cos x, \sin x\}$

In Problems 19–22, a particular solution and a fundamental solution set are given for a nonhomogeneous equation and its corresponding homogeneous equation. (a) Find a general solution to the nonhomogeneous equation. (b) Find the solution that satisfies the specified initial conditions.

19. $y''' + y'' + 3y' - 5y = 2 + 6x - 5x^2\,;$
 $y(0) = -1\,, \quad y'(0) = 1\,, \quad y''(0) = -3\,;$
 $y_p = x^2\,; \quad \{e^x, e^{-x}\cos 2x, e^{-x}\sin 2x\}$
20. $xy''' - y'' = -2\,; \quad y(1) = 2\,, \quad y'(1) = -1\,,$
 $y''(1) = -4\,; \quad y_p = x^2\,; \quad \{1, x, x^3\}$
21. $x^3 y''' + xy' - y = 3 - \ln x\,, \qquad x > 0\,;$
 $y(1) = 3\,, \quad y'(1) = 3\,, \quad y''(1) = 0\,;$
 $y_p = \ln x\,; \quad \{x, x\ln x, x(\ln x)^2\}$
22. $y^{(4)} + 4y = 5\cos x\,;$
 $y(0) = 2\,, \quad y'(0) = 1\,, \quad y''(0) = -1\,,$
 $y'''(0) = -2\,; \quad y_p = \cos x\,;$
 $\{e^x\cos x, e^x\sin x, e^{-x}\cos x, e^{-x}\sin x\}$

23. Let $L[y] := y''' + y' + xy$, $y_1(x) := \sin x$, and $y_2(x) := x$. Verify that $L[y_1](x) = x\sin x$ and $L[y_2](x) = x^2 + 1$. Then use the superposition principle (linearity) to find a solution to the differential equation:
 (a) $L[y] = 2x\sin x - x^2 - 1\,.$
 (b) $L[y] = 4x^2 + 4 - 6x\sin x\,.$
24. Let $L[y] := y''' - xy'' + 4y' - 3xy$, $y_1(x) = \cos 2x$, and $y_2(x) := -1/3$. Verify that $L[y_1](x) = x\cos 2x$ and $L[y_2](x) = x$. Then use the superposition principle (linearity) to find a solution to the differential equation:
 (a) $L[y] = 7x\cos 2x - 3x\,.$
 (b) $L[y] = -6x\cos 2x + 11x\,.$
25. Prove that L defined in (7) is a linear operator by verifying that properties (9) and (10) hold for any n-times differentiable functions $y, y_1, \ldots, y_m$ on (a, b).
26. **Existence of Fundamental Solution Sets.** By Theorem 1, for each $j = 1, 2, \ldots, n$ there is a unique solution $y_j(x)$ to equation (17) satisfying the initial conditions
 $$y_j^{(k)}(x_0) = \begin{cases} 1\,, & \text{for } k = j-1\,, \\ 0\,, & \text{for } k \neq j-1, \quad 0 \le k \le n-1\,. \end{cases}$$
 (a) Show that $\{y_1, y_2, \ldots, y_n\}$ is a fundamental solution set for (17). [*Hint:* Write out the Wronskian at x_0.]
 (b) For given initial values $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$, express the solution $y(x)$ to (17) satisfying $y^{(k)}(x_0) = \gamma_k$, $k = 0, \ldots, n-1$, [as in equations (4)] in terms of this fundamental solution set.
27. Show that the set of functions $\{1, x, x^2, \ldots, x^n\}$, where n is a positive integer, is linearly independent on every open interval (a, b). [*Hint:* Use the fact that a polynomial of degree at most n has no more than n zeros unless it is identically zero.]
28. The set of functions
 $$\{1, \cos x, \sin x, \ldots, \cos nx, \sin nx\},$$
 where n is a positive integer, is linearly independent on every interval (a, b). Prove this in the special case $n = 2$ and $(a, b) = (-\infty, \infty)$.

29. **(a)** Show that if $f_1, \ldots, f_m$ are linearly independent on $(-1, 1)$, then they are linearly independent on $(-\infty, \infty)$.

(b) Give an example to show that if $f_1, \ldots, f_m$ are linearly independent on $(-\infty, \infty)$, then they need not be linearly independent on $(-1, 1)$.

30. To prove Abel's identity (26) for $n = 3$, proceed as follows:

(a) Let $W(x) := W[y_1, y_2, y_3](x)$. Use the product rule for differentiation to show

$$W'(x) = \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} .$$

(b) Show that the above expression reduces to

$$W'(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} . \tag{32}$$

(c) Since each y_i satisfies (17), show that

$$y_i^{(3)}(x) = -\sum_{k=1}^{3} p_k(x) y_i^{(3-k)}(x) \quad (i = 1, 2, 3) . \tag{33}$$

(d) Substituting the expressions in (33) into (32), show that

$$W'(x) = -p_1(x)W(x) . \tag{34}$$

(e) Deduce Abel's identity by solving the first-order differential equation (34).

31. Reduction of Order. If a nontrivial solution $f(x)$ is known for the *homogeneous* equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0 ,$$

the substitution $y(x) = v(x)f(x)$ can be used to reduce the order of the equation, as was shown in Section 4.7 for second-order equations. By completing the following steps, demonstrate the method for the third-order equation

$$y''' - 2y'' - 5y' + 6y = 0 , \tag{35}$$

given that $f(x) = e^x$ is a solution.

(a) Set $y(x) = v(x)e^x$ and compute y', y'', and y'''.

(b) Substitute your expressions from (a) into (35) to obtain a *second-order* equation in $w := v'$.

(c) Solve the second-order equation in part (b) for w and integrate to find v. Determine two linearly independent choices for v, say, v_1 and v_2.

(d) By part (c), the functions $y_1(x) = v_1(x)e^x$ and $y_2(x) = v_2(x)e^x$ are two solutions to (35). Verify that the three solutions e^x, $y_1(x)$, and $y_2(x)$ are linearly independent on $(-\infty, \infty)$.

32. Given that the function $f(x) = x$ is a solution to $y''' - x^2y' + xy = 0$, show that the substitution $y(x) = v(x)f(x) = v(x)x$ reduces this equation to $xw'' + 3w' - x^3w = 0$, where $w = v'$.

33. Use the reduction of order method described in Problem 31 to find three linearly independent solutions to $y''' - 2y'' + y' - 2y = 0$, given that $f(x) = e^{2x}$ is a solution.

34. Constructing Differential Equations. Given three functions $f_1(x), f_2(x), f_3(x)$ that are each three times differentiable and whose Wronskian is never zero on (a, b), show that the equation

$$\begin{vmatrix} f_1(x) & f_2(x) & f_3(x) & y \\ f_1'(x) & f_2'(x) & f_3'(x) & y' \\ f_1''(x) & f_2''(x) & f_3''(x) & y'' \\ f_1'''(x) & f_2'''(x) & f_3'''(x) & y''' \end{vmatrix} = 0$$

is a third-order linear differential equation for which $\{f_1, f_2, f_3\}$ is a fundamental solution set. What is the coefficient of y''' in this equation?

35. Use the result of Problem 34 to construct a third-order differential equation for which $\{x, \sin x, \cos x\}$ is a fundamental solution set.

6.2 Homogeneous Linear Equations with Constant Coefficients

Our goal in this section is to obtain a general solution to an nth-order linear differential equation with constant coefficients. Based on the experience gained with second-order equations in Section 4.2, you should have little trouble guessing the form of such a solution. However, our interest here is to help you understand *why* these techniques work. This is done using an operator approach—a technique that is useful in tackling many other problems in analysis such as solving partial differential equations.

Let's consider the homogeneous linear nth-order differential equation

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = 0 , \tag{1}$$

where $a_n(\neq 0), a_{n-1}, \ldots, a_0$ are real constants.† Since constant functions are everywhere continuous, equation (1) has solutions defined for all x in $(-\infty, \infty)$ (recall Theorem 1 in Section 6.1). If we can find n linearly independent solutions to (1) on $(-\infty, \infty)$, say, $y_1, \ldots, y_n$, then we can express a general solution to (1) in the form

$$y(x) = C_1 y_1(x) + \cdots + C_n y_n(x) , \tag{2}$$

with $C_1, \ldots, C_n$ as arbitrary constants.

To find these n linearly independent solutions, we capitalize on our previous success with second-order equations. Namely, experience suggests that we begin by trying a function of the form $y = e^{rx}$.

If we let L be the differential operator defined by the left-hand side of (1), that is,

$$L[y] := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y , \tag{3}$$

then we can write (1) in the operator form

$$L[y](x) = 0 . \tag{4}$$

For $y = e^{rx}$, we find

$$\begin{aligned} L[e^{rx}](x) &= a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_0 e^{rx} \\ &= e^{rx}(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0) = e^{rx} P(r) , \end{aligned} \tag{5}$$

where $P(r)$ is the polynomial $a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0$. Thus, e^{rx} is a solution to equation (4), provided r is a root of the **auxiliary** (or **characteristic**) **equation**

$$P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0 . \tag{6}$$

According to the fundamental theorem of algebra, the auxiliary equation has n roots (counting multiplicities), which may be either real or complex. However, there are no formulas for determining the zeros of an arbitrary polynomial of degree greater than four, although if we can determine one zero r_1, then we can divide out the factor $r - r_1$ and be left with a polynomial of lower degree. (For convenience, we have chosen most of our examples and exercises so that 0, ± 1, or ± 2 are zeros of any polynomial of degree greater than two that we must factor.) When a zero cannot be exactly determined, numerical algorithms such as Newton's method or the quotient-difference algorithm can be used to compute approximate roots of the polynomial equation.‡ Some pocket calculators even have these algorithms built in.

We proceed to discuss the various possibilities.

Distinct Real Roots

If the roots $r_1, \ldots, r_n$ of the auxiliary equation (6) are real and distinct, then n solutions to equation (1) are

$$y_1(x) = e^{r_1 x}, y_2(x) = e^{r_2 x}, \ldots, y_n(x) = e^{r_n x} . \tag{7}$$

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†*Historical Footnote:* In a letter to John Bernoulli dated September 15, 1739, Leonhard Euler claimed to have solved the general case of the homogeneous linear nth-order equation with constant coefficients.

‡See, for example, *Applied and Computational Complex Analysis*, by P. Henrici (Wiley-Interscience, New York, 1993), Volume 1, or *Numerical Analysis*, 9th ed., by R. L. Burden and J. D. Faires (Brooks/Cole Cengage Learning, 2011).

As stated in the previous section, these functions are linearly independent on $(-\infty, \infty)$, a fact that we now officially verify. Let's assume that $c_1, \ldots, c_n$ are constants such that

$$c_1 e^{r_1 x} + \cdots + c_n e^{r_n x} = 0 \tag{8}$$

for all x in $(-\infty, \infty)$. Our goal is to prove that $c_1 = c_2 = \cdots = c_n = 0$.

One way to show this is to construct a linear operator L_k that annihilates (maps to zero) everything on the left-hand side of (8) except the kth term. For this purpose, we note that since $r_1, \ldots, r_n$ are the zeros of the auxiliary polynomial $P(r)$, then $P(r)$ can be factored as

$$P(r) = a_n(r - r_1)\cdots(r - r_n)\ . \tag{9}$$

Consequently, the operator $L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y$ can be expressed in terms of the differentiation operator D as the following composition:[†]

$$L = P(D) = a_n(D - r_1)\cdots(D - r_n)\ . \tag{10}$$

We now construct the polynomial $P_k(r)$ by deleting the factor $(r - r_k)$ from $P(r)$. Then we set $L_k := P_k(D)$; that is,

$$L_k := P_k(D) = a_n(D - r_1)\cdots(D - r_{k-1})(D - r_{k+1})\cdots(D - r_n)\ . \tag{11}$$

Applying L_k to both sides of (8), we get, via linearity,

$$c_1 L_k[e^{r_1 x}] + \cdots + c_n L_k[e^{r_n x}] = 0\ . \tag{12}$$

Also, since $L_k = P_k(D)$, we find [just as in equation (5)] that $L_k[e^{rx}](x) = e^{rx} P_k(r)$ for all r. Thus (12) can be written as

$$c_1 e^{r_1 x} P_k(r_1) + \cdots + c_n e^{r_n x} P_k(r_n) = 0\ ,$$

which simplifies to

$$c_k e^{r_k x} P_k(r_k) = 0\ , \tag{13}$$

because $P_k(r_i) = 0$ for $i \neq k$. Since r_k is not a root of $P_k(r)$, then $P_k(r_k) \neq 0$. It now follows from (13) that $c_k = 0$. But as k is arbitrary, all the constants $c_1, \ldots, c_n$ must be zero. Thus, $y_1(x), \ldots, y_n(x)$ as given in (7) are linearly independent. (See Problem 26 for an alternative proof.)

We have proved that, in the case of n distinct real roots, a general solution to (1) is

$$\boldsymbol{y(x) = C_1 e^{r_1 x} + \cdots + C_n e^{r_n x}}\ , \tag{14}$$

where $C_1, \ldots, C_n$ are arbitrary constants.

Example 1 Find a general solution to

$$y''' - 2y'' - 5y' + 6y = 0\ . \tag{15}$$

Solution The auxiliary equation is

$$r^3 - 2r^2 - 5r + 6 = 0\ . \tag{16}$$

By inspection we find that $r = 1$ is a root. Then, using polynomial division, we get

$$r^3 - 2r^2 - 5r + 6 = (r - 1)(r^2 - r - 6)\ ,$$

which further factors into $(r - 1)(r + 2)(r - 3)$. Hence the roots of equation (16) are $r_1 = 1$, $r_2 = -2$, $r_3 = 3$. Since these roots are real and distinct, a general solution to (15) is

$$y(x) = C_1 e^{x} + C_2 e^{-2x} + C_3 e^{3x}\ . \quad \blacklozenge$$

[†] *Historical Footnote:* The symbolic notation $P(D)$ was introduced by Augustin Cauchy in 1827.

Complex Roots

If $\alpha + i\beta$ $(\alpha, \beta$ real$)$ is a complex root of the auxiliary equation (6), then so is its complex conjugate $\alpha - i\beta$, since the coefficients of $P(r)$ are real-valued (see Problem 24). If we accept complex-valued functions as solutions, then both $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$ are solutions to (1). Moreover, if there are no repeated roots, then a general solution to (1) is again given by (14). To find two real-valued solutions corresponding to the roots $\alpha \pm i\beta$, we can just take the real and imaginary parts of $e^{(\alpha+i\beta)x}$. That is, since

$$e^{(\alpha+i\beta)x} = e^{\alpha x}\cos\beta x + ie^{\alpha x}\sin\beta x\,, \tag{17}$$

then two linearly independent solutions to (1) are

$$e^{\alpha x}\cos\beta x\,, \qquad e^{\alpha x}\sin\beta x\,. \tag{18}$$

In fact, using these solutions in place of $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$ in (14) preserves the linear independence of the set of n solutions. Thus, treating each of the conjugate pairs of roots in this manner, we obtain a real-valued general solution to (1).

Example 2 Find a general solution to

$$y''' + y'' + 3y' - 5y = 0\,. \tag{19}$$

Solution The auxiliary equation is

$$r^3 + r^2 + 3r - 5 = (r-1)(r^2 + 2r + 5) = 0\,, \tag{20}$$

which has distinct roots $r_1 = 1$, $r_2 = -1 + 2i$, $r_3 = -1 - 2i$. Thus, a general solution is

$$y(x) = C_1e^x + C_2e^{-x}\cos 2x + C_3e^{-x}\sin 2x\,. \tag{21}$$ ◆

Repeated Roots

If r_1 is a root of multiplicity m, then the n solutions given in (7) are not even distinct, let alone linearly independent. Recall that for a second-order equation, when we had a repeated root r_1 to the auxiliary equation, we obtained two linearly independent solutions by taking e^{r_1x} and xe^{r_1x}. So if r_1 is a root of (6) of multiplicity m, we might expect that m linearly independent solutions are

$$e^{r_1x}\,, \quad xe^{r_1x}\,, \quad x^2e^{r_1x}\,, \quad \ldots\,, \quad x^{m-1}e^{r_1x}\,. \tag{22}$$

To see that this is the case, observe that if r_1 is a root of multiplicity m, then the auxiliary equation can be written in the form

$$a_n(r-r_1)^m(r-r_{m+1})\cdots(r-r_n) = (r-r_1)^m\widetilde{P}(r) = 0\,, \tag{23}$$

where $\widetilde{P}(r) \coloneqq a_n(r-r_{m+1})\cdots(r-r_n)$ and $\widetilde{P}(r_1) \neq 0$. With this notation, we have the identity

$$L[e^{rx}](x) = e^{rx}(r-r_1)^m\widetilde{P}(r) \tag{24}$$

[see (5) on page 328]. Setting $r = r_1$ in (24), we again see that e^{r_1x} is a solution to $L[y] = 0$.

To find other solutions, we take the kth partial derivative with respect to r of both sides of (24):

$$\frac{\partial^k}{\partial r^k}L[e^{rx}](x) = \frac{\partial^k}{\partial r^k}[e^{rx}(r-r_1)^m\widetilde{P}(r)]\,. \tag{25}$$

Carrying out the differentiation on the right-hand side of (25), we find that the resulting expression will still have $(r-r_1)$ as a factor, provided $k \leq m-1$. Thus, setting $r = r_1$ in (25) gives

(26) $$\frac{\partial^k}{\partial r^k} L[e^{rx}](x)\bigg|_{r=r_1} = 0 \quad \text{if} \quad k \le m-1 \,.$$

Now notice that the function e^{rx} has continuous partial derivatives of all orders with respect to r and x. Hence, for mixed partial derivatives of e^{rx}, it makes no difference whether the differentiation is done first with respect to x, then with respect to r, or vice versa. Since L involves derivatives with respect to x, this means we can interchange the order of differentiation in (26) to obtain

$$L\left[\frac{\partial^k}{\partial r^k}(e^{rx})\bigg|_{r=r_1}\right](x) = 0 \,.$$

Thus,

(27) $$\frac{\partial^k}{\partial r^k}(e^{rx})\bigg|_{r=r_1} = x^k e^{r_1 x}$$

will be a solution to (1) for $k = 0, 1, \ldots, m-1$. So m distinct solutions to (1), due to the root $r = r_1$ of multiplicity m, are indeed given by (22). We leave it as an exercise to show that the m functions in (22) are linearly independent on $(-\infty, \infty)$ (see Problem 25).

If $\alpha + i\beta$ is a repeated complex root of multiplicity m, then we can replace the $2m$ complex-valued functions

$$\begin{aligned} &e^{(\alpha+i\beta)x}, \quad xe^{(\alpha+i\beta)x}, \quad \ldots, \quad x^{m-1}e^{(\alpha+i\beta)x}, \\ &e^{(\alpha-i\beta)x}, \quad xe^{(\alpha-i\beta)x}, \quad \ldots, \quad x^{m-1}e^{(\alpha-i\beta)x} \end{aligned}$$

by the $2m$ linearly independent real-valued functions

(28) $$\begin{aligned} &e^{\alpha x}\cos\beta x\,, \quad xe^{\alpha x}\cos\beta x\,, \quad \ldots, \quad x^{m-1}e^{\alpha x}\cos\beta x\,, \\ &e^{\alpha x}\sin\beta x\,, \quad xe^{\alpha x}\sin\beta x\,, \quad \ldots, \quad x^{m-1}e^{\alpha x}\sin\beta x\,. \end{aligned}$$

Using the results of the three cases discussed above, we can obtain a set of n linearly independent solutions that yield a real-valued general solution for (1).

Example 3 Find a general solution to

(29) $$y^{(4)} - y^{(3)} - 3y'' + 5y' - 2y = 0 \,.$$

Solution The auxiliary equation is

$$r^4 - r^3 - 3r^2 + 5r - 2 = (r-1)^3(r+2) = 0 \,,$$

which has roots $r_1 = 1$, $r_2 = 1$, $r_3 = 1$, $r_4 = -2$. Because the root at 1 has multiplicity 3, a general solution is

(30) $$y(x) = C_1e^x + C_2xe^x + C_3x^2e^x + C_4e^{-2x} \,. \;\blacklozenge$$

Example 4 Find a general solution to

(31) $$y^{(4)} - 8y^{(3)} + 26y'' - 40y' + 25y = 0 \,,$$

whose auxiliary equation can be factored as

(32) $$r^4 - 8r^3 + 26r^2 - 40r + 25 = (r^2 - 4r + 5)^2 = 0 \,.$$

Solution The auxiliary equation (32) has repeated complex roots: $r_1 = 2 + i$, $r_2 = 2 + i$, $r_3 = 2 - i$, and $r_4 = 2 - i$. Hence, a general solution is

$$y(x) = C_1e^{2x}\cos x + C_2xe^{2x}\cos x + C_3e^{2x}\sin x + C_4xe^{2x}\sin x \,. \;\blacklozenge$$

6.2 EXERCISES

In Problems 1–14, find a general solution for the differential equation with x as the independent variable.

1. $y''' + 2y'' - 8y' = 0$
2. $y''' - 3y'' - y' + 3y = 0$
3. $6z''' + 7z'' - z' - 2z = 0$
4. $y''' + 2y'' - 19y' - 20y = 0$
5. $y''' + 3y'' + 28y' + 26y = 0$
6. $y''' - y'' + 2y = 0$
7. $2y''' - y'' - 10y' - 7y = 0$
8. $y''' + 5y'' - 13y' + 7y = 0$
9. $u''' - 9u'' + 27u' - 27u = 0$
10. $y''' + 3y'' - 4y' - 6y = 0$
11. $y^{(4)} + 4y''' + 6y'' + 4y' + y = 0$
12. $y''' + 5y'' + 3y' - 9y = 0$
13. $y^{(4)} + 4y'' + 4y = 0$
14. $y^{(4)} + 2y''' + 10y'' + 18y' + 9y = 0$
    [*Hint:* $y(x) = \sin 3x$ is a solution.]

In Problems 15–18, find a general solution to the given homogeneous equation.

15. $(D-1)^2(D+3)(D^2+2D+5)^2[y] = 0$
16. $(D+1)^2(D-6)^3(D+5)(D^2+1)(D^2+4)[y] = 0$
17. $(D+4)(D-3)(D+2)^3(D^2+4D+5)^2D^5[y] = 0$
18. $(D-1)^3(D-2)(D^2+D+1)$
    $\cdot (D^2+6D+10)^3[y] = 0$

In Problems 19–21, solve the given initial value problem.

19. $y''' - y'' - 4y' + 4y = 0\,;$
    $y(0) = -4\,, \quad y'(0) = -1\,, \quad y''(0) = -19$
20. $y''' + 7y'' + 14y' + 8y = 0\,;$
    $y(0) = 1\,, \quad y'(0) = -3\,, \quad y''(0) = 13$
21. $y''' - 4y'' + 7y' - 6y = 0\,;$
    $y(0) = 1\,, \quad y'(0) = 0\,, \quad y''(0) = 0$

In Problems 22 and 23, find a general solution for the given linear system using the elimination method of Section 5.2.

22. $d^2x/dt^2 - x + 5y = 0\,,$
    $2x + d^2y/dt^2 + 2y = 0$
23. $d^3x/dt^3 - x + dy/dt + y = 0\,,$
    $dx/dt - x + y = 0$
24. Let $P(r) = a_n r^n + \cdots + a_1 r + a_0$ be a polynomial with real coefficients $a_n, \ldots, a_0$. Prove that if r_1 is a zero of $P(r)$, then so is its complex conjugate $\bar{r}_1$. [*Hint:* Show that $\overline{P(r)} = P(\bar{r})$, where the bar denotes complex conjugation.]
25. Show that the m functions $e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}$ are linearly independent on $(-\infty, \infty)$. [*Hint:* Show that these functions are linearly independent if and only if $1, x, \ldots, x^{m-1}$ are linearly independent.]
26. As an alternative proof that the functions $e^{r_1x}, e^{r_2x}, \ldots, e^{r_nx}$ are linearly independent on $(-\infty, \infty)$ when $r_1, r_2, \ldots, r_n$ are distinct, assume

    $$C_1e^{r_1x} + C_2e^{r_2x} + \cdots + C_ne^{r_nx} = 0 \tag{33}$$

    holds for all x in $(-\infty, \infty)$ and proceed as follows:

    (a) Because the r_i's are distinct we can (if necessary) relabel them so that

    $$r_1 > r_2 > \cdots > r_n\,.$$

    Divide equation (33) by e^{r_1x} to obtain

    $$C_1 + C_2\frac{e^{r_2x}}{e^{r_1x}} + \cdots + C_n\frac{e^{r_nx}}{e^{r_1x}} = 0\,.$$

    Now let $x \to +\infty$ on the left-hand side to obtain $C_1 = 0$.

    (b) Since $C_1 = 0$, equation (33) becomes

    $$C_2e^{r_2x} + C_3e^{r_3x} + \cdots + C_ne^{r_nx} = 0$$

    for all x in $(-\infty, \infty)$. Divide this equation by e^{r_2x} and let $x \to +\infty$ to conclude that $C_2 = 0$.

    (c) Continuing in the manner of (b), argue that all the coefficients, $C_1, C_2, \ldots, C_n$ are zero and hence $e^{r_1x}, e^{r_2x}, \ldots, e^{r_nx}$ are linearly independent on $(-\infty, \infty)$.
27. Find a general solution to

    $$y^{(4)} + 2y''' - 3y'' - y' + \frac{1}{2}y = 0$$

    by using Newton's method (Appendix B) or some other numerical procedure to approximate the roots of the auxiliary equation.
28. Find a general solution to $y''' - 3y' - y = 0$ by using Newton's method or some other numerical procedure to approximate the roots of the auxiliary equation.
29. Find a general solution to

    $$y^{(4)} + 2y^{(3)} + 4y'' + 3y' + 2y = 0$$

by using Newton's method to approximate numerically the roots of the auxiliary equation. [*Hint:* To find complex roots, use the Newton recursion formula $z_{n+1} = z_n - f(z_n)/f'(z_n)$ and start with a *complex* initial guess z_0.]

30. **(a)** Derive the form

$$y(x) = A_1 e^x + A_2 e^{-x} + A_3 \cos x + A_4 \sin x$$

for the general solution to the equation $y^{(4)} = y$, from the observation that the fourth roots of unity are $1, -1, i$, and $-i$.

(b) Derive the form

$$y(x) = A_1 e^x + A_2 e^{-x/2} \cos\left(\sqrt{3}x/2\right) + A_3 e^{-x/2} \sin\left(\sqrt{3}x/2\right)$$

for the general solution to the equation $y^{(3)} = y$, from the observation that the cube roots of unity are $1, e^{i2\pi/3}$, and $e^{-i2\pi/3}$.

31. **Higher-Order Cauchy–Euler Equations.** A differential equation that can be expressed in the form

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_0 y(x) = 0,$$

where $a_n, a_{n-1}, \ldots, a_0$ are constants, is called a homogeneous **Cauchy–Euler** equation. (The second-order case is discussed in Section 4.7.) Use the substitution $y = x^r$ to help determine a fundamental solution set for the following Cauchy–Euler equations:

(a) $x^3 y''' + x^2 y'' - 2xy' + 2y = 0, \; x > 0$.

(b) $x^4 y^{(4)} + 6x^3 y''' + 2x^2 y'' - 4xy' + 4y = 0, \; x > 0$

(c) $x^3 y''' - 2x^2 y'' + 13xy' - 13y = 0, \; x > 0$

[*Hint:* $x^{\alpha + i\beta} = e^{(\alpha + i\beta)\ln x} = x^{\alpha}\{\cos(\beta \ln x) + i \sin(\beta \ln x)\}$.]

32. Let $y(x) = Ce^{rx}$, where $C\ (\neq 0)$ and r are real numbers, be a solution to a differential equation. Suppose we cannot determine r exactly but can only approximate it by $\tilde{r}$. Let $\tilde{y}(x) := Ce^{\tilde{r}x}$ and consider the error $|y(x) - \tilde{y}(x)|$.

(a) If r and $\tilde{r}$ are positive, $r \neq \tilde{r}$, show that the error grows exponentially large as x approaches $+\infty$.

(b) If r and $\tilde{r}$ are negative, $r \neq \tilde{r}$, show that the error goes to zero exponentially as x approaches $+\infty$.

33. On a smooth horizontal surface, a mass of m_1 kg is attached to a fixed wall by a spring with spring constant k_1 N/m. Another mass of m_2 kg is attached to the first object by a spring with spring constant k_2 N/m. The objects are aligned horizontally so that the springs are their natural lengths. As we showed in Section 5.6, this coupled mass–spring system is governed by the system of differential equations

$$m_1 \frac{d^2x}{dt^2} + (k_1 + k_2)x - k_2 y = 0, \tag{34}$$

$$m_2 \frac{d^2y}{dt^2} - k_2 x + k_2 y = 0. \tag{35}$$

Let's assume that $m_1 = m_2 = 1$, $k_1 = 3$, and $k_2 = 2$. If both objects are displaced 1 m to the right of their equilibrium positions (compare Figure 5.26, page 283) and then released, determine the equations of motion for the objects as follows:

(a) Show that $x(t)$ satisfies the equation

$$x^{(4)}(t) + 7x''(t) + 6x(t) = 0. \tag{36}$$

(b) Find a general solution $x(t)$ to (36).

(c) Substitute $x(t)$ back into (34) to obtain a general solution for $y(t)$.

(d) Use the initial conditions to determine the solutions, $x(t)$ and $y(t)$, which are the equations of motion.

34. Suppose the two springs in the coupled mass–spring system discussed in Problem 33 are switched, giving the new data $m_1 = m_2 = 1$, $k_1 = 2$, and $k_2 = 3$. If both objects are now displaced 1 m to the right of their equilibrium positions and then released, determine the equations of motion of the two objects.

35. **Vibrating Beam.** In studying the transverse vibrations of a beam, one encounters the homogeneous equation

$$EI\frac{d^4y}{dx^4} - ky = 0,$$

where $y(x)$ is related to the displacement of the beam at position x, the constant E is Young's modulus, I is the area moment of inertia, and k is a parameter. Assuming E, I, and k are positive constants, find a general solution in terms of sines, cosines, hyperbolic sines, and hyperbolic cosines.

6.3 Undetermined Coefficients and the Annihilator Method

In Sections 4.4 and 4.5 we mastered an easy method for obtaining a particular solution to a nonhomogeneous linear second-order constant-coefficient equation,

$$L[y] = (aD^2 + bD + c)[y] = f(x) , \tag{1}$$

when the nonhomogeneity $f(x)$ had a particular form (namely, a product of a polynomial, an exponential, and a sinusoid). Roughly speaking, we were motivated by the observation that if a function f, of this type, resulted from operating on y with an operator L of the form $(aD^2 + bD + c)$, then we must have started with a y of the same type. So we solved (1) by postulating a solution form y_p that resembled f, but with *undetermined coefficients,* and we inserted this form into the equation to fix the values of these coefficients. Eventually, we realized that we had to make certain accommodations when f was a solution to the homogeneous equation $L[y] = 0$.

In this section we are going to reexamine the method of undetermined coefficients from another, more rigorous, point of view—partly with the objective of tying up the loose ends in our previous exposition and more importantly with the goal of extending the method to higher-order equations (with constant coefficients). At the outset we'll describe the new point of view that will be adopted for the analysis. Then we illustrate its implications and ultimately derive a simplified set of rules for its implementation: rules that justify and extend the procedures of Section 4.4. The rigorous approach is known as the **annihilator method**.

The first premise of the annihilator method is the observation, gleaned from the analysis of the previous section, that all of the "suitable types" of nonhomogeneities $f(x)$ (products of polynomials times exponentials times sinusoids) are themselves *solutions to* ***homogeneous*** *differential equations with constant coefficients*. Observe the following:

(i) Any nonhomogeneous term of the form $f(x) = e^{rx}$ satisfies $(D - r)[f] = 0$.
(ii) Any nonhomogeneous term of the form $f(x) = x^k e^{rx}$ satisfies $(D - r)^m[f] = 0$ for $k = 0, 1, \ldots, m - 1$.
(iii) Any nonhomogeneous term of the form $f(x) = \cos\beta x$ or $\sin\beta x$ satisfies $(D^2 + \beta^2)[f] = 0$.
(iv) Any nonhomogeneous term of the form $f(x) = x^k e^{\alpha x}\cos\beta x$ or $x^k e^{\alpha x}\sin\beta x$ satisfies $[(D - \alpha)^2 + \beta^2]^m[f] = 0$ for $k = 0, 1, \ldots, m - 1$.

In other words, each of these nonhomogeneities is *annihilated* by a differential operator with constant coefficients.

Annihilator

Definition 3. A linear differential operator A is said to **annihilate** a function f if

$$A[f](x) = 0 , \tag{2}$$

for all x. That is, A annihilates f if f is a solution to the homogeneous linear differential equation (2) on $(-\infty, \infty)$.

Example 1 Find a differential operator that annihilates

$$6xe^{-4x} + 5e^{x}\sin 2x\,. \tag{3}$$

Solution Consider the two functions whose sum appears in (3). Observe that $(D+4)^2$ annihilates the function $f_1(x) := 6xe^{-4x}$. Further, $f_2(x) := 5e^{x}\sin 2x$ is annihilated by the operator $(D-1)^2+4$. Hence, the composite operator

$$A := (D+4)^2[(D-1)^2+4]\,,$$

which is the same as the operator

$$[(D-1)^2+4](D+4)^2\,,$$

annihilates both f_1 and f_2. But then, by linearity, A also annihilates the sum f_1+f_2. ◆

We now show how annihilators can be used to determine particular solutions to certain *non-*homogeneous equations. Consider the nth-order differential equation with constant coefficients

$$a_n y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_0 y(x) = f(x)\,, \tag{4}$$

which can be written in the operator form

$$L[y](x) = f(x)\,, \tag{5}$$

where

$$L = a_n D^n + a_{n-1}D^{n-1} + \cdots + a_0\,.$$

Assume that A is a linear differential operator with constant coefficients that annihilates $f(x)$. Then

$$A\big[L[y]\big](x) = A[f](x) = 0\,,$$

so any solution to (5) is also a solution to the *homogeneous* equation

$$AL[y](x) = 0\,, \tag{6}$$

involving the composition of the operators A and L. But we are experts on *homogeneous* differential equations (with constant coefficients)! In particular, we can use the methods of Section 6.2 to write down a general solution of (6). From this we can deduce the *form* of a particular solution to (5). Let's look at some examples and then summarize our findings. The differential equation in the next example is second order, so we will be able to see exactly how the annihilator method is related to the techniques of Sections 4.4 and 4.5.

Example 2 Find a general solution to

$$y'' - y = xe^{x} + \sin x\,. \tag{7}$$

Solution First let's solve this by the methods of Sections 4.4 and 4.5, to get a perspective for the annihilator method. The homogeneous equation corresponding to (7) is $y''-y=0$, with the general solution $C_1e^{-x}+C_2e^{x}$. Since e^{x} is a solution of the homogeneous equation, the nonhomogeneity xe^{x} demands a solution form $x(C_3+C_4x)e^{x}$. To accommodate the nonhomogeneity $\sin x$, we need an undetermined coefficient form $C_5\sin x + C_6\cos x$. Values for C_3 through C_6 in the *particular solution* are determined by substitution:

$$\begin{aligned} y_p'' - y_p = {} & [C_3xe^{x} + C_4x^2e^{x} + C_5\sin x + C_6\cos x]'' \\ & -[C_3xe^{x} + C_4x^2e^{x} + C_5\sin x + C_6\cos x] = \sin x + xe^{x}\,, \end{aligned}$$

eventually leading to the conclusion $C_3 = -1/4$, $C_4 = 1/4$, $C_5 = -1/2$, and $C_6 = 0$. Thus (for future reference), a general solution to (7) is

$$\textbf{(8)} \qquad y(x) = C_1 e^{-x} + C_2 e^{x} + x\left(-\frac{1}{4} + \frac{1}{4}x\right)e^{x} - \frac{1}{2}\sin x\,.$$

For the annihilator method, observe that (D^2+1) annihilates $\sin x$ and $(D-1)^2$ annihilates xe^x. Therefore, any solution to (7), expressed for convenience in operator form as $(D^2-1)[y](x) = xe^x + \sin x$, is annihilated by the composition $(D^2+1)(D-1)^2(D^2-1)$; that is, it satisfies the constant-coefficient homogeneous equation

$$\textbf{(9)} \qquad (D^2+1)(D-1)^2(D^2-1)[y] = (D+1)(D-1)^3(D^2+1)[y] = 0\,.$$

From Section 6.2 we deduce that the general solution to (9) is given by

$$\textbf{(10)} \qquad y = C_1 e^{-x} + C_2 e^{x} + C_3 x e^{x} + C_4 x^2 e^{x} + C_5 \sin x + C_6 \cos x\,.$$

This is precisely the solution form generated by the methods of Chapter 4; the first two terms are the general solution to the associated homogeneous equation, and the remaining four terms express the particular solution to the nonhomogeneous equation with undetermined coefficients. Substitution of (10) into (7) will lead to the quoted values for C_3 through C_6, and indeterminant values for C_1 and C_2; the latter are available to fit initial conditions.

Note how the annihilator method automatically accounts for the fact that the nonhomogeneity xe^x requires the form $C_3xe^x + C_4x^2e^x$ in the particular solution, by counting the total number of factors of $(D-1)$ in the annihilator and *the original differential operator.* ◆

Example 3 Find a general solution, using the annihilator method, to

$$\textbf{(11)} \qquad y''' - 3y'' + 4y = xe^{2x}\,.$$

Solution The associated homogeneous equation takes the operator form

$$\textbf{(12)} \qquad (D^3 - 3D^2 + 4)[y] = (D+1)(D-2)^2[y] = 0\,.$$

The nonhomogeneity xe^{2x} is annihilated by $(D-2)^2$. Therefore, every solution of (11) also satisfies

$$\textbf{(13)} \qquad (D-2)^2(D^3 - 3D^2 + 4)[y] = (D+1)(D-2)^4[y] = 0\,.$$

A general solution to (13) is

$$\textbf{(14)} \qquad y(x) = C_1 e^{-x} + C_2 e^{2x} + C_3 x e^{2x} + C_4 x^2 e^{2x} + C_5 x^3 e^{2x}\,.$$

Comparison with (12) shows that the first three terms of (14) give a general solution to the associated homogeneous equation and the last two terms constitute a particular solution form with undetermined coefficients. Direct substitution reveals $C_4 = -1/18$ and $C_5 = 1/18$ and so a general solution to (11) is

$$y(x) = C_1 e^{-x} + C_2 e^{2x} + C_3 x e^{2x} - \frac{1}{18}x^2 e^{2x} + \frac{1}{18}x^3 e^{2x}\,. \quad ◆$$

The annihilator method, then, rigorously justifies the method of undetermined coefficients of Section 4.4. It also tells us how to upgrade that procedure for higher-order equations with constant coefficients. Note that we don't have to implement the annihilator method directly; we simply need to introduce the following modifications to the method of undetermined coefficients for second-order equations that was described in the procedural box on page 178.

Method of Undetermined Coefficients

To find a particular solution to the constant-coefficient differential equation $L[y] = Cx^m e^{rx}$, where m is a nonnegative integer, use the form

$$y_p(x) = x^s[A_m x^m + \cdots + A_1 x + A_0]e^{rx}, \tag{15}$$

with $s = 0$ if r is not a root of the associated auxiliary equation; otherwise, take s equal to the multiplicity of this root.

To find a particular solution to the constant-coefficient differential equation $L[y] = Cx^m e^{\alpha x}\cos\beta x$ or $L[y] = Cx^m e^{\alpha x}\sin\beta x$, where $\beta \neq 0$, use the form

$$\begin{aligned} y_p(x) &= x^s[A_m x^m + \cdots + A_1 x + A_0]e^{\alpha x}\cos\beta x \\ &\quad + x^s[B_m x^m + \cdots + B_1 x + B_0]e^{\alpha x}\sin\beta x, \end{aligned} \tag{16}$$

with $s = 0$ if $\alpha + i\beta$ is not a root of the associated auxiliary equation; otherwise, take s equal to the multiplicity of this root.

As in Chapter 4, the superposition principle can be used to streamline the method for sums of nonhomogeneous terms of the above types.

6.3 EXERCISES

In Problems 1–4, use the method of undetermined coefficients to determine the form of a particular solution for the given equation.

1. $y''' - 2y'' - 5y' + 6y = e^x + x^2$
2. $y''' + y'' - 5y' + 3y = e^{-x} + \sin x$
3. $y''' + 3y'' - 4y = e^{-2x}$
4. $y''' + y'' - 2y = xe^x + 1$

In Problems 5–10, find a general solution to the given equation.

5. $y''' - 2y'' - 5y' + 6y = e^x + x^2$
6. $y''' + y'' - 5y' + 3y = e^{-x} + \sin x$
7. $y''' + 3y'' - 4y = e^{-2x}$
8. $y''' + y'' - 2y = xe^x + 1$
9. $y''' - 3y'' + 3y' - y = e^x$
10. $y''' + 4y'' + y' - 26y = e^{-3x}\sin 2x + x$

In Problems 11–20, find a differential operator that annihilates the given function.

11. $x^4 - x^2 + 11$
12. $3x^2 - 6x + 1$
13. e^{-7x}
14. e^{5x}
15. $e^{2x} - 6e^x$
16. $x^2 - e^x$
17. $x^2e^{-x}\sin 2x$
18. $xe^{3x}\cos 5x$
19. $xe^{-2x} + xe^{-5x}\sin 3x$
20. $x^2e^x - x\sin 4x + x^3$

In Problems 21–30, use the annihilator method to determine the form of a particular solution for the given equation.

21. $u'' - 5u' + 6u = \cos 2x + 1$
22. $y'' + 6y' + 8y = e^{3x} - \sin x$
23. $y'' - 5y' + 6y = e^{3x} - x^2$
24. $\theta'' - \theta = xe^x$
25. $y'' - 6y' + 9y = \sin 2x + x$
26. $y'' + 2y' + y = x^2 - x + 1$
27. $y'' + 2y' + 2y = e^{-x}\cos x + x^2$
28. $y'' - 6y' + 10y = e^{3x} - x$
29. $z''' - 2z'' + z' = x - e^x$
30. $y''' + 2y'' - y' - 2y = e^x - 1$

In Problems 31–33, solve the given initial value problem.

31. $y''' + 2y'' - 9y' - 18y = -18x^2 - 18x + 22$;
$y(0) = -2, \quad y'(0) = -8, \quad y''(0) = -12$
32. $y''' - 2y'' + 5y' = -24e^{3x}$;
$y(0) = 4, \quad y'(0) = -1, \quad y''(0) = -5$
33. $y''' - 2y'' - 3y' + 10y$
$= 34xe^{-2x} - 16e^{-2x} - 10x^2 + 6x + 34$;
$y(0) = 3, \quad y'(0) = 0, \quad y''(0) = 0$
34. Use the annihilator method to show that if $a_0 \neq 0$ in equation (4) and $f(x)$ has the form

$$f(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0, \tag{17}$$

then

$$y_p(x) = B_m x^m + B_{m-1}x^{m-1} + \cdots + B_1 x + B_0$$

is the form of a particular solution to equation (4).
35. Use the annihilator method to show that if $a_0 = 0$ and $a_1 \neq 0$ in (4) and $f(x)$ has the form given in (17), then equation (4) has a particular solution of the form

$$y_p(x) = x\{B_m x^m + B_{m-1}x^{m-1} + \cdots + B_1 x + B_0\}.$$

36. Use the annihilator method to show that if $f(x)$ in (4) has the form $f(x) = Be^{\alpha x}$, then equation (4) has a particular solution of the form $y_p(x) = x^s Be^{\alpha x}$, where s is chosen to be the smallest nonnegative integer such that $x^s e^{\alpha x}$ is not a solution to the corresponding homogeneous equation.

37. Use the annihilator method to show that if $f(x)$ in (4) has the form

$$f(x) = a\cos\beta x + b\sin\beta x\,,$$

then equation (4) has a particular solution of the form

$$\textbf{(18)}\qquad y_p(x) = x^s\{A\cos\beta x + B\sin\beta x\}\,,$$

where s is chosen to be the smallest nonnegative integer such that $x^s\cos\beta x$ and $x^s\sin\beta x$ are not solutions to the corresponding homogeneous equation.

In Problems 38 and 39, use the elimination method of Section 5.2 to find a general solution to the given system.

38. $x - d^2y/dt^2 = t + 1\,,$
$dx/dt + dy/dt - 2y = e^t$

39. $d^2x/dt^2 - x + y = 0\,,$
$x + d^2y/dt^2 - y = e^{3t}$

40. The currents in the electrical network in Figure 6.1 satisfy the system

$$\frac{1}{9}I_1 + 64I_2'' = -2\sin\frac{t}{24}\,,$$
$$\frac{1}{64}I_3 + 9I_3'' - 64I_2'' = 0\,,$$
$$I_1 = I_2 + I_3\,,$$

where I_1, I_2, and I_3 are the currents through the different branches of the network. Using the elimination method of Section 5.2, determine the currents if initially $I_1(0) = I_2(0) = I_3(0) = 0$, $I_1'(0) = 73/12$, $I_2'(0) = 3/4$, and $I_3'(0) = 16/3$.

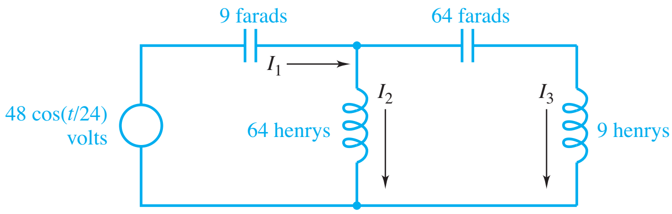

Figure 6.1 An electrical network

6.4 Method Of Variation of Parameters

In the previous section, we discussed the method of undetermined coefficients and the annihilator method. These methods work only for linear equations with constant coefficients *and* when the nonhomogeneous term is a solution to some homogeneous linear equation with constant coefficients. In this section we show how the method of **variation of parameters** discussed in Sections 4.6 and 4.7 generalizes to higher-order linear equations with variable coefficients.

Our goal is to find a particular solution to the standard form equation

$$\textbf{(1)}\qquad L[y](x) = g(x)\,,$$

where $L[y] := y^{(n)} + p_1y^{(n-1)} + \cdots + p_ny$ and the coefficient functions $p_1, \ldots, p_n$, as well as g, are continuous on (a, b). The method to be described requires that we already know a fundamental solution set $\{y_1, \ldots, y_n\}$ for the corresponding homogeneous equation

$$\textbf{(2)}\qquad L[y](x) = 0\,.$$

A general solution to (2) is then

$$\textbf{(3)}\qquad y_h(x) = C_1y_1(x) + \cdots + C_ny_n(x)\,,$$

where $C_1, \ldots, C_n$ are arbitrary constants. In the method of variation of parameters, we assume there exists a particular solution to (1) of the form

$$\textbf{(4)}\qquad y_p(x) = v_1(x)y_1(x) + \cdots + v_n(x)y_n(x)$$

and try to determine the functions $v_1, \ldots, v_n$.

There are n unknown functions, so we will need n conditions (equations) to determine them. These conditions are obtained as follows. Differentiating y_p in (4) gives

$$\textbf{(5)}\qquad y_p' = (v_1y_1' + \cdots + v_ny_n') + (v_1'y_1 + \cdots + v_n'y_n)\,.$$

To prevent second derivatives of the unknowns $v_1, \ldots, v_n$ from entering the formula for y_p'', we impose the condition

$$v_1'y_1 + \cdots + v_n'y_n = 0\ .$$

In a like manner, as we compute $y_p'', y_p''', \ldots, y_p^{(n-1)}$, we impose $(n-2)$ additional conditions involving $v_1', \ldots, v_n'$; namely,

$$v_1'y_1' + \cdots + v_n'y_n' = 0,\ \ldots,\ v_1'y_1^{(n-2)} + \cdots + v_n'y_n^{(n-2)} = 0\ .$$

Finally, the nth condition that we impose is that y_p satisfy the given equation (1). Using the previous conditions and the fact that $y_1, \ldots, y_n$ are solutions to the homogeneous equation, then $L[y_p] = g$ reduces to

(6) $$v_1'y_1^{(n-1)} + \cdots + v_n'y_n^{(n-1)} = g$$

(see Problem 12, page 341). We therefore seek n functions $v_1', \ldots, v_n'$ that satisfy the system

(7) $$\begin{aligned} y_1v_1' + \quad \cdots \quad + y_nv_n' &= 0\ , \\ \vdots \qquad\qquad\qquad \vdots \quad &\ \ \vdots \\ y_1^{(n-2)}v_1' + \cdots + y_n^{(n-2)}v_n' &= 0\ , \\ y_1^{(n-1)}v_1' + \cdots + y_n^{(n-1)}v_n' &= g\ . \end{aligned}$$

Caution. This system was derived under the assumption that the coefficient of the highest derivative $y^{(n)}$ in (1) is one. If, instead, the coefficient of this term is the constant a, then in the last equation in (7) the right-hand side becomes g/a.

A sufficient condition for the existence of a solution to system (7) for x in (a, b) is that the determinant of the matrix made up of the coefficients of $v_1', \ldots, v_n'$ be different from zero for all x in (a, b). But this determinant is just the Wronskian:

(8) $$\begin{vmatrix} y_1 & \cdots & y_n \\ \vdots & & \vdots \\ y_1^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = W[y_1, \ldots, y_n](x)\ ,$$

which is never zero on (a, b) because $\{y_1, \ldots, y_n\}$ is a fundamental solution set. Solving (7) via Cramer's rule (Appendix D), we find

(9) $$v_k'(x) = \frac{g(x)W_k(x)}{W[y_1, \ldots, y_n](x)}\ , \qquad k = 1, \ldots, n\ ,$$

where $W_k(x)$ is the determinant of the matrix obtained from the Wronskian $W[y_1, \ldots y_n](x)$ by replacing the kth column by $\operatorname{col}[0, \ldots, 0, 1]$. Using a cofactor expansion about this column, we can express $W_k(x)$ in terms of an $(n-1)$th-order Wronskian:

(10) $$W_k(x) = (-1)^{n-k}W[y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n](x)\ , \qquad k = 1, \ldots, n\ .$$

Integrating $v_k'(x)$ in (9) gives

(11) $$v_k(x) = \int \frac{g(x)W_k(x)}{W[y_1, \ldots, y_n](x)}dx\ , \qquad k = 1, \ldots, n\ .$$

Finally, substituting the v_k's back into (4), we obtain a particular solution to equation (1):

(12) $$y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{g(x)W_k(x)}{W[y_1, \ldots, y_n](x)}\, dx\ .$$

Note that in equation (1) we presumed that the coefficient of the leading term, $y^{(n)}$, was unity. If, instead, it is $p_0(x)$, we must replace $g(x)$ by $g(x)/p_0(x)$ in (12).

Although equation (12) gives a neat formula for a particular solution to (1), its implementation requires one to evaluate $n+1$ determinants and then perform n integrations. This may entail several tedious computations. However, the method does work in cases when the technique of undetermined coefficients does not apply (provided, of course, we know a fundamental solution set).

Example 1 Find a general solution to the Cauchy–Euler equation

$$x^3y''' + x^2y'' - 2xy' + 2y = x^3 \sin x\,, \qquad x > 0\,, \tag{13}$$

given that $\{x, x^{-1}, x^2\}$ is a fundamental solution set to the corresponding homogeneous equation.

Solution An important first step is to divide (13) by x^3 to obtain the standard form

$$y''' + \frac{1}{x}y'' - \frac{2}{x^2}y' + \frac{2}{x^3}y = \sin x\,, \qquad x > 0\,, \tag{14}$$

from which we see that $g(x) = \sin x$. Since $\{x, x^{-1}, x^2\}$ is a fundamental solution set, we can obtain a particular solution of the form

$$y_p(x) = v_1(x)x + v_2(x)x^{-1} + v_3(x)x^2\,. \tag{15}$$

To use formula (12), we must first evaluate the four determinants:

$$W[x, x^{-1}, x^2](x) = \begin{vmatrix} x & x^{-1} & x^2 \\ 1 & -x^{-2} & 2x \\ 0 & 2x^{-3} & 2 \end{vmatrix} = -6x^{-1}\,,$$

$$W_1(x) = (-1)^{(3-1)}W[x^{-1}, x^2](x) = (-1)^2\begin{vmatrix} x^{-1} & x^2 \\ -x^{-2} & 2x \end{vmatrix} = 3\,,$$

$$W_2(x) = (-1)^{(3-2)}\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = -x^2\,,$$

$$W_3(x) = (-1)^{(3-3)}\begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -2x^{-1}\,.$$

Substituting the above expressions into (12), we find

$$\begin{aligned} y_p(x) &= x\int \frac{(\sin x)3}{-6x^{-1}}\,dx + x^{-1}\int \frac{(\sin x)(-x^2)}{-6x^{-1}}\,dx + x^2\int \frac{(\sin x)(-2x^{-1})}{-6x^{-1}}\,dx \\ &= x\int\left(-\frac{1}{2}x\sin x\right)dx + x^{-1}\int \frac{1}{6}x^3\sin x\,dx + x^2\int \frac{1}{3}\sin x\,dx\,, \end{aligned}$$

which after some labor simplifies to

$$y_p(x) = \cos x - x^{-1}\sin x + C_1x + C_2x^{-1} + C_3x^2\,, \tag{16}$$

where C_1, C_2, and C_3 denote the constants of integration. Since $\{x, x^{-1}, x^2\}$ is a fundamental solution set for the homogeneous equation, we can take C_1, C_2, and C_3 to be arbitrary constants. The right-hand side of (16) then gives the desired general solution. ◆

In the preceding example, the fundamental solution set $\{x, x^{-1}, x^2\}$ can be derived by substituting $y = x^r$ into the homogeneous equation corresponding to (13) (see Problem 31, Exercises 6.2). However, in dealing with other equations that have variable coefficients, the determination of a fundamental set may be extremely difficult. In Chapter 8 we tackle this problem using power series methods.

6.4 EXERCISES

In Problems 1–6, use the method of variation of parameters to determine a particular solution to the given equation.

1. $y''' - 3y'' + 4y = e^{2x}$
2. $y''' - 2y'' + y' = x$
3. $z''' + 3z'' - 4z = e^{2x}$
4. $y''' - 3y'' + 3y' - y = e^{x}$
5. $y''' + y' = \tan x\,, \qquad 0 < x < \pi/2$
6. $y''' + y' = \sec\theta\tan\theta\,, \qquad 0 < \theta < \pi/2$

7. Find a general solution to the Cauchy–Euler equation
$$x^3y''' - 3x^2y'' + 6xy' - 6y = x^{-1}\,, \qquad x > 0\,,$$
given that $\{x, x^2, x^3\}$ is a fundamental solution set for the corresponding homogeneous equation.
8. Find a general solution to the Cauchy–Euler equation
$$x^3y''' - 2x^2y'' + 3xy' - 3y = x^2\,, \qquad x > 0\,,$$
given that $\{x, x\ln x, x^3\}$ is a fundamental solution set for the corresponding homogeneous equation.
9. Given that $\{e^x, e^{-x}, e^{2x}\}$ is a fundamental solution set for the homogeneous equation corresponding to the equation
$$y''' - 2y'' - y' + 2y = g(x)\,,$$
determine a formula involving integrals for a particular solution.
10. Given that $\{x, x^{-1}, x^4\}$ is a fundamental solution set for the homogeneous equation corresponding to the equation
$$x^3y''' - x^2y'' - 4xy' + 4y = g(x)\,, \qquad x > 0\,,$$
determine a formula involving integrals for a particular solution.
11. Find a general solution to the Cauchy–Euler equation
$$x^3y''' - 3xy' + 3y = x^4\cos x\,, \qquad x > 0$$
12. Derive the system (7) in the special case when $n = 3$. [*Hint:* To determine the last equation, require that $L[y_p] = g$ and use the fact that y_1, y_2, and y_3 satisfy the corresponding homogeneous equation.]
13. Show that
$$W_k(x) = (-1)^{(n-k)}W[y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n](x)\,.$$
14. **Deflection of a Beam Under Axial Force.** A uniform beam under a load and subject to a constant axial force is governed by the differential equation
$$y^{(4)}(x) - k^2y''(x) = q(x)\,, \qquad 0 < x < L,$$
where $y(x)$ is the deflection of the beam, L is the length of the beam, k^2 is proportional to the axial force, and $q(x)$ is proportional to the load (see Figure 6.2).
  (a) Show that a general solution can be written in the form
$$\begin{aligned} y(x) = {} & C_1 + C_2x + C_3e^{kx} + C_4e^{-kx} \\ & + \frac{1}{k^2}\int q(x)\,x\,dx - \frac{x}{k^2}\int q(x)\,dx \\ & + \frac{e^{kx}}{2k^3}\int q(x)e^{-kx}\,dx - \frac{e^{-kx}}{2k^3}\int q(x)e^{kx}\,dx\,. \end{aligned}$$
  (b) Show that the general solution in part (a) can be rewritten in the form
$$\begin{aligned} y(x) = {} & c_1 + c_2x + c_3e^{kx} + c_4e^{-kx} \\ & + \int_0^x q(s)G(s,x)\,ds\,, \end{aligned}$$
where
$$G(s,x) := \frac{s-x}{k^2} - \frac{\sinh[k(s-x)]}{k^3}\,.$$
  (c) Let $q(x) \equiv 1$. First compute the general solution using the formula in part (a) and then using the formula in part (b). Compare these two general solutions with the general solution
$$y(x) = B_1 + B_2x + B_3e^{kx} + B_4e^{-kx} - \frac{1}{2k^2}x^2\,,$$
which one would obtain using the method of undetermined coefficients.

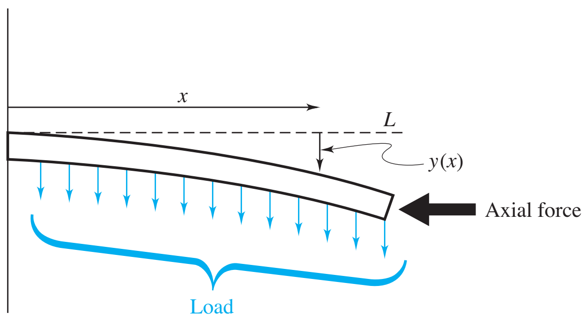

Figure 6.2 Deformation of a beam under axial force and load

7.2 Definition of the Laplace Transform

In earlier chapters we studied differential operators. These operators took a function and mapped or transformed it (via differentiation) into another function. The Laplace transform, which is an integral operator, is another such transformation.

Laplace Transform

Definition 1. Let $f(t)$ be a function on $[0, \infty)$. The **Laplace transform** of f is the function F defined by the integral

$$F(s) := \int_0^\infty e^{-st} f(t)\, dt\,. \tag{1}$$

The domain of $F(s)$ is all the values of s for which the integral in (1) exists.† The Laplace transform of f is denoted by both F and $\mathcal{L}\{f\}$.

Notice that the integral in (1) is an **improper** integral. More precisely,

$$\int_0^\infty e^{-st} f(t)\, dt := \lim_{N \to \infty} \int_0^N e^{-st} f(t)\, dt$$

whenever the limit exists.

Example 1 Determine the Laplace transform of the constant function $f(t) = 1,\ t \geq 0$.

Solution Using the definition of the transform, we compute

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{N \to \infty} \int_0^N e^{-st}\, dt \\ &= \lim_{N \to \infty} \frac{-e^{-st}}{s} \bigg|_{t=0}^{t=N} = \lim_{N \to \infty} \left[\frac{1}{s} - \frac{e^{-sN}}{s} \right]. \end{aligned}$$

Since $e^{-sN} \to 0$ when $s > 0$ is fixed and $N \to \infty$, we get

$$F(s) = \frac{1}{s} \qquad \text{for} \qquad s > 0\,.$$

When $s \leq 0$, the integral $\int_0^\infty e^{-st} dt$ diverges. (Why?) Hence $F(s) = 1/s$, with the domain of $F(s)$ being all $s > 0$. ◆

†We treat s as real-valued, but in certain applications s may be a complex variable. For a detailed treatment of complex-valued Laplace transforms, see *Complex Variables and the Laplace Transform for Engineers,* by Wilbur R. LePage (Dover Publications, New York, 2010), or *Fundamentals of Complex Analysis with Applications to Engineering and Science* (3rd ed.), by E. B. Saff and A. D. Snider (Pearson Education, Boston, MA, 2003).

Example 2 Determine the Laplace transform of $f(t) = e^{at}$, where a is a constant.

Solution Using the definition of the transform,

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt \\
&= \lim_{N\to\infty}\int_0^N e^{-(s-a)t}\,dt = \lim_{N\to\infty}\left.\frac{-e^{-(s-a)t}}{s-a}\right|_0^N \\
&= \lim_{N\to\infty}\left[\frac{1}{s-a} - \frac{e^{-(s-a)N}}{s-a}\right] \\
&= \frac{1}{s-a} \quad \text{for} \quad s > a\,.
\end{aligned}
$$

Again, if $s \le a$ the integral diverges, and hence the domain of $F(s)$ is all $s > a$. ◆

It is comforting to note from Example 2 that the transform of the constant function $f(t) = 1 = e^{0t}$ is $1/(s-0) = 1/s$, which agrees with the solution in Example 1.

Example 3 Find $\mathcal{L}\{\sin bt\}$, where b is a nonzero constant.

Solution We need to compute

$$
\mathcal{L}\{\sin bt\}(s) = \int_0^\infty e^{-st}\sin bt\,dt = \lim_{N\to\infty}\int_0^N e^{-st}\sin bt\,dt\,.
$$

Referring to the table of integrals at the back of the book, we see that

$$
\begin{aligned}
\mathcal{L}\{\sin bt\}(s) &= \lim_{N\to\infty}\left[\frac{e^{-st}}{s^2+b^2}(-s\sin bt - b\cos bt)\Big|_0^N\right] \\
&= \lim_{N\to\infty}\left[\frac{b}{s^2+b^2} - \frac{e^{-sN}}{s^2+b^2}(s\sin bN + b\cos bN)\right] \\
&= \frac{b}{s^2+b^2} \quad \text{for} \quad s > 0
\end{aligned}
$$

(since for such s we have $\lim_{N\to\infty} e^{-sN}(s\sin bN + b\cos bN) = 0$; see Problem 32). ◆

Example 4 Determine the Laplace transform of

$$
f(t) = \begin{cases} 2\,, & 0 < t < 5\,, \\ 0\,, & 5 < t < 10\,, \\ e^{4t}\,, & 10 < t\,. \end{cases}
$$

Solution Since $f(t)$ is defined by a different formula on different intervals, we begin by breaking up the integral in (1) into three separate parts.[†] Thus,

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,dt \\
&= \int_0^5 e^{-st}\cdot 2\,dt + \int_5^{10} e^{-st}\cdot 0\,dt + \int_{10}^\infty e^{-st}e^{4t}\,dt \\
&= 2\int_0^5 e^{-st}\,dt + \lim_{N\to\infty}\int_{10}^N e^{-(s-4)t}\,dt \\
&= \frac{2}{s} - \frac{2e^{-5s}}{s} + \lim_{N\to\infty}\left[\frac{e^{-10(s-4)}}{s-4} - \frac{e^{-(s-4)N}}{s-4}\right] \\
&= \frac{2}{s} - \frac{2e^{-5s}}{s} + \frac{e^{-10(s-4)}}{s-4} \qquad \text{for} \quad s > 4\,. \quad ◆
\end{aligned}
$$

Notice that the function $f(t)$ of Example 4 has jump discontinuities at $t = 5$ and $t = 10$. These values are reflected in the exponential terms e^{-5s} and e^{-10s} that appear in the formula for $F(s)$. We'll make this connection more precise when we discuss the unit step function in Section 7.6.

An important property of the Laplace transform is its **linearity.** That is, the Laplace transform $\mathscr{L}$ is a linear operator.

Linearity of the Transform

Theorem 1. Let f, f_1, and f_2 be functions whose Laplace transforms exist for $s > \alpha$ and let c be a constant. Then, for $s > \alpha$,

$$\mathscr{L}\{f_1 + f_2\} = \mathscr{L}\{f_1\} + \mathscr{L}\{f_2\}\,, \tag{2}$$

$$\mathscr{L}\{cf\} = c\mathscr{L}\{f\}\,. \tag{3}$$

Proof. Using the linearity properties of integration, we have for $s > \alpha$

$$
\begin{aligned}
\mathscr{L}\{f_1 + f_2\}(s) &= \int_0^\infty e^{-st}[f_1(t) + f_2(t)]\,dt \\
&= \int_0^\infty e^{-st} f_1(t)\,dt + \int_0^\infty e^{-st} f_2(t)\,dt \\
&= \mathscr{L}\{f_1\}(s) + \mathscr{L}\{f_2\}(s)\,.
\end{aligned}
$$

Hence, equation (2) is satisfied. In a similar fashion, we see that

$$
\begin{aligned}
\mathscr{L}\{cf\}(s) &= \int_0^\infty e^{-st}[cf(t)]\,dt = c\int_0^\infty e^{-st} f(t)\,dt \\
&= c\mathscr{L}\{f\}(s)\,. \quad ◆
\end{aligned}
$$

[†]Notice that $f(t)$ is not defined at the points $t = 0$, 5, and 10. Nevertheless, the integral in (1) is still meaningful and unaffected by the function's values at finitely many points.

Example 5 Determine $\mathscr{L}\{11 + 5e^{4t} - 6\sin 2t\}$.

Solution From the linearity property, we know that the Laplace transform of the sum of any finite number of functions is the sum of their Laplace transforms. Thus,

$$\begin{aligned}\mathscr{L}\{11 + 5e^{4t} - 6\sin 2t\} &= \mathscr{L}\{11\} + \mathscr{L}\{5e^{4t}\} + \mathscr{L}\{-6\sin 2t\}\\ &= 11\mathscr{L}\{1\} + 5\mathscr{L}\{e^{4t}\} - 6\mathscr{L}\{\sin 2t\} .\end{aligned}$$

In Examples 1, 2, and 3, we determined that

$$\mathscr{L}\{1\}(s) = \frac{1}{s}, \qquad \mathscr{L}\{e^{4t}\}(s) = \frac{1}{s-4}, \qquad \mathscr{L}\{\sin 2t\}(s) = \frac{2}{s^2+2^2} .$$

Using these results, we find

$$\begin{aligned}\mathscr{L}\{11 + 5e^{4t} - 6\sin 2t\}(s) &= 11\left(\frac{1}{s}\right) + 5\left(\frac{1}{s-4}\right) - 6\left(\frac{2}{s^2+4}\right)\\ &= \frac{11}{s} + \frac{5}{s-4} - \frac{12}{s^2+4} .\end{aligned}$$

Since $\mathscr{L}\{1\}, \mathscr{L}\{e^{4t}\}$, and $\mathscr{L}\{\sin 2t\}$ are all defined for $s > 4$, so is the transform $\mathscr{L}\{11 + 5e^{4t} - 6\sin 2t\}$. ◆

Table 7.1 lists the Laplace transforms of some of the elementary functions. You should become familiar with these, since they are frequently encountered in solving linear differential equations with constant coefficients. The entries in the table can be derived from the definition of the Laplace transform. A more elaborate table of transforms is given on the inside back cover of this book.

TABLE 7.1 Brief Table of Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$

Example 6 Use Table 7.1 to determine $\mathscr{L}\{5t^2e^{-3t} - e^{12t}\cos 8t\}$.

Solution From the table,

$$\mathscr{L}\{t^2e^{-3t}\} = \frac{2!}{[s-(-3)]^{2+1}} = \frac{2}{(s+3)^3} \quad \text{for } s > -3,$$

and

$$\mathscr{L}\{e^{12t}\cos 8t\} = \frac{s-12}{(s-12)^2+8^2} \quad \text{for } s > 12 .$$

Therefore, by linearity,

$$\mathscr{L}\{5t^2e^{-3t} - e^{12t}\cos 8t\} = \frac{10}{(s+3)^3} - \frac{s-12}{(s-12)^2+64} \quad \text{for } s > 12 . \ \blacklozenge$$

Existence of the Transform

There are functions for which the improper integral in (1) fails to converge for any value of s. For example, this is the case for the function $f(t) = 1/t$, which grows too fast near zero. Likewise, no Laplace transform exists for the function $f(t) = e^{t^2}$, which increases too rapidly as $t \to \infty$. Fortunately, the set of functions for which the Laplace transform is defined includes many of the functions that arise in applications involving linear differential equations. We now discuss some properties that will (collectively) ensure the existence of the Laplace transform.

A function $f(t)$ on $[a, b]$ is said to have a **jump discontinuity** at $t_0 \in (a, b)$ if $f(t)$ is discontinuous at t_0, but the one-sided limits

$$\lim_{t \to t_0^-} f(t) \quad \text{and} \quad \lim_{t \to t_0^+} f(t)$$

exist as finite numbers. We have encountered jump discontinuities in Example 4 (page 354) and in the input to the mixing tank in Section 7.1 (page 350). If the discontinuity occurs at an endpoint, $t_0 = a$ (or b), a jump discontinuity occurs if the one-sided limit of $f(t)$ as $t \to a^+$ $(t \to b^-)$ exists as a finite number. We can now define piecewise continuity.

Piecewise Continuity

Definition 2. A function $f(t)$ is said to be **piecewise continuous on a finite interval** $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

A function $f(t)$ is said to be **piecewise continuous on** $[\mathbf{0}, \boldsymbol{\infty})$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

Example 7 Show that

$$f(t) = \begin{cases} t, & 0 < t < 1, \\ 2, & 1 < t < 2, \\ (t-2)^2, & 2 \le t \le 3, \end{cases}$$

whose graph is sketched in Figure 7.4 (on page 358), is piecewise continuous on $[0, 3]$.

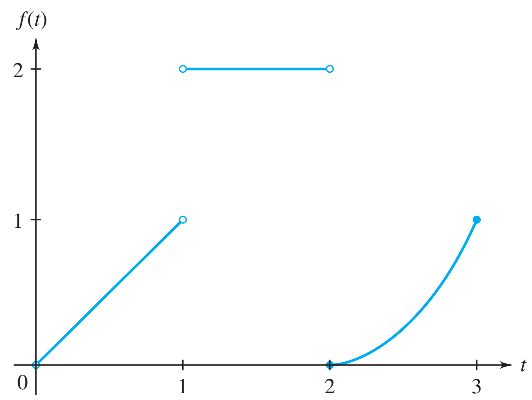

Figure 7.4 Graph of $f(t)$ in Example 7

Solution From the graph of $f(t)$ we see that $f(t)$ is continuous on the intervals (0, 1), (1, 2), and (2, 3]. Moreover, at the points of discontinuity, $t = 0$, 1, and 2, the function has jump discontinuities, since the one-sided limits exist as finite numbers. In particular, at $t = 1$, the left-hand limit is 1 and the right-hand limit is 2. Therefore $f(t)$ is piecewise continuous on $[0, 3]$. ◆

Observe that the function $f(t)$ of Example 4 on page 354 is piecewise continuous on $[0, \infty)$ because it is piecewise continuous on every finite interval of the form $[0, N]$, with $N > 0$. In contrast, the function $f(t) = 1/t$ is not piecewise continuous on any interval containing the origin, since it has an "infinite jump" at the origin (see Figure 7.5).

A function that is piecewise continuous on a *finite* interval is necessarily integrable over that interval. However, piecewise continuity on $[0, \infty)$ is not enough to guarantee the existence (as a finite number) of the improper integral over $[0, \infty)$; we also need to consider the

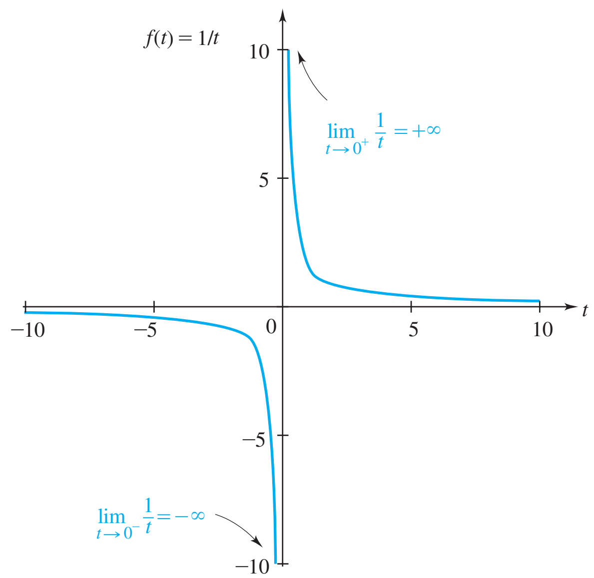

Figure 7.5 Infinite jump at origin

growth of the integrand for large t. Roughly speaking, we'll show that the Laplace transform of a piecewise continuous function exists, provided the function does not grow "faster than an exponential."

Exponential Order α

Definition 3. A function $f(t)$ is said to be of **exponential order α** if there exist positive constants T and M such that

$$|f(t)| \leq Me^{\alpha t}, \qquad \text{for all } t \geq T. \tag{4}$$

For example, $f(t) = e^{5t}\sin 2t$ is of exponential order $\alpha = 5$ since

$$|e^{5t}\sin 2t| \leq e^{5t},$$

and hence (4) holds with $M = 1$ and T any positive constant.

We use the phrase *$f(t)$ is of exponential order* to mean that for *some* value of α, the function $f(t)$ satisfies the conditions of Definition 3; that is, $f(t)$ grows no faster than a function of the form $Me^{\alpha t}$. The function e^{t^2} is *not* of exponential order. To see this, observe that

$$\lim_{t\to\infty} \frac{e^{t^2}}{e^{\alpha t}} = \lim_{t\to\infty} e^{t(t-\alpha)} = +\infty$$

for any α. Consequently, e^{t^2} grows faster than $e^{\alpha t}$ for every choice of α.

The functions usually encountered in solving linear differential equations with constant coefficients (e.g., polynomials, exponentials, sines, and cosines) are both piecewise continuous and of exponential order. As we now show, the Laplace transforms of such functions exist for large enough values of s.

Conditions for Existence of the Transform

Theorem 2. If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$.

Proof. We need to show that the integral

$$\int_0^\infty e^{-st}f(t)\,dt$$

converges for $s > \alpha$. We begin by breaking up this integral into two separate integrals:

$$\int_0^T e^{-st}f(t)\,dt + \int_T^\infty e^{-st}f(t)\,dt, \tag{5}$$

where T is chosen so that inequality (4) holds. The first integral in (5) exists because $f(t)$ and hence $e^{-st}f(t)$ are piecewise continuous on the interval $[0, T]$ for any fixed s. To see that the second integral in (5) converges, we use the **comparison test for improper integrals.**

Since $f(t)$ is of exponential order α, we have for $t \geq T$

$$|f(t)| \leq Me^{\alpha t},$$

and hence

$$|e^{-st}f(t)| = e^{-st}|f(t)| \le Me^{-(s-\alpha)t},$$

for all $t \ge T$. Now for $s > \alpha$.

$$\int_T^\infty Me^{-(s-\alpha)t}dt = M\int_T^\infty e^{-(s-\alpha)t}dt = \frac{Me^{-(s-\alpha)T}}{s-\alpha} < \infty .$$

Since $|e^{-st}f(t)| \le Me^{-(s-\alpha)t}$ for $t \ge T$ and the improper integral of the larger function converges for $s > \alpha$, then, by the comparison test, the integral

$$\int_T^\infty e^{-st}f(t)\,dt$$

converges for $s > \alpha$. Finally, because the two integrals in (5) exist, the Laplace transform $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$. ◆

7.2 EXERCISES

In Problems 1–12, use Definition 1 to determine the Laplace transform of the given function.

1. t

2. t^2

3. e^{6t}

4. te^{3t}

5. $\cos 2t$

6. $\cos bt$, b a constant

7. $e^{2t}\cos 3t$

8. $e^{-t}\sin 2t$

9. $f(t) = \begin{cases} 0, & 0 < t < 2, \\ t, & 2 < t \end{cases}$

10. $f(t) = \begin{cases} 1-t, & 0 < t < 1, \\ 0, & 1 < t \end{cases}$

11. $f(t) = \begin{cases} \sin t, & 0 < t < \pi, \\ 0, & \pi < t \end{cases}$

12. $f(t) = \begin{cases} e^{2t}, & 0 < t < 3, \\ 1, & 3 < t \end{cases}$

In Problems 13–20, use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

13. $\mathcal{L}\{6e^{-3t} - t^2 + 2t - 8\}$

14. $\mathcal{L}\{5 - e^{2t} + 6t^2\}$

15. $\mathcal{L}\{t^3 - te^t + e^{4t}\cos t\}$

16. $\mathcal{L}\{t^2 - 3t - 2e^{-t}\sin 3t\}$

17. $\mathcal{L}\{e^{3t}\sin 6t - t^3 + e^t\}$

18. $\mathcal{L}\{t^4 - t^2 - t + \sin\sqrt{2}t\}$

19. $\mathcal{L}\{t^4e^{5t} - e^t\cos\sqrt{7}t\}$

20. $\mathcal{L}\{e^{-2t}\cos\sqrt{3}t - t^2e^{-2t}\}$

In Problems 21–28, determine whether $f(t)$ is continuous, piecewise continuous, or neither on $[0, 10]$ and sketch the graph of $f(t)$.

21. $f(t) = \begin{cases} 1, & 0 \le t \le 1, \\ (t-2)^2, & 1 < t \le 10 \end{cases}$

22. $f(t) = \begin{cases} 0, & 0 \le t < 2, \\ t, & 2 \le t \le 10 \end{cases}$

23. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ t-1, & 1 < t < 3, \\ t^2-4, & 3 < t \le 10 \end{cases}$

24. $f(t) = \dfrac{t^2 - 3t + 2}{t^2 - 4}$

25. $f(t) = \dfrac{t^2 - t - 20}{t^2 + 7t + 10}$

26. $f(t) = \dfrac{t}{t^2 - 1}$

27. $f(t) = \begin{cases} 1/t, & 0 < t < 1, \\ 1, & 1 \le t \le 2, \\ 1-t, & 2 < t \le 10 \end{cases}$

28. $f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \ne 0, \\ 1, & t = 0 \end{cases}$

29. Which of the following functions are of exponential order?

(a) $t^3\sin t$ **(b)** $100e^{49t}$ **(c)** e^{t^3}

(d) $t\ln t$ **(e)** $\cosh(t^2)$ **(f)** $\dfrac{1}{t^2+1}$

(g) $\sin(t^2) + t^4e^{6t}$ **(h)** $3 - e^{t^2} + \cos 4t$

(i) $\exp\{t^2/(t+1)\}$ **(j)** $\sin(e^{t^2}) + e^{\sin t}$

30. For the transforms $F(s)$ in Table 7.1, what can be said about $\lim_{s\to\infty} F(s)$?

31. Thanks to Euler's formula (page 166) and the algebraic properties of complex numbers, several of the entries of Table 7.1 can be derived from a single formula; namely,

$$\mathcal{L}\left\{e^{(a+ib)t}\right\}(s) = \frac{s-a+ib}{(s-a)^2+b^2}, \qquad s > a. \tag{6}$$

(a) By computing the integral in the definition of the Laplace transform on page 353 with $f(t) = e^{(a+ib)t}$, show that

$$\mathcal{L}\left\{e^{(a+ib)t}\right\}(s) = \frac{1}{s-(a+ib)}, \qquad s > a.$$

(b) Deduce (6) from part (a) by showing that

$$\frac{1}{s-(a+ib)} = \frac{s-a+ib}{(s-a)^2+b^2}.$$

(c) By equating the real and imaginary parts in formula (6), deduce the last two entries in Table 7.1.

32. Prove that for fixed $s > 0$, we have

$$\lim_{N\to\infty} e^{-sN}(s\sin bN + b\cos bN) = 0.$$

33. Prove that if f is piecewise continuous on $[a, b]$ and g is continuous on $[a, b]$, then the product fg is piecewise continuous on $[a, b]$.

7.3 Properties of the Laplace Transform

In the previous section, we defined the Laplace transform of a function $f(t)$ as

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)\,dt.$$

Using this definition to get an explicit expression for $\mathcal{L}\{f\}$ requires the evaluation of the improper integral—frequently a tedious task! We have already seen how the linearity property of the transform can help relieve this burden. In this section we discuss some further properties of the Laplace transform that simplify its computation. These new properties will also enable us to use the Laplace transform to solve initial value problems.

Translation in s

Theorem 3. If the Laplace transform $\mathcal{L}\{f\}(s) = F(s)$ exists for $s > \alpha$, then

$$\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a) \tag{1}$$

for $s > \alpha + a$.

Proof. We simply compute

$$\begin{aligned}\mathcal{L}\{e^{at}f(t)\}(s) &= \int_0^\infty e^{-st}e^{at}f(t)\,dt \\ &= \int_0^\infty e^{-(s-a)t}f(t)\,dt \\ &= F(s-a). \quad \blacklozenge\end{aligned}$$

Theorem 3 illustrates the effect on the Laplace transform of multiplication of a function $f(t)$ by e^{at}.

Example 1 Determine the Laplace transform of $e^{at}\sin bt$.

Solution In Example 3 in Section 7.2, page 354, we found that

$$\mathscr{L}\{\sin bt\}(s) = F(s) = \frac{b}{s^2+b^2}.$$

Thus, by the translation property of $F(s)$, we have

$$\mathscr{L}\{e^{at}\sin bt\}(s) = F(s-a) = \frac{b}{(s-a)^2+b^2}. \quad \blacklozenge$$

Laplace Transform of the Derivative

Theorem 4. Let $f(t)$ be continuous on $[0, \infty)$ and $f'(t)$ be piecewise continuous on $[0, \infty)$, with both of exponential order α. Then, for $s > \alpha$,

(2) $$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0).$$

Proof. Since $\mathscr{L}\{f'\}$ exists, we can use integration by parts [with $u = e^{-st}$ and $dv = f'(t)dt$] to obtain

(3) $$\begin{aligned}\mathscr{L}\{f'\}(s) &= \int_0^\infty e^{-st}f'(t)\,dt = \lim_{N\to\infty}\int_0^N e^{-st}f'(t)\,dt \\ &= \lim_{N\to\infty}\left[e^{-st}f(t)\Big|_0^N + s\int_0^N e^{-st}f(t)\,dt\right] \\ &= \lim_{N\to\infty} e^{-sN}f(N) - f(0) + s\lim_{N\to\infty}\int_0^N e^{-st}f(t)\,dt \\ &= \lim_{N\to\infty} e^{-sN}f(N) - f(0) + s\mathscr{L}\{f\}(s).\end{aligned}$$

To evaluate $\lim_{N\to\infty} e^{-sN}f(N)$, we observe that since $f(t)$ is of exponential order α, there exists a constant M such that for N large,

$$|e^{-sN}f(N)| \le e^{-sN}Me^{\alpha N} = Me^{-(s-\alpha)N}.$$

Hence, for $s > \alpha$,

$$0 \le \lim_{N\to\infty}|e^{-sN}f(N)| \le \lim_{N\to\infty} Me^{-(s-\alpha)N} = 0,$$

so

$$\lim_{N\to\infty} e^{-sN}f(N) = 0$$

for $s > \alpha$. Equation (3) now reduces to

$$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0). \quad \blacklozenge$$

Using induction, we can extend the last theorem to higher-order derivatives of $f(t)$. For example,

$$\begin{aligned}\mathscr{L}\{f''\}(s) &= s\mathscr{L}\{f'\}(s) - f'(0) \\ &= s[s\mathscr{L}\{f\}(s) - f(0)] - f'(0),\end{aligned}$$

which simplifies to

$$\mathscr{L}\{f''\}(s) = s^2\mathscr{L}\{f\}(s) - sf(0) - f'(0) .$$

In general, we obtain the following result.

Laplace Transform of Higher-Order Derivatives

Theorem 5. Let $f(t), f'(t), \ldots, f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0, \infty)$, with all these functions of exponential order α. Then, for $s > \alpha$,

$$\mathscr{L}\{f^{(n)}\}(s) = s^n\mathscr{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) . \tag{4}$$

The last two theorems shed light on the reason why the Laplace transform is such a useful tool in solving initial value problems. Roughly speaking, they tell us that by using the Laplace transform we can replace "differentiation with respect to t" with "multiplication by s," thereby converting a differential equation into an algebraic one. This idea is explored in Section 7.5. For now, we show how Theorem 4 can be helpful in computing a Laplace transform.

Example 2 Using Theorem 4 and the fact that

$$\mathscr{L}\{\sin bt\}(s) = \frac{b}{s^2 + b^2} ,$$

determine $\mathscr{L}\{\cos bt\}$.

Solution Let $f(t) := \sin bt$. Then $f(0) = 0$ and $f'(t) = b \cos bt$. Substituting into equation (2), we have

$$\begin{aligned} \mathscr{L}\{f'\}(s) &= s\mathscr{L}\{f\}(s) - f(0) , \\ \mathscr{L}\{b \cos bt\}(s) &= s\mathscr{L}\{\sin bt\}(s) - 0 , \\ b\mathscr{L}\{\cos bt\}(s) &= \frac{sb}{s^2 + b^2} . \end{aligned}$$

Dividing by b gives

$$\mathscr{L}\{\cos bt\}(s) = \frac{s}{s^2 + b^2} . \quad ◆$$

Example 3 Prove the following identity for continuous functions $f(t)$ (assuming the transforms exist):

$$\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) = \frac{1}{s}\mathscr{L}\{f(t)\}(s) . \tag{5}$$

Use it to verify the solution to Example 2.

Solution Define the function $g(t)$ by the integral

$$g(t) := \int_0^t f(\tau)\,d\tau .$$

Observe that $g(0) = 0$ and $g'(t) = f(t)$. Thus, if we apply Theorem 4 to $g(t)$ [instead of $f(t)$], equation (2) on page 362 reads

$$\mathscr{L}\{f(t)\}(s) = s\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) - 0\,,$$

which is equivalent to equation (5).

Now since

$$\sin bt = \int_0^t b\cos b\tau\,d\tau\,,$$

equation (5) predicts

$$\mathscr{L}\{\sin bt\}(s) = \frac{1}{s}\mathscr{L}\{b\cos bt\}(s) = \frac{b}{s}\mathscr{L}\{\cos bt\}(s)\,.$$

This identity is indeed valid for the transforms in Example 2. ◆

Another question arises concerning the Laplace transform. If $F(s)$ is the Laplace transform of $f(t)$, is $F'(s)$ also a Laplace transform of some function of t? The answer is yes:

$$F'(s) = \mathscr{L}\{-tf(t)\}(s)\,.$$

In fact, the following more general assertion holds.

Derivatives of the Laplace Transform

Theorem 6. Let $F(s) = \mathscr{L}\{f\}(s)$ and assume $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α. Then, for $s > \alpha$,

$$\mathscr{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^nF}{ds^n}(s)\,. \tag{6}$$

Proof. Consider the identity

$$\frac{dF}{ds}(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt\,.$$

Because of the assumptions on $f(t)$, we can apply a theorem from advanced calculus (sometimes called **Leibniz's rule**) to interchange the order of integration and differentiation:

$$\begin{aligned}\frac{dF}{ds}(s) &= \int_0^\infty \frac{d}{ds}(e^{-st})f(t)\,dt \\ &= -\int_0^\infty e^{-st}tf(t)\,dt = -\mathscr{L}\{tf(t)\}(s)\,.\end{aligned}$$

Thus,

$$\mathscr{L}\{tf(t)\}(s) = (-1)\frac{dF}{ds}(s)\,.$$

The general result (6) now follows by induction on n. ◆

A consequence of the above theorem is that if $f(t)$ is piecewise continuous and of exponential order, then its transform $F(s)$ has derivatives of all orders.

Example 4 Determine $\mathscr{L}\{t \sin bt\}$.

Solution We already know that

$$\mathscr{L}\{\sin bt\}(s) = F(s) = \frac{b}{s^2 + b^2} .$$

Differentiating $F(s)$, we obtain

$$\frac{dF}{ds}(s) = \frac{-2bs}{(s^2 + b^2)^2} .$$

Hence, using formula (6), we have

$$\mathscr{L}\{t \sin bt\}(s) = -\frac{dF}{ds}(s) = \frac{2bs}{(s^2 + b^2)^2} . \quad \blacklozenge$$

For easy reference, Table 7.2 lists some of the basic properties of the Laplace transform derived so far.

TABLE 7.2 Properties of Laplace Transforms

$\mathscr{L}\{f + g\} = \mathscr{L}\{f\} + \mathscr{L}\{g\}$.
$\mathscr{L}\{cf\} = c\mathscr{L}\{f\}$ for any constant c .
$\mathscr{L}\{e^{at}f(t)\}(s) = \mathscr{L}\{f\}(s - a)$.
$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0)$.
$\mathscr{L}\{f''\}(s) = s^2\mathscr{L}\{f\}(s) - sf(0) - f'(0)$.
$\mathscr{L}\{f^{(n)}\}(s) = s^n\mathscr{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$.
$\mathscr{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathscr{L}\{f\}(s)\right)$.

7.3 EXERCISES

In Problems 1–20, determine the Laplace transform of the given function using Table 7.1 on page 356 and the properties of the transform given in Table 7.2. [Hint: In Problems 12–20, use an appropriate trigonometric identity.]

1. $t^2 + e^t \sin 2t$

2. $3t^2 - e^{2t}$

3. $e^{-t}\cos 3t + e^{6t} - 1$

4. $3t^4 - 2t^2 + 1$

5. $2t^2e^{-t} - t + \cos 4t$

6. $e^{-2t}\sin 2t + e^{3t}t^2$

7. $(t - 1)^4$

8. $(1 + e^{-t})^2$

9. $e^{-t}t \sin 2t$

10. $te^{2t}\cos 5t$

11. $\cosh bt$

12. $\sin 3t \cos 3t$

13. $\sin^2 t$

14. $e^{7t}\sin^2 t$

15. $\cos^3 t$

16. $t \sin^2 t$

17. $\sin 2t \sin 5t$

18. $\cos nt \cos mt$, $m \neq n$

19. $\cos nt \sin mt$, $m \neq n$

20. $t \sin 2t \sin 5t$

21. Given that $\mathscr{L}\{\cos bt\}(s) = s/(s^2 + b^2)$, use the translation property to compute $\mathscr{L}\{e^{at}\cos bt\}$.

22. Starting with the transform $\mathscr{L}\{1\}(s) = 1/s$, use formula (6) for the derivatives of the Laplace transform to show that $\mathscr{L}\{t\}(s) = 1/s^2$, $\mathscr{L}\{t^2\}(s) = 2!/s^3$, and, by using induction, that $\mathscr{L}\{t^n\}(s) = n!/s^{n+1}$, $n = 1, 2, \ldots$.

23. Use Theorem 4 on page 362 to show how entry 32 follows from entry 31 in the Laplace transform table on the inside back cover of the text.

24. Show that $\mathscr{L}\{e^{at}t^n\}(s) = n!/(s - a)^{n+1}$ in two ways:

(a) Use the translation property for $F(s)$.

(b) Use formula (6) for the derivatives of the Laplace transform.

25. Use formula (6) to help determine

(a) $\mathcal{L}\{t \cos bt\}$. **(b)** $\mathcal{L}\{t^2 \cos bt\}$.

26. Let $f(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order.

(a) Show that there exist constants K and α such that

$$|f(t)| \le Ke^{\alpha t} \qquad \text{for all } t \ge 0 .$$

(b) By using the definition of the transform and estimating the integral with the help of part (a), prove that

$$\lim_{s \to \infty} \mathcal{L}\{f\}(s) = 0 .$$

27. Let $f(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order α and assume $\lim_{t \to 0^+}[f(t)/t]$ exists. Show that

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(u)\,du ,$$

where $F(s) = \mathcal{L}\{f\}(s)$. [*Hint:* First show that $\frac{d}{ds}\mathcal{L}\{f(t)/t\}(s) = -F(s)$ and then use the result of Problem 26.]

28. Verify the identity in Problem 27 for the following functions. (Use the table of Laplace transforms on the inside back cover.)

(a) $f(t) = t^5$ **(b)** $f(t) = t^{3/2}$

29. The **transfer function** of a linear system is defined as the ratio of the Laplace transform of the output function $y(t)$ to the Laplace transform of the input function $g(t)$, when all initial conditions are zero. If a linear system is governed by the differential equation

$$y''(t) + 6y'(t) + 10y(t) = g(t) , \qquad t > 0 ,$$

use the linearity property of the Laplace transform and Theorem 5 on page 363 on the Laplace transform of higher-order derivatives to determine the transfer function $H(s) = Y(s)/G(s)$ for this system.

30. Find the transfer function, as defined in Problem 29, for the linear system governed by

$$y''(t) + 5y'(t) + 6y(t) = g(t) , \qquad t > 0 .$$

31. **Translation in *t*.** Show that for $c > 0$, the translated function

$$g(t) = \begin{cases} 0 , & 0 < t < c , \\ f(t-c) , & c < t \end{cases}$$

has Laplace transform

$$\mathcal{L}\{g\}(s) = e^{-cs}\mathcal{L}\{f\}(s) .$$

In Problems 32–35, let $g(t)$ be the given function $f(t)$ translated to the right by c units. Sketch $f(t)$ and $g(t)$ and find $\mathcal{L}\{g(t)\}(s)$. (See Problem 31.)

32. $f(t) \equiv 1 , \quad c = 2$

33. $f(t) = t , \quad c = 1$

34. $f(t) = \sin t , \quad c = \pi$

35. $f(t) = \sin t , \quad c = \pi/2$

36. Use equation (5) to provide another derivation of the formula $\mathcal{L}\{t^n\}(s) = n!/s^{n+1}$. [*Hint*: Start with $\mathcal{L}\{1\}(s) = 1/s$ and use induction.]

37. **Initial Value Theorem.** Apply the relation

$$\mathcal{L}\{f'\}(s) = \int_0^\infty e^{-st} f'(t)\,dt = s\mathcal{L}\{f\}(s) - f(0) \tag{7}$$

to argue that for any function $f(t)$ whose derivative is piecewise continuous and of exponential order on $[0, \infty)$,

$$f(0) = \lim_{s \to \infty} s\mathcal{L}\{f\}(s) .$$

38. Verify the initial value theorem (Problem 37) for the following functions. (Use the table of Laplace transforms on the inside back cover.)

(a) 1 **(b)** e^t **(c)** e^{-t} **(d)** $\cos t$
(e) $\sin t$ **(f)** t^2 **(g)** $t \cos t$

7.4 Inverse Laplace Transform

In Section 7.2 we defined the Laplace transform as an integral operator that maps a function $f(t)$ into a function $F(s)$. In this section we consider the problem of finding the function $f(t)$ when we are given the transform $F(s)$. That is, we seek an **inverse mapping** for the Laplace transform.

To see the usefulness of such an inverse, let's consider the simple initial value problem

$$y'' - y = -t ; \qquad y(0) = 0 , \qquad y'(0) = 1 . \tag{1}$$

If we take the transform of both sides of equation (1) and use the linearity property of the transform, we find

$$\mathcal{L}\{y''\}(s) - Y(s) = -\frac{1}{s^2} ,$$

where $Y(s) := \mathscr{L}\{y\}(s)$. We know the initial values of the solution $y(t)$, so we can use Theorem 5, page 363, on the Laplace transform of higher-order derivatives to express

$$\mathscr{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 1 .$$

Substituting for $\mathscr{L}\{y''\}(s)$ yields

$$s^2Y(s) - 1 - Y(s) = -\frac{1}{s^2} .$$

Solving this algebraic equation for $Y(s)$ gives

$$Y(s) = \frac{1 - \left(\frac{1}{s^2}\right)}{s^2 - 1} = \frac{s^2 - 1}{s^2(s^2 - 1)} = \frac{1}{s^2} .$$

We now recall that $\mathscr{L}\{t\}(s) = 1/s^2$, and since $Y(s) = \mathscr{L}\{y\}(s)$, we have

$$\mathscr{L}\{y\}(s) = 1/s^2 = \mathscr{L}\{t\}(s) .$$

It therefore seems reasonable to conclude that $y(t) = t$ is the solution to the initial value problem (1). A quick check confirms this!

Notice that in the above procedure, a crucial step is to determine $y(t)$ from its Laplace transform $Y(s) = 1/s^2$. As we noted, $y(t) = t$ is such a function, but it is *not* the only function whose Laplace function is $1/s^2$. For example, the transform of

$$g(t) := \begin{cases} t, & t \neq 6 , \\ 0, & t = 6 \end{cases}$$

is also $1/s^2$. This is because the transform is an integral, and integrals are not affected by changing a function's values at isolated points. The significant difference between $y(t)$ and $g(t)$ as far as we are concerned is that $y(t)$ is continuous on $[0, \infty)$, whereas $g(t)$ is not. Naturally, we prefer to work with continuous functions, since solutions to differential equations are continuous. Fortunately, it can be shown that if two different functions have the same Laplace transform, at most one of them can be continuous.† With this in mind we give the following definition.

Inverse Laplace Transform

Definition 4. Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies

(2) $\quad \mathscr{L}\{f\} = F ,$

then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and employ the notation $f = \mathscr{L}^{-1}\{F\}$.

In case every function $f(t)$ satisfying (2) is discontinuous (and hence not a solution of a differential equation), one could choose any one of them to be the inverse transform; the distinction among them has no physical significance. [Indeed, two *piecewise* continuous functions satisfying (2) can only differ at their points of discontinuity.]

†For this result and further properties of the Laplace transform and its inverse, we refer you to *Operational Mathematics,* 3rd ed., by R. V. Churchill (McGraw-Hill, New York, 1971).

Naturally the Laplace transform tables will be a great help in determining the inverse Laplace transform of a given function $F(s)$.

Example 1 Determine $\mathscr{L}^{-1}\{F\}$, where

(a) $F(s) = \dfrac{2}{s^3}$. **(b)** $F(s) = \dfrac{3}{s^2+9}$. **(c)** $F(s) = \dfrac{s-1}{s^2-2s+5}$.

Solution To compute $\mathscr{L}^{-1}\{F\}$, we refer to the Laplace transform table on page 356.

(a) $\mathscr{L}^{-1}\left\{\dfrac{2}{s^3}\right\}(t) = \mathscr{L}^{-1}\left\{\dfrac{2!}{s^3}\right\}(t) = t^2$

(b) $\mathscr{L}^{-1}\left\{\dfrac{3}{s^2+9}\right\}(t) = \mathscr{L}^{-1}\left\{\dfrac{3}{s^2+3^2}\right\}(t) = \sin 3t$

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-1}{s^2-2s+5}\right\}(t) = \mathscr{L}^{-1}\left\{\dfrac{s-1}{(s-1)^2+2^2}\right\}(t) = e^t\cos 2t$

In part (c) we used the technique of completing the square to rewrite the denominator in a form that we could find in the table. ◆

In practice, we do not always encounter a transform $F(s)$ that exactly corresponds to an entry in the second column of the Laplace transform table. To handle more complicated functions $F(s)$, we use properties of $\mathscr{L}^{-1}$, just as we used properties of $\mathscr{L}$. One such tool is the linearity of the inverse Laplace transform, a property that is inherited from the linearity of the operator $\mathscr{L}$.

Linearity of the Inverse Transform

Theorem 7. Assume that $\mathscr{L}^{-1}\{F\}$, $\mathscr{L}^{-1}\{F_1\}$, and $\mathscr{L}^{-1}\{F_2\}$ exist and are continuous on $[0, \infty)$ and let c be any constant. Then

(3) $$\mathscr{L}^{-1}\{F_1+F_2\} = \mathscr{L}^{-1}\{F_1\} + \mathscr{L}^{-1}\{F_2\}\ ,$$

(4) $$\mathscr{L}^{-1}\{cF\} = c\mathscr{L}^{-1}\{F\}\ .$$

The proof of Theorem 7 is outlined in Problem 37. We illustrate the usefulness of this theorem in the next example.

Example 2 Determine $\mathscr{L}^{-1}\left\{\dfrac{5}{s-6} - \dfrac{6s}{s^2+9} + \dfrac{3}{2s^2+8s+10}\right\}$.

Solution We begin by using the linearity property. Thus,

$$\mathscr{L}^{-1}\left\{\frac{5}{s-6} - \frac{6s}{s^2+9} + \frac{3}{2(s^2+4s+5)}\right\}$$
$$= 5\mathscr{L}^{-1}\left\{\frac{1}{s-6}\right\} - 6\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{3}{2}\mathscr{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\}.$$

Referring to the Laplace transform tables, we see that

$$\mathscr{L}^{-1}\left\{\frac{1}{s-6}\right\}(t) = e^{6t} \qquad \text{and} \qquad \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}(t) = \cos 3t\,.$$

This gives us the first two terms. To determine $\mathscr{L}^{-1}\{1/(s^2+4s+5)\}$, we complete the square of the denominator to obtain $s^2+4s+5 = (s+2)^2+1$. We now recognize from the tables that

$$\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2+1^2}\right\}(t) = e^{-2t}\sin t\,.$$

Hence,

$$\mathscr{L}^{-1}\left\{\frac{5}{s-6} - \frac{6s}{s^2+9} + \frac{3}{2s^2+8s+10}\right\}(t) = 5e^{6t} - 6\cos 3t + \frac{3e^{-2t}}{2}\sin t\,. \ ◆$$

Example 3 Determine $\mathscr{L}^{-1}\left\{\dfrac{5}{(s+2)^4}\right\}$.

Solution The $(s+2)^4$ in the denominator suggests that we work with the formula

$$\mathscr{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\}(t) = e^{at}t^n\,.$$

Here we have $a = -2$ and $n = 3$, so $\mathscr{L}^{-1}\{6/(s+2)^4\}(t) = e^{-2t}t^3$. Using the linearity property, we find

$$\mathscr{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\}(t) = \frac{5}{6}\mathscr{L}^{-1}\left\{\frac{3!}{(s+2)^4}\right\}(t) = \frac{5}{6}e^{-2t}t^3\,. \ ◆$$

Example 4 Determine $\mathscr{L}^{-1}\left\{\dfrac{3s+2}{s^2+2s+10}\right\}$.

Solution By completing the square, the quadratic in the denominator can be written as

$$s^2+2s+10 = s^2+2s+1+9 = (s+1)^2+3^2\,.$$

The form of $F(s)$ now suggests that we use one or both of the formulas

$$\mathscr{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\}(t) = e^{at}\cos bt\,,$$

$$\mathscr{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\}(t) = e^{at}\sin bt\,.$$

In this case, $a = -1$ and $b = 3$. The next step is to express

$$\text{(5)} \qquad \frac{3s+2}{s^2+2s+10} = A\frac{s+1}{(s+1)^2+3^2} + B\frac{3}{(s+1)^2+3^2}\,,$$

where A, B are constants to be determined. Multiplying both sides of (5) by $s^2+2s+10$ leaves

$$3s+2 = A(s+1)+3B = As+(A+3B)\,,$$

which is an identity between two polynomials in s. Equating the coefficients of like terms gives

$$A = 3\,, \qquad A+3B = 2\,,$$

so $A = 3$ and $B = -1/3$. Finally, from (5) and the linearity property, we find

$$\mathscr{L}^{-1}\left\{\frac{3s+2}{s^2+2s+10}\right\}(t) = 3\mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\}(t) - \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{(s+1)^2+3^2}\right\}(t)$$
$$= 3e^{-t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t\,. \;\blacklozenge$$

Given the choice of finding the inverse Laplace transform of

$$F_1(s) = \frac{7s^2+10s-1}{s^3+3s^2-s-3}$$

or of

$$F_2(s) = \frac{2}{s-1} + \frac{1}{s+1} + \frac{4}{s+3}\,,$$

which would you select? No doubt $F_2(s)$ is the easier one. Actually, the two functions $F_1(s)$ and $F_2(s)$ are identical. This can be checked by combining the simple fractions that form $F_2(s)$. Thus, if we are faced with the problem of computing $\mathscr{L}^{-1}$ of a rational function such as $F_1(s)$, we will first express it, as we did $F_2(s)$, as a sum of simple rational functions. This is accomplished by the **method of partial fractions.**

We briefly review this method. Recall from calculus that a rational function of the form $P(s)/Q(s)$, where $P(s)$ and $Q(s)$ are polynomials with the degree of P less than the degree of Q, has a partial fraction expansion whose form is based on the linear and quadratic factors of $Q(s)$. (We assume the coefficients of the polynomials to be real numbers.) There are three cases to consider:

1. Nonrepeated linear factors.
2. Repeated linear factors.
3. Quadratic factors.

1. Nonrepeated Linear Factors

If $Q(s)$ can be factored into a product of distinct linear factors,

$$Q(s) = (s-r_1)(s-r_2)\cdots(s-r_n)\,,$$

where the r_i's are all distinct real numbers, then the partial fraction expansion has the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-r_1} + \frac{A_2}{s-r_2} + \cdots + \frac{A_n}{s-r_n}\,,$$

where the A_i's are real numbers. There are various ways of determining the constants $A_1, \ldots, A_n$. In the next example, we demonstrate two such methods.

Example 5 Determine $\mathscr{L}^{-1}\{F\}$, where

$$F(s) = \frac{7s-1}{(s+1)(s+2)(s-3)}\,.$$

Solution We begin by finding the partial fraction expansion for $F(s)$. The denominator consists of three distinct linear factors, so the expansion has the form

$$\frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3}\,, \tag{6}$$

where A, B, and C are real numbers to be determined.

One procedure that works for all partial fraction expansions is first to multiply the expansion equation by the denominator of the given rational function. This leaves us with two identical polynomials. Equating the coefficients of s^k leads to a system of linear equations that we can solve to determine the unknown constants. In this example, we multiply (6) by $(s+1)(s+2)(s-3)$ and find

$$7s - 1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)\,,^{\dagger} \tag{7}$$

which reduces to

$$7s - 1 = (A+B+C)s^2 + (-A-2B+3C)s + (-6A-3B+2C)\,.$$

Equating the coefficients of s^2, s, and 1 gives the system of linear equations

$$\begin{aligned} A + B + C &= 0\,, \\ -A - 2B + 3C &= 7\,, \\ -6A - 3B + 2C &= -1\,. \end{aligned}$$

Solving this system yields $A = 2$, $B = -3$, and $C = 1$. Hence,

$$\frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3}\,. \tag{8}$$

An alternative method for finding the constants A, B, and C from (7) is to choose three values for s and substitute them into (7) to obtain three linear equations in the three unknowns. If we are careful in our choice of the values for s, the system is easy to solve. In this case, equation (7) obviously simplifies if $s = -1, -2$, or 3. Putting $s = -1$ gives

$$\begin{aligned} -7 - 1 &= A(1)(-4) + B(0) + C(0)\,, \\ -8 &= -4A\,. \end{aligned}$$

Hence $A = 2$. Next, setting $s = -2$ gives

$$\begin{aligned} -14 - 1 &= A(0) + B(-1)(-5) + C(0)\,, \\ -15 &= 5B\,, \end{aligned}$$

and so $B = -3$. Finally, letting $s = 3$, we similarly find that $C = 1$. In the case of nonrepeated linear factors, the alternative method is easier to use.

Now that we have obtained the partial fraction expansion (8), we use linearity to compute

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\}(t) &= \mathscr{L}^{-1}\left\{\frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3}\right\}(t) \\ &= 2\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - 3\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) \\ &\quad + \mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\}(t) \\ &= 2e^{-t} - 3e^{-2t} + e^{3t}\,. \quad \blacklozenge \end{aligned}$$

† Rigorously speaking, equation (7) was derived for s different from -1, -2, and 3, but by continuity it holds for these values as well.

2. Repeated Linear Factors

If $s - r$ is a factor of $Q(s)$ and $(s - r)^m$ is the highest power of $s - r$ that divides $Q(s)$, then the portion of the partial fraction expansion of $P(s)/Q(s)$ that corresponds to the term $(s - r)^m$ is

$$\frac{A_1}{s - r} + \frac{A_2}{(s - r)^2} + \cdots + \frac{A_m}{(s - r)^m},$$

where the A_i's are real numbers.

Example 6 Determine $\mathscr{L}^{-1}\left\{\dfrac{s^2 + 9s + 2}{(s - 1)^2(s + 3)}\right\}$.

Solution Since $s - 1$ is a repeated linear factor with multiplicity two and $s + 3$ is a nonrepeated linear factor, the partial fraction expansion has the form

$$\frac{s^2 + 9s + 2}{(s - 1)^2(s + 3)} = \frac{A}{s - 1} + \frac{B}{(s - 1)^2} + \frac{C}{s + 3}.$$

We begin by multiplying both sides by $(s - 1)^2(s + 3)$ to obtain

$$s^2 + 9s + 2 = A(s - 1)(s + 3) + B(s + 3) + C(s - 1)^2. \tag{9}$$

Now observe that when we set $s = 1$ (or $s = -3$), two terms on the right-hand side of (9) vanish, leaving a linear equation that we can solve for B (or C). Setting $s = 1$ in (9) gives

$$\begin{aligned} 1 + 9 + 2 &= A(0) + 4B + C(0), \\ 12 &= 4B, \end{aligned}$$

and, hence, $B = 3$. Similarly, setting $s = -3$ in (9) gives

$$\begin{aligned} 9 - 27 + 2 &= A(0) + B(0) + 16C \\ -16 &= 16C. \end{aligned}$$

Thus, $C = -1$. Finally, to find A, we pick a different value for s, say $s = 0$. Then, since $B = 3$ and $C = -1$, plugging $s = 0$ into (9) yields

$$2 = -3A + 3B + C = -3A + 9 - 1$$

so that $A = 2$. Hence,

$$\frac{s^2 + 9s + 2}{(s - 1)^2(s + 3)} = \frac{2}{s - 1} + \frac{3}{(s - 1)^2} - \frac{1}{s + 3}. \tag{10}$$

We could also have determined the constants A, B, and C by first rewriting equation (9) in the form

$$s^2 + 9s + 2 = (A + C)s^2 + (2A + B - 2C)s + (-3A + 3B + C).$$

Then, equating the corresponding coefficients of s^2, s, and 1 and solving the resulting system, we again find $A = 2$, $B = 3$, and $C = -1$.

Now that we have derived the partial fraction expansion (10) for the given rational function, we can determine its inverse Laplace transform:

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}(t) &= \mathscr{L}^{-1}\left\{\frac{2}{s-1}+\frac{3}{(s-1)^2}-\frac{1}{s+3}\right\}(t)\\ &= 2\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}(t)+3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}(t)\\ &\quad -\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\}(t)\\ &= 2e^t+3te^t-e^{-3t}\,. \quad \blacklozenge\end{aligned}$$

3. Quadratic Factors

If $(s-\alpha)^2+\beta^2$ is a quadratic factor of $Q(s)$ that cannot be reduced to linear factors with real coefficients and m is the highest power of $(s-\alpha)^2+\beta^2$ that divides $Q(s)$, then the portion of the partial fraction expansion that corresponds to $(s-\alpha)^2+\beta^2$ is

$$\frac{C_1s+D_1}{(s-\alpha)^2+\beta^2}+\frac{C_2s+D_2}{[(s-\alpha)^2+\beta^2]^2}+\cdots+\frac{C_ms+D_m}{[(s-\alpha)^2+\beta^2]^m}\,.$$

As we saw in Example 4, page 369, it is more convenient to express C_is+D_i in the form $A_i(s-\alpha)+\beta B_i$ when we look up the Laplace transforms. So let's agree to write this portion of the partial fraction expansion in the equivalent form

$$\frac{A_1(s-\alpha)+\beta B_1}{(s-\alpha)^2+\beta^2}+\frac{A_2(s-\alpha)+\beta B_2}{[(s-\alpha)^2+\beta^2]^2}+\cdots+\frac{A_m(s-\alpha)+\beta B_m}{[(s-\alpha)^2+\beta^2]^m}\,.$$

Example 7 Determine $\mathscr{L}^{-1}\left\{\dfrac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$.

Solution We first observe that the quadratic factor s^2-2s+5 is irreducible (check the sign of the discriminant in the quadratic formula). Next we write the quadratic in the form $(s-\alpha)^2+\beta^2$ by completing the square:

$$s^2-2s+5 = (s-1)^2+2^2\,.$$

Since s^2-2s+5 and $s+1$ are nonrepeated factors, the partial fraction expansion has the form

$$\frac{2s^2+10s}{(s^2-2s+5)(s+1)} = \frac{A(s-1)+2B}{(s-1)^2+2^2}+\frac{C}{s+1}\,.$$

When we multiply both sides by the common denominator, we obtain

$$2s^2+10s = [A(s-1)+2B](s+1)+C(s^2-2s+5)\,. \tag{11}$$

In equation (11), let's put $s=-1$, 1, and 0. With $s=-1$, we find

$$\begin{aligned}2-10 &= [A(-2)+2B](0)+C(8)\,,\\ -8 &= 8C\,,\end{aligned}$$

and, hence, $C = -1$. With $s = 1$ in (11), we obtain

$$2 + 10 = [A(0) + 2B](2) + C(4) ,$$

and since $C = -1$, the last equation becomes $12 = 4B - 4$. Thus $B = 4$. Finally, setting $s = 0$ in (11) and using $C = -1$ and $B = 4$ gives

$$\begin{aligned} 0 &= [A(-1) + 2B](1) + C(5) , \\ 0 &= -A + 8 - 5 , \\ A &= 3 . \end{aligned}$$

Hence, $A = 3$, $B = 4$, and $C = -1$ so that

$$\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} = \frac{3(s - 1) + 2(4)}{(s - 1)^2 + 2^2} - \frac{1}{s + 1} .$$

With this partial fraction expansion in hand, we can immediately determine the inverse Laplace transform:

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)}\right\}(t) &= \mathscr{L}^{-1}\left\{\frac{3(s - 1) + 2(4)}{(s - 1)^2 + 2^2} - \frac{1}{s + 1}\right\}(t) \\ &= 3\mathscr{L}^{-1}\left\{\frac{s - 1}{(s - 1)^2 + 2^2}\right\}(t) \\ &\quad + 4\mathscr{L}^{-1}\left\{\frac{2}{(s - 1)^2 + 2^2}\right\}(t) - \mathscr{L}^{-1}\left\{\frac{1}{s + 1}\right\}(t) \\ &= 3e^t \cos 2t + 4e^t \sin 2t - e^{-t} . \end{aligned}$$ ◆

In Section 7.8, we discuss a different method (involving convolutions) for computing inverse transforms that does not require partial fraction decompositions. Moreover, the convolution method is convenient in the case of a rational function with a repeated quadratic factor in the denominator. Other helpful tools are described in Problems 33–36 and 38–43.

7.4 EXERCISES

In Problems 1–10, determine the inverse Laplace transform of the given function.

1. $\dfrac{6}{(s - 1)^4}$
2. $\dfrac{2}{s^2 + 4}$
3. $\dfrac{s + 1}{s^2 + 2s + 10}$
4. $\dfrac{4}{s^2 + 9}$
5. $\dfrac{1}{s^2 + 4s + 8}$
6. $\dfrac{3}{(2s + 5)^3}$
7. $\dfrac{2s + 16}{s^2 + 4s + 13}$
8. $\dfrac{1}{s^5}$
9. $\dfrac{3s - 15}{2s^2 - 4s + 10}$
10. $\dfrac{s - 1}{2s^2 + s + 6}$

In Problems 11–20, determine the partial fraction expansion for the given rational function.

11. $\dfrac{s^2 - 26s - 47}{(s - 1)(s + 2)(s + 5)}$
12. $\dfrac{-s - 7}{(s + 1)(s - 2)}$
13. $\dfrac{-2s^2 - 3s - 2}{s(s + 1)^2}$
14. $\dfrac{-8s^2 - 5s + 9}{(s + 1)(s^2 - 3s + 2)}$
15. $\dfrac{8s - 2s^2 - 14}{(s + 1)(s^2 - 2s + 5)}$
16. $\dfrac{-5s - 36}{(s + 2)(s^2 + 9)}$
17. $\dfrac{3s + 5}{s(s^2 + s - 6)}$
18. $\dfrac{3s^2 + 5s + 3}{s^4 + s^3}$

19. $\dfrac{1}{(s-3)(s^2+2s+2)}$

20. $\dfrac{s}{(s-1)(s^2-1)}$

In Problems 21–30, determine $\mathcal{L}^{-1}\{F\}$.

21. $F(s) = \dfrac{6s^2-13s+2}{s(s-1)(s-6)}$

22. $F(s) = \dfrac{s+11}{(s-1)(s+3)}$

23. $F(s) = \dfrac{5s^2+34s+53}{(s+3)^2(s+1)}$

24. $F(s) = \dfrac{7s^2-41s+84}{(s-1)(s^2-4s+13)}$

25. $F(s) = \dfrac{7s^2+23s+30}{(s-2)(s^2+2s+5)}$

26. $F(s) = \dfrac{7s^3-2s^2-3s+6}{s^3(s-2)}$

27. $s^2F(s) - 4F(s) = \dfrac{5}{s+1}$

28. $s^2F(s) + sF(s) - 6F(s) = \dfrac{s^2+4}{s^2+s}$

29. $sF(s) + 2F(s) = \dfrac{10s^2+12s+14}{s^2-2s+2}$

30. $sF(s) - F(s) = \dfrac{2s+5}{s^2+2s+1}$

31. Determine the Laplace transform of each of the following functions:

(a) $f_1(t) = \begin{cases} 0, & t = 2, \\ t, & t \neq 2. \end{cases}$

(b) $f_2(t) = \begin{cases} 5, & t = 1, \\ 2, & t = 6, \\ t, & t \neq 1, 6. \end{cases}$

(c) $f_3(t) = t$.

Which of the preceding functions is the inverse Laplace transform of $1/s^2$?

32. Determine the Laplace transform of each of the following functions:

(a) $f_1(t) = \begin{cases} t, & t = 1, 2, 3, \ldots, \\ e^t, & t \neq 1, 2, 3, \ldots. \end{cases}$

(b) $f_2(t) = \begin{cases} e^t, & t \neq 5, 8, \\ 6, & t = 5, \\ 0, & t = 8. \end{cases}$

(c) $f_3(t) = e^t$.

Which of the preceding functions is the inverse Laplace transform of $1/(s-1)$?

Theorem 6 in Section 7.3 on page 364 can be expressed in terms of the inverse Laplace transform as

$$\mathcal{L}^{-1}\left\{\frac{d^nF}{ds^n}\right\}(t) = (-t)^n f(t),$$

where $f = \mathcal{L}^{-1}\{F\}$. *Use this equation in Problems 33–36 to compute* $\mathcal{L}^{-1}\{F\}$.

33. $F(s) = \ln\left(\dfrac{s+2}{s-5}\right)$

34. $F(s) = \ln\left(\dfrac{s-4}{s-3}\right)$

35. $F(s) = \ln\left(\dfrac{s^2+9}{s^2+1}\right)$

36. $F(s) = \arctan(1/s)$

37. Prove Theorem 7, page 368, on the linearity of the inverse transform. [*Hint:* Show that the right-hand side of equation (3) is a continuous function on $[0, \infty)$ whose Laplace transform is $F_1(s) + F_2(s)$.]

38. Residue Computation. Let $P(s)/Q(s)$ be a rational function with deg $P <$ deg Q and suppose $s - r$ is a nonrepeated linear factor of $Q(s)$. Prove that the portion of the partial fraction expansion of $P(s)/Q(s)$ corresponding to $s - r$ is

$$\frac{A}{s-r},$$

where A (called the **residue**) is given by the formula

$$A = \lim_{s \to r} \frac{(s-r)P(s)}{Q(s)} = \frac{P(r)}{Q'(r)}.$$

39. Use the residue computation formula derived in Problem 38 to determine quickly the partial fraction expansion for

$$F(s) = \frac{2s+1}{s(s-1)(s+2)}.$$

40. Heaviside's Expansion Formula.[†] Let $P(s)$ and $Q(s)$ be polynomials with the degree of $P(s)$ less than the degree of $Q(s)$. Let

$$Q(s) = (s-r_1)(s-r_2)\cdots(s-r_n),$$

where the r_i's are distinct real numbers. Show that

$$\mathcal{L}^{-1}\left\{\frac{P}{Q}\right\}(t) = \sum_{i=1}^{n} \frac{P(r_i)}{Q'(r_i)} e^{r_i t}.$$

[†]*Historical Footnote:* This formula played an important role in the "operational solution" to ordinary differential equations developed by Oliver Heaviside in the 1890s.

41. Use Heaviside's expansion formula derived in Problem 40 to determine the inverse Laplace transform of

$$F(s) = \frac{3s^2 - 16s + 5}{(s+1)(s-3)(s-2)} .$$

42. Complex Residues. Let $P(s)/Q(s)$ be a rational function with deg $P <$ deg Q and suppose $(s-\alpha)^2 + \beta^2$ is a nonrepeated quadratic factor of Q. (That is, $\alpha \pm i\beta$ are complex conjugate zeros of Q.) Prove that the portion of the partial fraction expansion of $P(s)/Q(s)$ corresponding to $(s-\alpha)^2 + \beta^2$ is

$$\frac{A(s-\alpha) + \beta B}{(s-\alpha)^2 + \beta^2} ,$$

where the **complex residue** $\beta B + i\beta A$ is given by the formula

$$\beta B + i\beta A = \lim_{s \to \alpha + i\beta} \frac{[(s-\alpha)^2 + \beta^2]P(s)}{Q(s)} .$$

(Thus we can determine B and A by taking the real and imaginary parts of the limit and dividing them by β.)

43. Use the residue formulas derived in Problems 38 and 42 to determine the partial fraction expansion for

$$F(s) = \frac{6s^2 + 28}{(s^2 - 2s + 5)(s+2)} .$$

7.5 Solving Initial Value Problems

Our goal is to show how Laplace transforms can be used to solve initial value problems for linear differential equations. Recall that we have already studied ways of solving such initial value problems in Chapter 4. These previous methods required that we first find a *general solution* of the differential equation and then use the initial conditions to determine the desired solution. As we will see, the method of Laplace transforms leads to the solution of the initial value problem *without* first finding a general solution.

Other advantages to the transform method are worth noting. For example, the technique can easily handle equations involving forcing functions having jump discontinuities, as illustrated in Section 7.1. Further, the method can be used for certain linear differential equations with variable coefficients, a special class of integral equations, systems of differential equations, and partial differential equations.

Method of Laplace Transforms

To solve an initial value problem:

(a) Take the Laplace transform of both sides of the equation.
(b) Use the properties of the Laplace transform and the initial conditions to obtain an equation for the Laplace transform of the solution and then solve this equation for the transform.
(c) Determine the inverse Laplace transform of the solution by looking it up in a table or by using a suitable method (such as partial fractions) in combination with the table.

In step (a) we are tacitly assuming the solution is piecewise continuous on $[0, \infty)$ and of exponential order. Once we have obtained the inverse Laplace transform in step (c), we can verify that these tacit assumptions are satisfied.

Example 1 Solve the initial value problem

$$y'' - 2y' + 5y = -8e^{-t} ; \qquad y(0) = 2 , \qquad y'(0) = 12 . \tag{1}$$

Solution The differential equation in (1) is an identity between two functions of t. Hence equality holds for the Laplace transforms of these functions:

$$\mathscr{L}\{y'' - 2y' + 5y\} = \mathscr{L}\{-8e^{-t}\} .$$

Using the linearity property of $\mathscr{L}$ and the previously computed transform of the exponential function, we can write

$$\mathscr{L}\{y''\}(s) - 2\mathscr{L}\{y'\}(s) + 5\mathscr{L}\{y\}(s) = \frac{-8}{s+1} . \tag{2}$$

Now let $Y(s) \coloneqq \mathscr{L}\{y\}(s)$. From the formulas for the Laplace transform of higher-order derivatives (see Section 7.3) and the initial conditions in (1), we find

$$\begin{aligned}\mathscr{L}\{y'\}(s) &= sY(s) - y(0) = sY(s) - 2 ,\\ \mathscr{L}\{y''\}(s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 12 .\end{aligned}$$

Substituting these expressions into (2) and solving for $Y(s)$ yields

$$\begin{aligned}[s^2Y(s) - 2s - 12] - 2[sY(s) - 2] + 5Y(s) &= \frac{-8}{s+1}\\ (s^2 - 2s + 5)Y(s) &= 2s + 8 - \frac{8}{s+1}\\ (s^2 - 2s + 5)Y(s) &= \frac{2s^2 + 10s}{s+1}\\ Y(s) &= \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s+1)} .\end{aligned}$$

Our remaining task is to compute the inverse transform of the rational function $Y(s)$. This was done in Example 7 of Section 7.4, page 373, where, using a partial fraction expansion, we found

$$y(t) = 3e^t \cos 2t + 4e^t \sin 2t - e^{-t} , \tag{3}$$

which is the solution to the initial value problem (1). ◆

As a quick check on the accuracy of our computations, the reader is advised to verify that the computed solution satisfies the given initial conditions.

The reader is probably questioning the wisdom of using the Laplace transform method to solve an initial value problem that can be easily handled by the methods discussed in Chapter 4. The objective of the first few examples in this section is simply to make the reader familiar with the Laplace transform procedure. We will see in Example 4 and in later sections that the method is applicable to problems that cannot be readily handled by the techniques discussed in the previous chapters.

Example 2 Solve the initial value problem

$$y'' + 4y' - 5y = te^t ; \qquad y(0) = 1 , \qquad y'(0) = 0 . \tag{4}$$

Solution Let $Y(s) \coloneqq \mathscr{L}\{y\}(s)$. Taking the Laplace transform of both sides of the differential equation in (4) gives

$$\mathscr{L}\{y''\}(s) + 4\mathscr{L}\{y'\}(s) - 5Y(s) = \frac{1}{(s-1)^2} . \tag{5}$$

Using the initial conditions, we can express $\mathscr{L}\{y'\}(s)$ and $\mathscr{L}\{y''\}(s)$ in terms of $Y(s)$. That is,

$$\mathscr{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 1\,,$$
$$\mathscr{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s\,.$$

Substituting back into (5) and solving for $Y(s)$ gives

$$\begin{aligned}
[s^2Y(s) - s] + 4[sY(s) - 1] - 5Y(s) &= \frac{1}{(s-1)^2} \\
(s^2 + 4s - 5)Y(s) &= s + 4 + \frac{1}{(s-1)^2} \\
(s+5)(s-1)Y(s) &= \frac{s^3 + 2s^2 - 7s + 5}{(s-1)^2} \\
Y(s) &= \frac{s^3 + 2s^2 - 7s + 5}{(s+5)(s-1)^3}\,.
\end{aligned}$$

The partial fraction expansion for $Y(s)$ has the form

(6) $$\frac{s^3 + 2s^2 - 7s + 5}{(s+5)(s-1)^3} = \frac{A}{s+5} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3}\,.$$

Solving for the numerators, we ultimately obtain $A = 35/216$, $B = 181/216$, $C = -1/36$, and $D = 1/6$. Substituting these values into (6) gives

$$Y(s) = \frac{35}{216}\left(\frac{1}{s+5}\right) + \frac{181}{216}\left(\frac{1}{s-1}\right) - \frac{1}{36}\left(\frac{1}{(s-1)^2}\right) + \frac{1}{12}\left(\frac{2}{(s-1)^3}\right),$$

where we have written $D = 1/6 = (1/12)2$ to facilitate the final step of taking the inverse transform. From the tables, we now obtain

(7) $$y(t) = \frac{35}{216}e^{-5t} + \frac{181}{216}e^{t} - \frac{1}{36}te^{t} + \frac{1}{12}t^2e^{t}$$

as the solution to the initial value problem (4). ◆

Example 3 Solve the initial value problem

(8) $$w''(t) - 2w'(t) + 5w(t) = -8e^{\pi - t}\,; \quad w(\pi) = 2\,, \quad w'(\pi) = 12\,.$$

Solution To use the method of Laplace transforms, we first move the initial conditions to $t = 0$. This can be done by setting $y(t) := w(t + \pi)$. Then

$$y'(t) = w'(t + \pi)\,, \qquad y''(t) = w''(t + \pi)\,.$$

Replacing t by $t + \pi$ in the differential equation in (8), we have

(9) $$w''(t + \pi) - 2w'(t + \pi) + 5w(t + \pi) = -8e^{\pi - (t + \pi)} = -8e^{-t}\,.$$

Substituting $y(t) = w(t + \pi)$ in (9), the initial value problem in (8) becomes

$$y''(t) - 2y'(t) + 5y(t) = -8e^{-t}\,; \qquad y(0) = 2\,, \qquad y'(0) = 12\,.$$

Because the initial conditions are now given at the origin, the Laplace transform method is applicable. In fact, we carried out the procedure in Example 1, page 376, where we found

(10) $$y(t) = 3e^{t}\cos 2t + 4e^{t}\sin 2t - e^{-t}\,.$$

Since $w(t+\pi) = y(t)$, then $w(t) = y(t-\pi)$. Hence, replacing t by $t-\pi$ in (10) gives

$$\begin{aligned} w(t) = y(t-\pi) &= 3e^{t-\pi}\cos[2(t-\pi)] + 4e^{t-\pi}\sin[2(t-\pi)] - e^{-(t-\pi)} \\ &= 3e^{t-\pi}\cos 2t + 4e^{t-\pi}\sin 2t - e^{\pi-t} . \quad ◆ \end{aligned}$$

Thus far we have applied the Laplace transform method only to linear equations with constant coefficients. Yet several important equations in mathematical physics involve linear equations whose coefficients are polynomials in t. To solve such equations using Laplace transforms, we apply Theorem 6, page 364, where we proved that

(11) $$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n \mathcal{L}\{f\}}{ds^n}(s) .$$

If we let $n = 1$ and $f(t) = y'(t)$, we find

$$\begin{aligned} \mathcal{L}\{ty'(t)\}(s) &= -\frac{d}{ds}\mathcal{L}\{y'\}(s) \\ &= -\frac{d}{ds}[sY(s) - y(0)] = -sY'(s) - Y(s) . \end{aligned}$$

Similarly, with $n = 1$ and $f(t) = y''(t)$, we obtain from (11)

$$\begin{aligned} \mathcal{L}\{ty''(t)\}(s) &= -\frac{d}{ds}\mathcal{L}\{y''\}(s) \\ &= -\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] \\ &= -s^2Y'(s) - 2sY(s) + y(0) . \end{aligned}$$

Thus, we see that for a linear differential equation in $y(t)$ whose coefficients are polynomials in t, the method of Laplace transforms will convert the given equation into a linear differential equation in $Y(s)$ whose coefficients are polynomials in s. Moreover, if the coefficients of the given equation are polynomials of degree 1 in t, then (regardless of the order of the given equation) the differential equation for $Y(s)$ is just a linear *first-order* equation. Since we know how to solve this first-order equation, the only serious obstacle we may encounter is obtaining the inverse Laplace transform of $Y(s)$. [This problem may be insurmountable, since the solution $y(t)$ may *not* have a Laplace transform.]

In illustrating the technique, we make use of the following fact. *If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order, then*

(12) $$\lim_{s\to\infty}\mathcal{L}\{f\}(s) = 0 .$$

(You may have already guessed this from the entries in Table 7.1, page 356.) An outline of the proof of (12) is given in Exercises 7.3, page 366, Problem 26.

Example 4 Solve the initial value problem

(13) $$y'' + 2ty' - 4y = 1 , \qquad y(0) = y'(0) = 0 .$$

Solution Let $Y(s) = \mathcal{L}\{y\}(s)$ and take the Laplace transform of both sides of the equation in (13):

(14) $$\mathcal{L}\{y''\}(s) + 2\mathcal{L}\{ty'(t)\}(s) - 4Y(s) = \frac{1}{s} .$$

Using the initial conditions, we find

$$\mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$$

and

$$\mathcal{L}\{ty'(t)\}(s) = -\frac{d}{ds}\mathcal{L}\{y'\}(s)$$
$$= -\frac{d}{ds}[sY(s) - y(0)] = -sY'(s) - Y(s) .$$

Substituting these expressions into (14) gives

$$s^2Y(s) + 2[-sY'(s) - Y(s)] - 4Y(s) = \frac{1}{s}$$
$$-2sY'(s) + (s^2 - 6)Y(s) = \frac{1}{s}$$

(15)
$$Y'(s) + \left(\frac{3}{s} - \frac{s}{2}\right)Y(s) = \frac{-1}{2s^2} .$$

Equation (15) is a linear first-order equation and has the integrating factor

$$\mu(s) = e^{\int(3/s - s/2)ds} = e^{\ln s^3 - s^2/4} = s^3e^{-s^2/4}$$

(see Section 2.3). Multiplying (15) by $\mu(s)$, we obtain

$$\frac{d}{ds}\{\mu(s)Y(s)\} = \frac{d}{ds}\left\{s^3e^{-s^2/4}Y(s)\right\} = -\frac{s}{2}e^{-s^2/4} .$$

Integrating and solving for $Y(s)$ yields

$$s^3e^{-s^2/4}Y(s) = -\int\frac{s}{2}e^{-s^2/4}\,ds = e^{-s^2/4} + C$$

(16)
$$Y(s) = \frac{1}{s^3} + C\frac{e^{s^2/4}}{s^3} .$$

Now if $Y(s)$ is the Laplace transform of a piecewise continuous function of exponential order, then it follows from equation (12) that

$$\lim_{s\to\infty} Y(s) = 0 .$$

For this to occur, the constant C in equation (16) must be zero. Hence, $Y(s) = 1/s^3$, and taking the inverse transform gives $y(t) = t^2/2$. We can easily verify that $y(t) = t^2/2$ is the solution to the given initial value problem by substituting it into (13). ◆

We end this section with an application from **control theory.** Let's consider a servomechanism that models an automatic pilot. Such a mechanism applies a torque to the steering control shaft so that a plane or boat will follow a prescribed course. If we let $y(t)$ be the true direction (angle) of the craft at time t and $g(t)$ be the desired direction at time t, then

$$e(t) \coloneqq y(t) - g(t)$$

denotes the **error** or **deviation** between the desired direction and the true direction.

Let's assume that the servomechanism can measure the error $e(t)$ and feed back to the steering shaft a component of torque that is proportional to $e(t)$ but opposite in sign (see Figure 7.6 on page 381). Newton's second law, expressed in terms of torques, states that

$$(\textbf{moment of inertia}) \times (\textbf{angular acceleration}) = \textbf{total torque.}$$

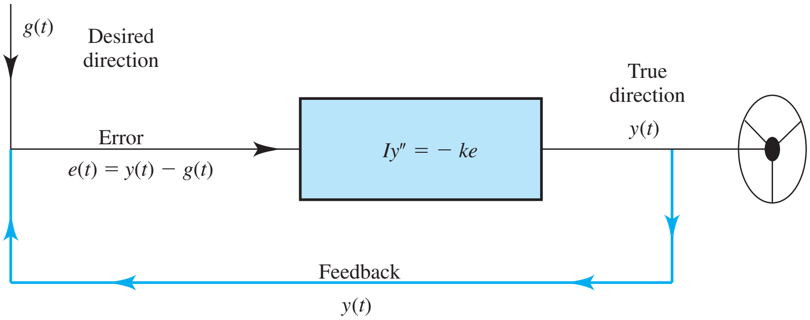

Figure 7.6 Servomechanism with feedback

For the servomechanism described, this becomes

(17) $$Iy''(t) = -ke(t)\ ,$$

where I is the moment of inertia of the steering shaft and k is a positive proportionality constant.

Example 5 Determine the error $e(t)$ for the automatic pilot if the steering shaft is initially at rest in the zero direction and the desired direction is given by $g(t) = at$, where a is a constant.

Solution Based on the discussion leading to equation (17), a model for the mechanism is given by the initial value problem

(18) $$Iy''(t) = -ke(t)\ ; \qquad y(0) = 0\ , \qquad y'(0) = 0\ ,$$

where $e(t) = y(t) - g(t) = y(t) - at$. We begin by taking the Laplace transform of both sides of (18):

$$I\mathcal{L}\{y''\}(s) = -k\mathcal{L}\{e\}(s)$$
$$I[s^2Y(s) - sy(0) - y'(0)] = -kE(s)$$

(19) $$s^2IY(s) = -kE(s)\ ,$$

where $Y(s) = \mathcal{L}\{y\}(s)$ and $E(s) = \mathcal{L}\{e\}(s)$. Since

$$E(s) = \mathcal{L}\{y(t) - at\}(s) = Y(s) - \mathcal{L}\{at\}(s) = Y(s) - as^{-2}\ ,$$

we find from (19) that

$$s^2IE(s) + aI = -kE(s)\ .$$

Solving this equation for $E(s)$ gives

$$E(s) = -\frac{aI}{s^2I + k} = \frac{-a}{\sqrt{k/I}}\,\frac{\sqrt{k/I}}{s^2 + k/I}\ .$$

Hence, on taking the inverse Laplace transform, we obtain the error

(20) $$e(t) = -\frac{a}{\sqrt{k/I}}\sin\left(\sqrt{k/I}\,t\right)\ . \quad ◆$$

As we can see from equation (20), the automatic pilot will oscillate back and forth about the desired course, always "oversteering" by the factor $a/\sqrt{k/I}$. Clearly, we can make the

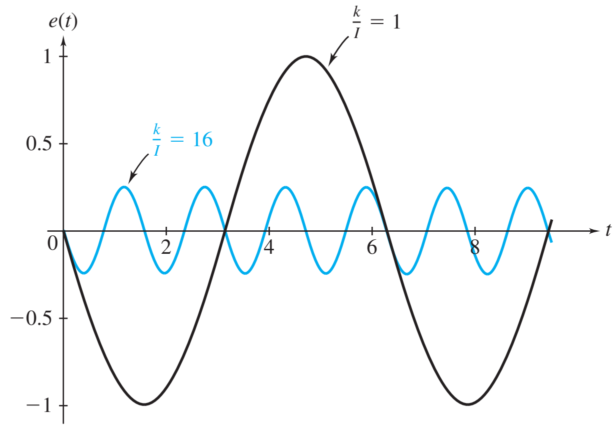

Figure 7.7 Error for automatic pilot when $k/I = 1$ and when $k/I = 16$

error small by making k large relative to I, but then the term $\sqrt{k/I}$ becomes large, causing the error to oscillate more rapidly. (See Figure 7.7.) As with vibrations, the oscillations or oversteering can be controlled by introducing a damping torque proportional to $e'(t)$ but opposite in sign (see Problem 40).

7.5 EXERCISES

In Problems 1–14, solve the given initial value problem using the method of Laplace transforms.

1. $y'' - 2y' + 5y = 0$; $\quad y(0) = 2$, $\quad y'(0) = 4$
2. $y'' - y' - 2y = 0$; $\quad y(0) = -2$, $\quad y'(0) = 5$
3. $y'' + 6y' + 9y = 0$; $\quad y(0) = -1$, $\quad y'(0) = 6$
4. $y'' + 6y' + 5y = 12e^t$; $\quad y(0) = -1$, $\quad y'(0) = 7$
5. $w'' + w = t^2 + 2$; $\quad w(0) = 1$, $\quad w'(0) = -1$
6. $y'' - 4y' + 5y = 4e^{3t}$; $\quad y(0) = 2$, $\quad y'(0) = 7$
7. $y'' - 7y' + 10y = 9\cos t + 7\sin t$; $y(0) = 5$, $\quad y'(0) = -4$
8. $y'' + 4y = 4t^2 - 4t + 10$; $y(0) = 0$, $\quad y'(0) = 3$
9. $z'' + 5z' - 6z = 21e^{t-1}$; $z(1) = -1$, $\quad z'(1) = 9$
10. $y'' - 4y = 4t - 8e^{-2t}$; $\quad y(0) = 0$, $\quad y'(0) = 5$
11. $y'' - y = t - 2$; $\quad y(2) = 3$, $\quad y'(2) = 0$
12. $w'' - 2w' + w = 6t - 2$; $w(-1) = 3$; $\quad w'(-1) = 7$
13. $y'' - y' - 2y = -8\cos t - 2\sin t$; $y(\pi/2) = 1$, $\quad y'(\pi/2) = 0$
14. $y'' + y = t$; $\quad y(\pi) = 0$, $\quad y'(\pi) = 0$

In Problems 15–24, solve for $Y(s)$, the Laplace transform of the solution $y(t)$ to the given initial value problem.

15. $y'' - 3y' + 2y = \cos t$; $\quad y(0) = 0$, $\quad y'(0) = -1$
16. $y'' + 6y = t^2 - 1$; $\quad y(0) = 0$, $\quad y'(0) = -1$
17. $y'' + y' - y = t^3$; $\quad y(0) = 1$, $\quad y'(0) = 0$
18. $y'' - 2y' - y = e^{2t} - e^t$; $\quad y(0) = 1$, $\quad y'(0) = 3$
19. $y'' + 5y' - y = e^t - 1$; $\quad y(0) = 1$, $\quad y'(0) = 1$
20. $y'' + 3y = t^3$; $\quad y(0) = 0$, $\quad y'(0) = 0$
21. $y'' - 2y' + y = \cos t - \sin t$; $\quad y(0) = 1$, $\quad y'(0) = 3$

22. $y'' - 6y' + 5y = te^t\,;\quad y(0) = 2\,,\quad y'(0) = -1$

23. $y'' + 4y = g(t)\,;\quad y(0) = -1\,,\quad y'(0) = 0\,,$
where
$$g(t) = \begin{cases} t\,, & t < 2\,, \\ 5\,, & t > 2 \end{cases}$$

24. $y'' - y = g(t)\,;\quad y(0) = 1\,,\quad y'(0) = 2\,,$
where
$$g(t) = \begin{cases} 1\,, & t < 3\,, \\ t\,, & t > 3 \end{cases}$$

In Problems 25–28, solve the given third-order initial value problem for $y(t)$ using the method of Laplace transforms.

25. $y''' - y'' + y' - y = 0\,;$
$y(0) = 1\,,\quad y'(0) = 1\,,\quad y''(0) = 3$

26. $y''' + 4y'' + y' - 6y = -12\,;$
$y(0) = 1\,,\quad y'(0) = 4\,,\quad y''(0) = -2$

27. $y''' + 3y'' + 3y' + y = 0\,;$
$y(0) = -4\,,\quad y'(0) = 4\,,\quad y''(0) = -2$

28. $y''' + y'' + 3y' - 5y = 16e^{-t}\,;$
$y(0) = 0\,,\quad y'(0) = 2\,,\quad y''(0) = -4$

In Problems 29–32, use the method of Laplace transforms to find a general solution to the given differential equation by assuming $y(0) = a$ and $y'(0) = b$, where a and b are arbitrary constants.

29. $y'' - 4y' + 3y = 0$

30. $y'' + 6y' + 5y = t$

31. $y'' + 2y' + 2y = 5$

32. $y'' - 5y' + 6y = -6te^{2t}$

33. Use Theorem 6 in Section 7.3, page 364, to show that
$$\mathscr{L}\{t^2y'(t)\}(s) = sY''(s) + 2Y'(s)\,,$$
where $Y(s) = \mathscr{L}\{y\}(s)$.

34. Use Theorem 6 in Section 7.3, page 364, to show that
$$\mathscr{L}\{t^2y''(t)\}(s) = s^2Y''(s) + 4sY'(s) + 2Y(s)\,,$$
where $Y(s) = \mathscr{L}\{y\}(s)$.

In Problems 35–38, find solutions to the given initial value problem.

35. $y'' + 3ty' - 6y = 1\,;\quad y(0) = 0\,,\quad y'(0) = 0$

36. $ty'' - ty' + y = 2\,;\quad y(0) = 2\,,\quad y'(0) = -1$

37. $ty'' - 2y' + ty = 0\,;\quad y(0) = 1\,,\quad y'(0) = 0$
[*Hint:* $\mathscr{L}^{-1}\{1/(s^2+1)^2\}(t) = (\sin t - t\cos t)/2.$]

38. $y'' + ty' - y = 0\,;$
$y(0) = 0\,,\quad y'(0) = 3$

39. Determine the error $e(t)$ for the automatic pilot in Example 5, page 381, if the shaft is initially at rest in the zero direction and the desired direction is $g(t) = a$, where a is a constant.

40. In Example 5 assume that in order to control oscillations, a component of torque proportional to $e'(t)$, but opposite in sign, is also fed back to the steering shaft. Show that equation (17) is now replaced by
$$Iy''(t) = -ke(t) - \mu e'(t)\,,$$
where μ is a positive constant. Determine the error $e(t)$ for the automatic pilot with mild damping (i.e., $\mu < 2\sqrt{Ik}$) if the steering shaft is initially at rest in the zero direction and the desired direction is given by $g(t) = a$, where a is a constant.

41. In Problem 40 determine the error $e(t)$ when the desired direction is given by $g(t) = at$, where a is a constant.

7.6 Transforms of Discontinuous Functions

In this section we study special functions that often arise when the method of Laplace transforms is applied to physical problems. Of particular interest are methods for handling functions with jump discontinuities. As we saw in the mixing problem of Section 7.1, jump discontinuities occur naturally in any physical situation that involves switching. Finding the Laplace transforms of such functions is straightforward; however, we need some theory for inverting these transforms. To facilitate this, Oliver Heaviside introduced the following step function.

Unit Step Function

Definition 5. The **unit step function** $u(t)$ is defined by

$$\textbf{(1)} \qquad u(t) := \begin{cases} 0\,, & t < 0\,, \\ 1\,, & 0 < t\,. \end{cases}$$

(Any Riemann integral, like the Laplace transform, of a function is unaffected if the integrand's value at a single point is changed by a finite amount. Therefore, we do not specify a value for $u(t)$ at $t = 0$.)

By shifting the argument of $u(t)$, the jump can be moved to a different location. That is,

$$\textbf{(2)} \qquad u(t-a) = \begin{cases} 0\,, & t-a < 0\,, \\ 1\,, & 0 < t-a \end{cases} = \begin{cases} 0\,, & t < a \\ 1\,, & a < t \end{cases}$$

has its jump at $t = a$. By multiplying by a constant M, the height of the jump can also be modified:

$$Mu(t-a) = \begin{cases} 0\,, & t < a\,, \\ M\,, & a < t\,. \end{cases}$$

See Figure 7.8.

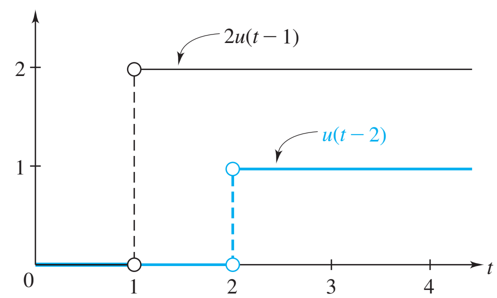

Figure 7.8 Two-step functions expressed using the unit step function

To simplify the formulas for piecewise continuous functions, we employ the rectangular window, which turns the step function on and then turns it back off.

Rectangular Window Function

Definition 6. The **rectangular window function** $\Pi_{a,b}(t)$ is defined by†

$$\textbf{(3)} \qquad \Pi_{a,b}(t) := u(t-a) - u(t-b) = \begin{cases} 0\,, & t < a\,, \\ 1\,, & a < t < b\,, \\ 0\,, & b < t\,. \end{cases}$$

†Also known as the **square pulse**, or the **boxcar function**.

The function $\Pi_{a,b}(t)$ is displayed in Figure 7.9, and Figure 7.10, illustrating multiplication of a function by $\Pi_{a,b}(t)$, justifies its name.

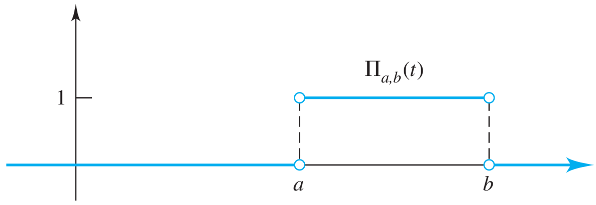

Figure 7.9 The rectangular window

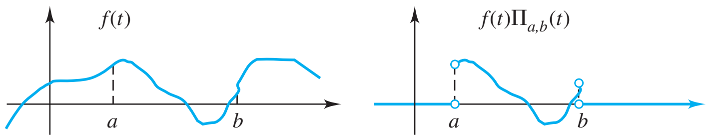

Figure 7.10 The windowing effect of $\Pi_{a,b}(t)$

Any piecewise continuous function can be expressed in terms of window and step functions.

Example 1 Write the function

$$f(t) = \begin{cases} 3\,, & t < 2\,, \\ 1\,, & 2 < t < 5\,, \\ t\,, & 5 < t < 8\,, \\ t^2/10\,, & 8 < t \end{cases} \tag{4}$$

(see Figure 7.11 on page 386) in terms of window and step functions.

Solution Clearly, from the figure we want to window the function in the intervals (0, 2), (2, 5), and (5, 8), and to introduce a step for $t > 8$. From (5) we read off the desired representation as

$$f(t) = 3\Pi_{0,2}(t) + 1\Pi_{2,5}(t) + t\Pi_{5,8}(t) + (t^2/10)u(t-8)\,. \tag{5}$$ ◆

The Laplace transform of $u(t-a)$ with $a \geq 0$ is

$$\boxed{\mathscr{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s}\,,} \tag{6}$$

since, for $s > 0$,

$$\mathscr{L}\{u(t-a)\}(s) = \int_0^\infty e^{-st}u(t-a)\,dt = \int_a^\infty e^{-st}dt$$

$$= \lim_{N\to\infty} \frac{-e^{-st}}{s}\bigg|_a^N = \frac{e^{-as}}{s}\,.$$

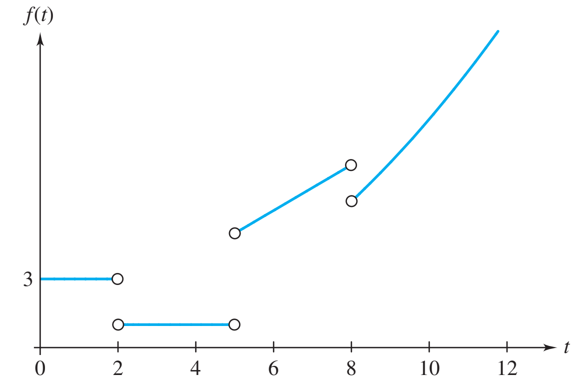

Figure 7.11 Graph of $f(t)$ in equation (4)

Conversely, for $a > 0$, we say that the piecewise continuous function $u(t-a)$ is an inverse Laplace transform for e^{-as}/s and we write

$$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\}(t) = u(t-a) .$$

For the rectangular window function, we deduce from (6) that

$$\mathscr{L}\{\Pi_{a,b}(t)\}(s) = \mathscr{L}\{u(t-a) - u(t-b)\}(s) = [e^{-sa} - e^{-sb}]/s\,, \; 0 < a < b. \tag{7}$$

The translation property of $F(s)$ discussed in Section 7.3 described the effect on the Laplace transform of multiplying a function by e^{at}. The next theorem illustrates an analogous effect of multiplying the Laplace transform of a function by e^{-as}.

Translation in t

Theorem 8. Let $F(s) = \mathscr{L}\{f\}(s)$ exist for $s > \alpha \geq 0$. If a is a positive constant, then

$$\mathscr{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)\,, \tag{8}$$

and, conversely, an inverse Laplace transform† of $e^{-as}F(s)$ is given by

$$\mathscr{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a)\,. \tag{9}$$

Proof. By the definition of the Laplace transform,

$$\begin{aligned} \mathscr{L}\{f(t-a)u(t-a)\}(s) &= \int_0^\infty e^{-st}f(t-a)u(t-a)\,dt \\ &= \int_a^\infty e^{-st}f(t-a)\,dt\,, \end{aligned} \tag{10}$$

†This inverse transform is in fact a *continuous* function of t if $f(0) = 0$ and $f(t)$ is continuous for $t \geq 0$; the values of $f(t)$ for $t < 0$ are of no consequence, since the factor $u(t-a)$ is zero there.

where, in the last equation, we used the fact that $u(t-a)$ is zero for $t<a$ and equals 1 for $t>a$. Now let $v=t-a$. Then we have $dv=dt$, and equation (10) becomes

$$\mathscr{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty e^{-as}e^{-sv}f(v)\,dv$$
$$= e^{-as}\int_0^\infty e^{-sv}f(v)\,dv = e^{-as}F(s)\,. \quad \blacklozenge$$

Notice that formula (8) includes as a special case the formula for $\mathscr{L}\{u(t-a)\}$; indeed, if we take $f(t)\equiv 1$, then $F(s)=1/s$ and (8) becomes $\mathscr{L}\{u(t-a)\}(s)=e^{-as}/s$.

In practice it is more common to be faced with the problem of computing the transform of a function expressed as $g(t)u(t-a)$ rather than $f(t-a)u(t-a)$. To compute $\mathscr{L}\{g(t)u(t-a)\}$, we simply identify $g(t)$ with $f(t-a)$ so that $f(t)=g(t+a)$. Equation (8) then gives

(11) $$\mathscr{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathscr{L}\{g(t+a)\}(s)\,.$$

Example 2 Determine the Laplace transform of $t^2u(t-1)$.

Solution To apply equation (11), we take $g(t)=t^2$ and $a=1$. Then

$$g(t+a) = g(t+1) = (t+1)^2 = t^2+2t+1\,.$$

Now the Laplace transform of $g(t+a)$ is

$$\mathscr{L}\{g(t+a)\}(s) = \mathscr{L}\{t^2+2t+1\}(s) = \frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\,.$$

So, by formula (11), we have

$$\mathscr{L}\{t^2u(t-1)\}(s) = e^{-s}\left\{\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right\}. \quad \blacklozenge$$

Example 3 Determine $\mathscr{L}\{(\cos t)u(t-\pi)\}$.

Solution Here $g(t)=\cos t$ and $a=\pi$. Hence,

$$g(t+a) = g(t+\pi) = \cos(t+\pi) = -\cos t\,,$$

and so the Laplace transform of $g(t+a)$ is

$$\mathscr{L}\{g(t+a)\}(s) = -\mathscr{L}\{\cos t\}(s) = -\frac{s}{s^2+1}\,.$$

Thus, from formula (11), we get

$$\mathscr{L}\{(\cos t)u(t-\pi)\}(s) = -e^{-\pi s}\frac{s}{s^2+1}\,. \quad \blacklozenge$$

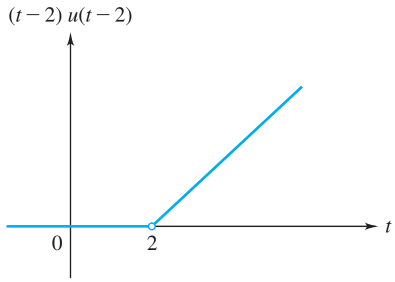

Figure 7.12 Graph of solution to Example 4

In Examples 2 and 3, we could also have computed the Laplace transform directly from the definition. In dealing with inverse transforms, however, we do not have a simple alternative formula† upon which to rely, and so formula (9) is especially useful whenever the transform has e^{-as} as a factor.

Example 4 Determine $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\}$ and sketch its graph.

Solution To use the translation property (9), we first express e^{-2s}/s^2 as the product $e^{-as}F(s)$. For this purpose, we put $e^{-as} = e^{-2s}$ and $F(s) = 1/s^2$. Thus, $a = 2$ and

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) = t\,.$$

It now follows from the translation property that

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}(t) = f(t-2)u(t-2) = (t-2)u(t-2)\,.$$

See Figure 7.12. ◆

As we anticipated in the beginning of this section, step functions arise in the modeling of on/off switches, changes in polarity, etc.

Example 5 The current I in an LC series circuit is governed by the initial value problem

$$I''(t) + 4I(t) = g(t)\,; \qquad I(0) = 0\,, \qquad I'(0) = 0\,, \tag{12}$$

where

$$g(t) := \begin{cases} 1\,, & 0 < t < 1\,, \\ -1\,, & 1 < t < 2\,, \\ 0\,, & 2 < t\,. \end{cases}$$

Determine the current as a function of time t.

Solution Let $J(s) := \mathscr{L}\{I\}(s)$. Then we have $\mathscr{L}\{I''\}(s) = s^2J(s)$.

---

†Under certain conditions, the inverse transform is given by the contour integral

$$\mathscr{L}^{-1}\{F\}(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}F(s)\,ds\,.$$

See, for example, *Complex Variables and the Laplace Transform for Engineers,* by Wilbur R. LePage (Dover Publications, New York, 2010), or *Fundamentals of Complex Analysis with Applications to Engineering and Science,* 3rd ed., by E. B. Saff and A. D. Snider (Pearson Education, Boston. MA, 2003).

Writing $g(t)$ in terms of the rectangular window function $\Pi_{a,b}(t) = u(t-a) - u(t-b)$, we get

$$\begin{aligned} g(t) &= \Pi_{0,1}(t) + (-1)\Pi_{1,2}(t) = u(t) - u(t-1) - [u(t-1) - u(t-2)] \\ &= 1 - 2u(t-1) + u(t-2) , \end{aligned}$$

and so

$$\mathcal{L}\{g\}(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} .$$

Thus, when we take the Laplace transform of both sides of (12), we obtain

$$\begin{aligned} \mathcal{L}\{I''\}(s) + 4\mathcal{L}\{I\}(s) &= \mathcal{L}\{g\}(s) \\ s^2 J(s) + 4J(s) &= \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} \\ J(s) &= \frac{1}{s(s^2+4)} - \frac{2e^{-s}}{s(s^2+4)} + \frac{e^{-2s}}{s(s^2+4)} . \end{aligned}$$

To find $I = \mathcal{L}^{-1}\{J\}$, we first observe that

$$J(s) = F(s) - 2e^{-s}F(s) + e^{-2s}F(s) ,$$

where

$$F(s) := \frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) .$$

Computing the inverse transform of $F(s)$ gives

$$f(t) := \mathcal{L}^{-1}\{F\}(t) = \frac{1}{4} - \frac{1}{4}\cos 2t .$$

Hence, via the translation property (9), we find

$$\begin{aligned} I(t) &= \mathcal{L}^{-1}\{F(s) - 2e^{-s}F(s) + e^{-2s}F(s)\}(t) \\ &= f(t) - 2f(t-1)u(t-1) + f(t-2)u(t-2) \\ &= \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right) - \left[\frac{1}{2} - \frac{1}{2}\cos 2(t-1)\right]u(t-1) \\ &\quad + \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-2)\right]u(t-2) . \end{aligned}$$

The current is graphed in Figure 7.13. Note that $I(t)$ is smoother than $g(t)$; the former has discontinuities in its second derivative at the points where the latter has jumps. ◆

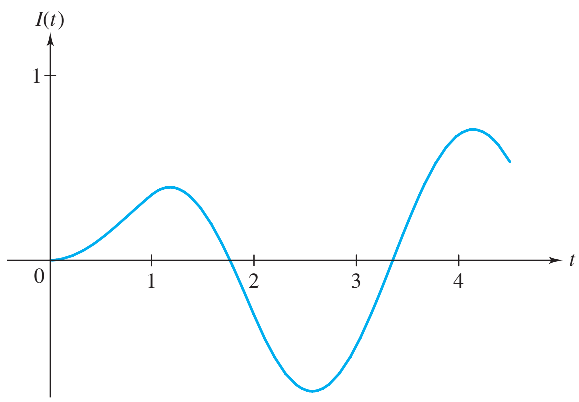

Figure 7.13 Solution to Example 5

7.6 EXERCISES

In Problems 1–4, sketch the graph of the given function and determine its Laplace transform.

1. $(t-1)^2u(t-1)$ **2.** $u(t-1)-u(t-4)$

3. $t^2u(t-2)$ **4.** $tu(t-1)$

In Problems 5–10, express the given function using window and step functions and compute its Laplace transform.

5. $$g(t) = \begin{cases} 0, & 0<t<1, \\ 2, & 1<t<2, \\ 1, & 2<t<3, \\ 3, & 3<t \end{cases}$$

6. $$g(t) = \begin{cases} 0, & 0<t<2, \\ t+1, & 2<t \end{cases}$$

7.

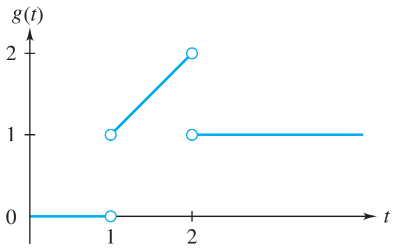

Figure 7.14 Function in Problem 7

8.

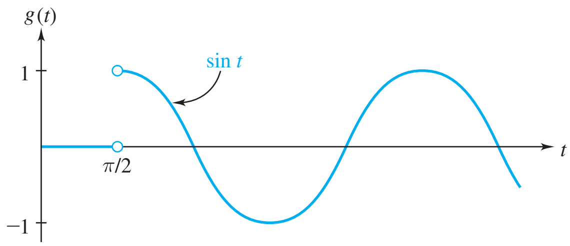

Figure 7.15 Function in Problem 8

9.

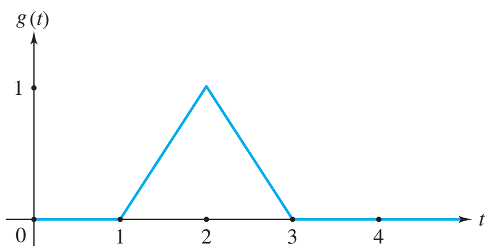

Figure 7.16 Function in Problem 9

10.

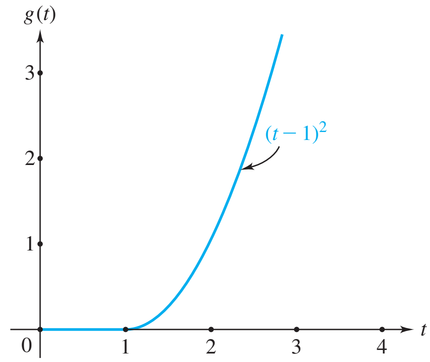

Figure 7.17 Function in Problem 10

In Problems 11–18, determine an inverse Laplace transform of the given function.

11. $\dfrac{e^{-2s}}{s-1}$ **12.** $\dfrac{e^{-3s}}{s^2}$

13. $\dfrac{e^{-2s}-3e^{-4s}}{s+2}$ **14.** $\dfrac{e^{-3s}}{s^2+9}$

15. $\dfrac{se^{-3s}}{s^2+4s+5}$ **16.** $\dfrac{e^{-s}}{s^2+4}$

17. $\dfrac{e^{-3s}(s-5)}{(s+1)(s+2)}$ **18.** $\dfrac{e^{-s}(3s^2-s+2)}{(s-1)(s^2+1)}$

19. The current $I(t)$ in an *RLC* series circuit is governed by the initial value problem

$$I''(t)+2I'(t)+2I(t) = g(t);$$
$$I(0) = 10, \quad I'(0) = 0,$$

where

$$g(t) := \begin{cases} 20, & 0<t<3\pi, \\ 0, & 3\pi<t<4\pi, \\ 20, & 4\pi<t. \end{cases}$$

Determine the current as a function of time *t*. Sketch $I(t)$ for $0<t<8\pi$.

20. The current $I(t)$ in an *LC* series circuit is governed by the initial value problem

$$I''(t)+4I(t) = g(t);$$
$$I(0) = 1, \quad I'(0) = 3,$$

where

$$g(t) := \begin{cases} 3\sin t, & 0 \le t \le 2\pi, \\ 0, & 2\pi<t. \end{cases}$$

Determine the current as a function of time *t*.

In Problems 21–24, solve the given initial value problem using the method of Laplace transforms. Sketch the graph of the solution.

21. $y'' + y = u(t-3)\,;$
$y(0) = 0\,,\quad y'(0) = 1$

22. $w'' + w = u(t-2) - u(t-4)\,;$
$w(0) = 1\,,\quad w'(0) = 0$

23. $y'' + y = t - (t-4)u(t-2)\,;$
$y(0) = 0\,,\quad y'(0) = 1$

24. $y'' + y = 3\sin 2t - 3(\sin 2t)u(t-2\pi)\,;$
$y(0) = 1\,,\quad y'(0) = -2$

In Problems 25–32, solve the given initial value problem using the method of Laplace transforms.

25. $y'' + 2y' + 2y = u(t-2\pi) - u(t-4\pi)\,;$
$y(0) = 1\,,\quad y'(0) = 1$

26. $y'' + 4y' + 4y = u(t-\pi) - u(t-2\pi)\,;$
$y(0) = 0\,,\quad y'(0) = 0$

27. $z'' + 3z' + 2z = e^{-3t}u(t-2)\,;$
$z(0) = 2\,,\quad z'(0) = -3$

28. $y'' + 5y' + 6y = tu(t-2)\,;$
$y(0) = 0\,,\quad y'(0) = 1$

29. $y'' + 4y = g(t)\,;\quad y(0) = 1\,,\quad y'(0) = 3\,,$

where $g(t) = \begin{cases} \sin t\,, & 0 \le t \le 2\pi\,, \\ 0\,, & 2\pi < t \end{cases}$

30. $y'' + 2y' + 10y = g(t)\,;$
$y(0) = -1\,,\quad y'(0) = 0\,,$

where $g(t) = \begin{cases} 10\,, & 0 \le t \le 10\,, \\ 20\,, & 10 < t < 20\,, \\ 0\,, & 20 < t \end{cases}$

31. $y'' + 5y' + 6y = g(t)\,;$
$y(0) = 0\,,\quad y'(0) = 2\,,$

where $g(t) = \begin{cases} 0\,, & 0 \le t < 1\,, \\ t\,, & 1 < t < 5\,, \\ 1\,, & 5 < t \end{cases}$

32. $y'' + 3y' + 2y = g(t)\,;$
$y(0) = 2\,,\quad y'(0) = -1\,,$

where $g(t) = \begin{cases} e^{-t}\,, & 0 \le t < 3\,, \\ 1\,, & 3 < t \end{cases}$

33. The mixing tank in Figure 7.18 initially holds 500 L of a brine solution with a salt concentration of 0.02 kg/L. For the first 10 min of operation, valve *A* is open, adding 12 L/min of brine containing a 0.04 kg/L salt concentration. After 10 min, valve *B* is switched in, adding a 0.06 kg/L concentration at 12 L/min. The exit valve *C* removes 12 L/min, thereby keeping the volume constant. Find the concentration of salt in the tank as a function of time.

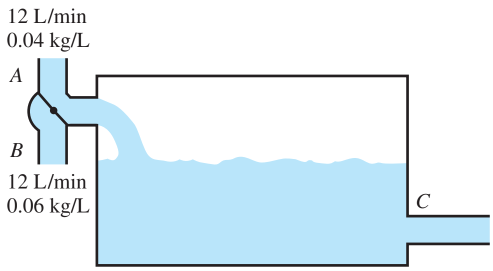

Figure 7.18 Mixing tank

34. Suppose in Problem 33 valve *B* is initially opened for 10 min and then valve *A* is switched in for 10 min. Finally, valve *B* is switched back in. Find the concentration of salt in the tank as a function of time.

35. Suppose valve *C* removes only 6 L/min in Problem 33. Can Laplace transforms be used to solve the problem? Discuss.

36. The **unit triangular pulse** $\Lambda(t)$ is defined by

$$\Lambda(t) := \begin{cases} 0\,, & t < 0\,, \\ 2t\,, & 0 < t < 1/2\,, \\ 2 - 2t\,, & 1/2 < t < 1\,, \\ 0\,, & t > 1\,. \end{cases}$$

(a) Sketch the graph of $\Lambda(t)$. Why is it so named? Why is its symbol appropriate?

(b) Show that $\Lambda(t) = \int_{-\infty}^{t} 2\{\Pi_{0,1/2}(\tau) - \Pi_{1/2,1}(\tau)\}\,d\tau$.

(c) Find the Laplace transform of $\Lambda(t)$.

7.7 Transforms of Periodic and Power Functions

Periodic functions arise frequently in physical situations such as sinusoidal vibrations in structures, and in electromagnetic oscillations in AC machinery and microwave transmission. *Power* functions (t^n) occur in more specialized applications: the square-cube law of biomechanics[*], the cube rule of electoral politics,[†] Coulomb's inverse-square force, and, most significantly, the Taylor series of Section 3.7 and Chapter 8. The manipulation of these functions' transforms (when they exist) is facilitated by the techniques described in this section.

Periodic Function

Definition 7. A function $f(t)$ is said to be **periodic of period T** $(\neq 0)$ if

$$f(t+T) = f(t)$$

for all t in the domain of f.

As we know, the sine and cosine functions are periodic with period 2π and the tangent function is periodic with period π.[‡] To specify a periodic function, it is sufficient to give its values over one period. For example, the square wave function in Figure 7.19 can be expressed as

$$f(t) := \begin{cases} 1\,, & 0 < t < 1\,, \\ -1\,, & 1 < t < 2\,, \end{cases} \qquad \text{and } f(t) \text{ has period 2.} \tag{1}$$

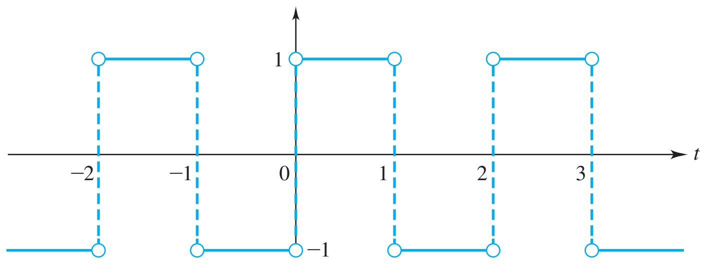

Figure 7.19 Graph of square wave function $f(t)$

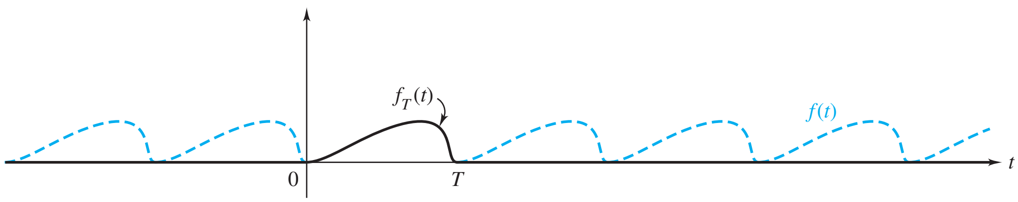

Figure 7.20 Windowed version of periodic function

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[*] The volume of a body increases as the cube of its length; its surface area increases as the square of the length. First formulated by Galileo in 1638 (*Discourses and Mathematical Demonstrations Relating to Two New Sciences*), this principle is useful in explaining the limitations on animal growth.

[†] In a two-party system, the ratio of the seats won equals the cube of the ratio of the votes cast. (G. Upton, "Blocks of voters and the cube law," *British Journal of Political Science*. Vol. 15, Issue 03 (1985): 388–398.)

[‡] A function that has period T will also have period $2T$, $3T$, etc. For example, the sine function has periods 2π, 4π, 6π, etc. Some authors refer to the smallest period as the **fundamental period** or just the period of the function.

It is convenient to introduce a notation for the "windowed" version of a periodic function $f(t)$, using a rectangular window whose width is the period:

$$f_T(t) := f(t)\Pi_{0,T}(t) = f(t)[u(t) - u(t-T)] = \begin{cases} f(t)\,, & 0 < t < T\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{2}$$

(See Figure 7.20 on page 392.) The Laplace transform of $f_T(t)$ is given by

$$F_T(s) = \int_0^\infty e^{-st} f_T(t)\,dt = \int_0^T e^{-st} f(t)\,dt\,.$$

It is related to the Laplace transform of $f(t)$ as follows.

Transform of Periodic Function

Theorem 9. If f has period T and is piecewise continuous on $[0, T]$, then the Laplace transforms

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt \quad \text{and} \quad F_T(s) = \int_0^\infty e^{-st} f_T(t)\,dt = \int_0^T e^{-st} f(t)\,dt$$

are related by

$$F_T(s) = F(s)\left[1 - e^{-sT}\right] \quad \text{or} \quad \boldsymbol{F(s) = \frac{F_T(s)}{1 - e^{-sT}}}. \tag{3}$$

Proof. From (2) and the periodicity of f, we have

$$f_T(t) = f(t)u(t) - f(t)u(t-T) = f(t)u(t) - f(t-T)u(t-T)\,, \tag{4}$$

so taking transforms and applying the translation-in-t property (Theorem 8, page 386) yields $F_T(s) = F(s) - e^{-sT}F(s)$, which is equivalent to (3). ◆

Example 1 Determine $\mathcal{L}\{f\}$, where f is the periodic square wave function in Figure 7.19.

Solution Here $T = 2$. Windowing the function results in $f_T(t) = \Pi_{0,1}(t) - \Pi_{1,2}(t)$, so from the formula for the transform of the window function (equation (7) in Section 7.6, page 386) we get $F_T(s) = (1 - e^{-s})/s - (e^{-s} - e^{-2s})/s = (1 - e^{-s})^2/s$. Therefore (3) implies

$$\mathcal{L}\{f\}(s) = \frac{(1 - e^{-s})^2/s}{1 - e^{-2s}} = \frac{1 - e^{-s}}{(1 + e^{-s})s}\,. \ ◆$$

We next turn to the problem of finding transforms of functions given by a power series. Our approach is simply to apply the formula $\mathcal{L}\{t^n\}(s) = n!/s^{n+1}$, $n = 0, 1, 2, \ldots$, to the terms of the series.

Example 2 Determine $\mathcal{L}\{f\}$, where

$$f(t) := \begin{cases} \dfrac{\sin t}{t}\,, & t \neq 0\,, \\ 1\,, & t = 0\,. \end{cases}$$

Solution We begin by expressing $f(t)$ in a Taylor series[†] about $t = 0$. Since

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots ,$$

then dividing by t, we obtain

$$f(t) = \frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots$$

for $t > 0$. This representation also holds at $t = 0$ since

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} \frac{\sin t}{t} = 1 .$$

Observe that $f(t)$ is continuous on $[0, \infty)$ and of exponential order. Hence, its Laplace transform exists for all s sufficiently large. Because of the linearity of the Laplace transform, we would expect that

$$\begin{aligned} \mathscr{L}\{f\}(s) &= \mathscr{L}\{1\}(s) - \frac{1}{3!}\mathscr{L}\{t^2\}(s) + \frac{1}{5!}\mathscr{L}\{t^4\}(s) + \cdots \\ &= \frac{1}{s} - \frac{2!}{3!s^3} + \frac{4!}{5!s^5} - \frac{6!}{7!s^7} + \cdots \\ &= \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \cdots . \end{aligned}$$

Indeed, using tools from analysis, it can be verified that this series representation is valid for all $s > 1$. Moreover, one can show that the series converges to the function $\arctan(1/s)$ (see Problem 22). Thus,

$$\text{(5)} \qquad \mathscr{L}\left\{\frac{\sin t}{t}\right\}(s) = \arctan \frac{1}{s} . \quad ◆$$

A similar procedure involving the series expansion for $F(s)$ in powers of $1/s$ can be used to compute $f(t) = \mathscr{L}^{-1}\{F\}(t)$ (see Problems 23–25).

We have previously shown, for every nonnegative integer n, that $\mathscr{L}\{t^n\}(s) = n!/s^{n+1}$. But what if the power of t is *not* an integer? Is this formula still valid? To answer this question, we need to extend the idea of "factorial." This is accomplished by the gamma function.[‡]

Gamma Function

Definition 8. The **gamma function** $\Gamma(r)$ is defined by

$$\text{(6)} \qquad \Gamma(r) := \int_0^{\infty} e^{-u} u^{r-1} du , \qquad r > 0 .$$

It can be shown that the integral in (6) converges for $r > 0$. A useful property of the gamma function is the recursive relation

$$\text{(7)} \qquad \Gamma(r + 1) = r\Gamma(r) .$$

[†]For a discussion of Taylor series, see Sections 8.1 and 8.2.

[‡]*Historical Footnote:* The gamma function was introduced by Leonhard Euler.

This identity follows from the definition (6) after performing an integration by parts:

$$\begin{aligned}\Gamma(r+1) &= \int_0^\infty e^{-u}u^r\,du = \lim_{N\to\infty}\int_0^N e^{-u}u^r\,du \\ &= \lim_{N\to\infty}\left\{ e^{-u}u^r\Big|_0^N + \int_0^N re^{-u}u^{r-1}\,du \right\} \\ &= \lim_{N\to\infty}(e^{-N}N^r) + r\lim_{N\to\infty}\int_0^N e^{-u}u^{r-1}\,du \\ &= 0 + r\Gamma(r) = r\Gamma(r)\ .\end{aligned}$$

When r is a positive integer, say $r = n$, then the recursive relation (7) can be repeatedly applied to obtain

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots \\ &= n(n-1)(n-2)\cdots 2\Gamma(1)\ .\end{aligned}$$

It follows from the definition (6) that $\Gamma(1) = 1$, so we find

$$\boldsymbol{\Gamma(n+1) = n!\ .}$$

Thus, the gamma function extends the notion of factorial.

As an application of the gamma function, let's return to the problem of determining the Laplace transform of an arbitrary power of t. We will verify that the formula

(8) $$\mathscr{L}\{t^r\}(s) = \frac{\Gamma(r+1)}{s^{r+1}}$$

holds for every constant $r > -1$.

By definition,

$$\mathscr{L}\{t^r\}(s) = \int_0^\infty e^{-st}t^r\,dt\ .$$

Let's make the substitution $u = st$. Then $du = s\,dt$, and we find

$$\begin{aligned}\mathscr{L}\{t^r\}(s) &= \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^r\left(\frac{1}{s}\right)du \\ &= \frac{1}{s^{r+1}}\int_0^\infty e^{-u}u^r\,du = \frac{\Gamma(r+1)}{s^{r+1}}\ .\end{aligned}$$

Notice that when $r = n$ is a nonnegative integer, then $\Gamma(n+1) = n!$, and so formula (8) reduces to the familiar formula for $\mathscr{L}\{t^n\}$.

Example 3 Given that $\Gamma(1/2) = \sqrt{\pi}$ (see Problem 26), find the Laplace transform of $f(t) = t^{3/2}e^{2t}$.

Solution We'll apply the translation-in-s property (Theorem 3, page 361) to the transform for $t^{3/2}$, which from (8) is given by $\Gamma\left(\frac{3}{2}+1\right)/s^{\frac{3}{2}+1}$. Thanks to the basic gamma function property (7), we can write

$$\Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}\ .$$

Hence $\mathscr{L}\{t^{3/2}\}(s) = \dfrac{3\sqrt{\pi}}{4s^{5/2}}$, and so

$$\mathscr{L}\left\{t^{3/2}e^{2t}\right\}(s) = \frac{3\sqrt{\pi}}{4(s-2)^{5/2}}\ .\ \blacklozenge$$

7.7 EXERCISES

In Problems 1–4, determine $\mathscr{L}\{f\}$, where $f(t)$ is periodic with the given period. Also graph $f(t)$.

1. $f(t) = t$, $0 < t < 2$, and $f(t)$ has period 2.

2. $f(t) = e^t$, $0 < t < 1$, and $f(t)$ has period 1.

3. $f(t) = \begin{cases} e^{-t}, & 0 < t < 1, \\ 1, & 1 < t < 2, \end{cases}$ and $f(t)$ has period 2.

4. $f(t) = \begin{cases} t, & 0 < t < 1, \\ 1 - t, & 1 < t < 2, \end{cases}$ and $f(t)$ has period 2.

In Problems 5–8, determine $\mathscr{L}\{f\}$, where the periodic function is described by its graph.

5.

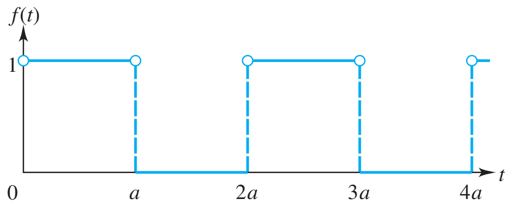

Figure 7.21 Square wave

6.

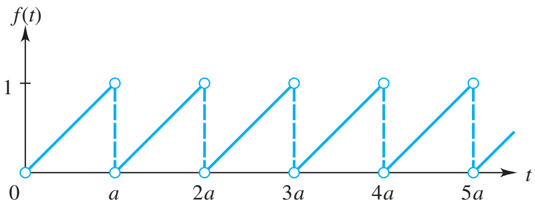

Figure 7.22 Sawtooth wave

7.

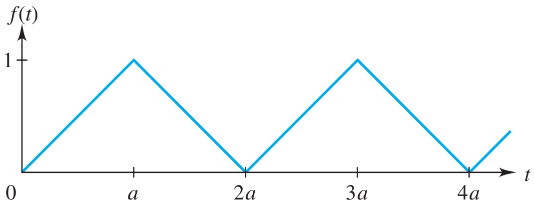

Figure 7.23 Triangular wave

8.

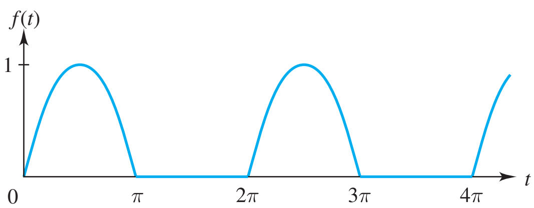

Figure 7.24 Half-rectified sine wave

9. Show that if $\mathscr{L}\{g\}(s) = [(s+\alpha)(1-e^{-Ts})]^{-1}$, where $T > 0$ is fixed, then

$$g(t) = e^{-\alpha t} + e^{-\alpha(t-T)}u(t-T) + e^{-\alpha(t-2T)}u(t-2T) + e^{-\alpha(t-3T)}u(t-3T) + \cdots. \tag{9}$$

[*Hint:* Use the fact that $1 + x + x^2 + \cdots = 1/(1-x)$.]

10. The function $g(t)$ in (9) can be expressed in a more convenient form as follows:

(a) Show that for each $n = 0, 1, 2, \ldots,$

$$g(t) = e^{-\alpha t}\left[\frac{e^{(n+1)\alpha T} - 1}{e^{\alpha T} - 1}\right] \quad \text{for } nT < t < (n+1)T.$$

[*Hint:* Use the fact that $1 + x + x^2 + \cdots + x^n = (x^{n+1} - 1)/(x - 1)$.]

(b) Let $v = t - (n+1)T$. Show that when $nT < t < (n+1)T$, then $-T < v < 0$ and

$$g(t) = \frac{e^{-\alpha v}}{e^{\alpha T} - 1} - \frac{e^{-\alpha t}}{e^{\alpha T} - 1}. \tag{10}$$

(c) Use the facts that the first term in (10) is periodic with period T and the second term is independent of n to sketch the graph of $g(t)$ in (10) for $\alpha = 1$ and $T = 2$.

11. Show that if $\mathscr{L}\{g\}(s) = \beta[(s^2 + \beta^2)(1 - e^{-Ts})]^{-1}$, then

$$g(t) = \sin\beta t + [\sin\beta(t-T)]u(t-T) + [\sin\beta(t-2T)]u(t-2T) + [\sin\beta(t-3T)]u(t-3T) + \cdots.$$

12. Use the result of Problem 11 to show that

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)(1-e^{-\pi s})}\right\}(t) = g(t),$$

where $g(t)$ is periodic with period 2π and

$$g(t) := \begin{cases} \sin t, & 0 \le t \le \pi, \\ 0, & \pi \le t \le 2\pi. \end{cases}$$

In Problems 13 and 14, use the method of Laplace transforms and the results of Problems 9 and 10 to solve the initial value problem.

$$y'' + 3y' + 2y = f(t);$$
$$y(0) = 0, \qquad y'(0) = 0,$$

where $f(t)$ is the periodic function defined in the stated problem.

13. Problem 2

14. Problem 5 with $a = 1$

In Problems 15–18, find a Taylor series for $f(t)$ about $t = 0$. Assuming the Laplace transform of $f(t)$ can be computed term by term, find an expansion for $\mathscr{L}\{f\}(s)$ in powers of $1/s$. If possible, sum the series.

15. $f(t) = e^t$

16. $f(t) = \sin t$

17. $f(t) = \dfrac{1 - \cos t}{t}$

18. $f(t) = e^{-t^2}$

19. Using the recursive relation (7) and the fact that $\Gamma(1/2) = \sqrt{\pi}$, determine

(a) $\mathscr{L}\{t^{-1/2}\}$.

(b) $\mathscr{L}\{t^{7/2}\}$.

20. Use the recursive relation (7) and the fact that $\Gamma(1/2) = \sqrt{\pi}$ to show that

$$\mathscr{L}^{-1}\{s^{-(n+1/2)}\}(t) = \frac{2^n t^{n-1/2}}{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}},$$

where n is a positive integer.

21. Verify (3) in Theorem 9 for the function $f(t) = \sin t$, taking the period as 2π. Repeat, taking the period as 4π.

22. By replacing s by $1/s$ in the Maclaurin series expansion for arctan s, show that

$$\arctan\frac{1}{s} = \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \cdots.$$

23. Find an expansion for $e^{-1/s}$ in powers of $1/s$. Use the expansion for $e^{-1/s}$ to obtain an expansion for $s^{-1/2}e^{-1/s}$ in terms of $1/s^{n+1/2}$. Assuming the inverse Laplace transform can be computed term by term, show that

$$\mathscr{L}^{-1}\{s^{-1/2}e^{-1/s}\}(t) = \frac{1}{\sqrt{\pi t}}\cos 2\sqrt{t}.$$

[*Hint:* Use the result of Problem 20.]

24. Use the procedure discussed in Problem 23 to show that

$$\mathscr{L}^{-1}\{s^{-3/2}e^{-1/s}\}(t) = \frac{1}{\sqrt{\pi}}\sin 2\sqrt{t}.$$

25. Find an expansion for $\ln[1 + (1/s^2)]$ in powers of $1/s$. Assuming the inverse Laplace transform can be computed term by term, show that

$$\mathscr{L}^{-1}\left\{\ln\left(1 + \frac{1}{s^2}\right)\right\}(t) = \frac{2}{t}(1 - \cos t).$$

26. Evaluate $\Gamma(1/2)$ by setting $r = x^2$ in (6) and relating it to the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$. (The latter formula is proved by using polar coordinates to evaluate its square; type "Gaussian integral" into your web browser.)

27. Which of these periodic functions coincides with the square wave in Figure 7.19?

(a) $f(t) = -1, \quad -1 < t < 0; \quad f(t) = 1, \quad 0 < t < 1;$ and f has period 2.

(b) $f(t) = 1, \quad 2 < t < 3; \quad f(t) = -1, \quad 3 < t < 4;$ and f has period 2.

(c) $f(t) = 1, \quad 3 < t < 4; \quad f(t) = -1, \quad 4 < t < 5;$ and f has period 2.

7.8 Convolution

Consider the initial value problem

$$y'' + y = g(t); \qquad y(0) = 0, \qquad y'(0) = 0. \tag{1}$$

If we let $Y(s) = \mathscr{L}\{y\}(s)$ and $G(s) = \mathscr{L}\{g\}(s)$, then taking the Laplace transform of both sides of (1) yields

$$s^2Y(s) + Y(s) = G(s),$$

and hence

$$Y(s) = \left(\frac{1}{s^2+1}\right)G(s). \tag{2}$$

That is, the Laplace transform of the solution to (1) is the product of the Laplace transform of $\sin t$ and the Laplace transform of the forcing term $g(t)$. What we would now like to have is a simple formula for $y(t)$ in terms of $\sin t$ and $g(t)$. Just as the integral of a product is not the product of the integrals, $y(t)$ is not the product of $\sin t$ and $g(t)$. However, we can express $y(t)$ as the "convolution" of $\sin t$ and $g(t)$.

Convolution

Definition 9. Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$. The **convolution** of $f(t)$ and $g(t)$, denoted $f * g$, is defined by

$$(f * g)(t) := \int_0^t f(t - v)g(v)\,dv\,. \tag{3}$$

For example, the convolution of t and t^2 is

$$\begin{aligned} t * t^2 &= \int_0^t (t - v)v^2\,dv = \int_0^t (tv^2 - v^3)\,dv \\ &= \left(\frac{tv^3}{3} - \frac{v^4}{4}\right)\Bigg|_0^t = \frac{t^4}{3} - \frac{t^4}{4} = \frac{t^4}{12}\,. \end{aligned}$$

Convolution is certainly different from ordinary multiplication. For example, $1 * 1 = t \neq 1$ and in general $1 * f \neq f$. However, convolution does satisfy some of the same properties as multiplication.

Properties of Convolution

Theorem 10. Let $f(t)$, $g(t)$, and $h(t)$ be piecewise continuous on $[0, \infty)$. Then

$$f * g = g * f\,, \tag{4}$$

$$f * (g + h) = (f * g) + (f * h)\,, \tag{5}$$

$$(f * g) * h = f * (g * h)\,, \tag{6}$$

$$f * 0 = 0\,. \tag{7}$$

Proof. To prove equation (4), we begin with the definition

$$(f * g)(t) := \int_0^t f(t - v)g(v)\,dv\,.$$

Using the change of variables $w = t - v$, we have

$$(f * g)(t) = \int_t^0 f(w)g(t - w)(-dw) = \int_0^t g(t - w)f(w)\,dw = (g * f)(t)\,,$$

which proves (4). The proofs of equations (5) and (6) are left to the exercises (see Problems 33 and 34). Equation (7) is obvious, since $f(t - v) \cdot 0 \equiv 0$. ◆

Returning to our original goal, we now prove that if $Y(s)$ is the product of the Laplace transforms $F(s)$ and $G(s)$, then $y(t)$ is the convolution $(f * g)(t)$.

Convolution Theorem

Theorem 11. Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order α and set $F(s) = \mathscr{L}\{f\}(s)$ and $G(s) = \mathscr{L}\{g\}(s)$. Then

(8) $$\mathscr{L}\{f * g\}(s) = F(s)G(s),$$

or, equivalently,

(9) $$\mathscr{L}^{-1}\{F(s)G(s)\}(t) = (f * g)(t).$$

Proof. Starting with the left-hand side of (8), we use the definition of convolution to write for $s > \alpha$

$$\mathscr{L}\{f * g\}(s) = \int_0^\infty e^{-st}\left[\int_0^t f(t-v)g(v)\,dv\right]dt.$$

To simplify the evaluation of this iterated integral, we introduce the unit step function $u(t-v)$ and write

$$\mathscr{L}\{f * g\}(s) = \int_0^\infty e^{-st}\left[\int_0^\infty u(t-v)f(t-v)g(v)\,dv\right]dt,$$

where we have used the fact that $u(t-v) = 0$ if $v > t$. Reversing the order of integration† gives

(10) $$\mathscr{L}\{f * g\}(s) = \int_0^\infty g(v)\left[\int_0^\infty e^{-st}u(t-v)f(t-v)\,dt\right]dv.$$

Recall from the translation property in Section 7.6 that the integral in brackets in equation (10) equals $e^{-sv}F(s)$. Hence,

$$\mathscr{L}\{f * g\}(s) = \int_0^\infty g(v)e^{-sv}F(s)\,dv = F(s)\int_0^\infty e^{-sv}g(v)\,dv = F(s)G(s).$$

This proves formula (8). ◆

For the initial value problem (1), recall that we found

$$Y(s) = \left(\frac{1}{s^2+1}\right)G(s) = \mathscr{L}\{\sin t\}(s)\,\mathscr{L}\{g\}(s).$$

It now follows from the convolution theorem that

$$y(t) = \sin t * g(t) = \int_0^t \sin(t-v)g(v)\,dv.$$

Thus we have obtained an integral representation for the solution to the initial value problem (1) for any forcing function $g(t)$ that is piecewise continuous on $[0, \infty)$ and of exponential order.

Example 1 Use the convolution theorem to solve the initial value problem

(11) $$y'' - y = g(t); \qquad y(0) = 1, \qquad y'(0) = 1,$$

where $g(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order.

†This is permitted since, for each $s > \alpha$, the absolute value of the integrand is integrable on $(0, \infty) \times (0, \infty)$.

Solution Let $Y(s) = \mathscr{L}\{y\}(s)$ and $G(s) = \mathscr{L}\{g\}(s)$. Taking the Laplace transform of both sides of the differential equation in (11) and using the initial conditions gives

$$s^2Y(s) - s - 1 - Y(s) = G(s) .$$

Solving for $Y(s)$, we have

$$Y(s) = \frac{s+1}{s^2-1} + \left(\frac{1}{s^2-1}\right)G(s) = \frac{1}{s-1} + \left(\frac{1}{s^2-1}\right)G(s) .$$

Hence,

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) + \mathscr{L}^{-1}\left\{\frac{1}{s^2-1}\,G(s)\right\}(t) \\ &= e^t + \mathscr{L}^{-1}\left\{\frac{1}{s^2-1}\,G(s)\right\}(t) . \end{aligned}$$

Referring to the table of Laplace transforms on the inside back cover, we find

$$\mathscr{L}\{\sinh t\}(s) = \frac{1}{s^2-1} ,$$

so we can now express

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2-1}\,G(s)\right\}(t) = \sinh t * g(t) .$$

Thus,

$$y(t) = e^t + \int_0^t \sinh(t-v)g(v)\,dv$$

is the solution to the initial value problem (11). ◆

Example 2 Use the convolution theorem to find $\mathscr{L}^{-1}\{1/(s^2+1)^2\}$.

Solution Write

$$\frac{1}{(s^2+1)^2} = \left(\frac{1}{s^2+1}\right)\left(\frac{1}{s^2+1}\right) .$$

Since $\mathscr{L}\{\sin t\}(s) = 1/(s^2+1)$, it follows from the convolution theorem that

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}(t) &= \sin t * \sin t = \int_0^t \sin(t-v)\sin v\,dv \\ &= \frac{1}{2}\int_0^t [\cos(2v-t) - \cos t]dv^{\dagger} \\ &= \frac{1}{2}\left[\frac{\sin(2v-t)}{2}\right]_0^t - \frac{1}{2}t\cos t \\ &= \frac{1}{2}\left[\frac{\sin t}{2} - \frac{\sin(-t)}{2}\right] - \frac{1}{2}t\cos t \\ &= \frac{\sin t - t\cos t}{2} . \end{aligned}$$ ◆

† Here we used the identity $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\beta-\alpha) - \cos(\beta+\alpha)]$.

As the preceding example attests, the convolution theorem is useful in determining the inverse transforms of rational functions of s. In fact, it provides an alternative to the method of partial fractions. For example,

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}(t) = \mathscr{L}^{-1}\left\{\left(\frac{1}{s-a}\right)\left(\frac{1}{s-b}\right)\right\}(t) = e^{at} * e^{bt},$$

and all that remains in finding the inverse is to compute the convolution $e^{at} * e^{bt}$.

In the early 1900s, V. Volterra introduced **integro-differential** equations in his study of population growth. These equations enabled him to take into account "hereditary influences." In certain cases, these equations involved a convolution. As the next example shows, the convolution theorem helps to solve such integro-differential equations.

Example 3 Solve the integro-differential equation

$$y'(t) = 1 - \int_0^t y(t-v)e^{-2v}\,dv, \qquad y(0) = 1. \tag{12}$$

Solution Equation (12) can be written as

$$y'(t) = 1 - y(t) * e^{-2t}. \tag{13}$$

Let $Y(s) = \mathscr{L}\{y\}(s)$. Taking the Laplace transform of (13) (with the help of the convolution theorem) and solving for $Y(s)$, we obtain

$$\begin{aligned}
sY(s) - 1 &= \frac{1}{s} - Y(s)\left(\frac{1}{s+2}\right)\\
sY(s) + \left(\frac{1}{s+2}\right)Y(s) &= 1 + \frac{1}{s}\\
\left(\frac{s^2+2s+1}{s+2}\right)Y(s) &= \frac{s+1}{s}\\
Y(s) &= \frac{(s+1)(s+2)}{s(s+1)^2} = \frac{s+2}{s(s+1)}\\
Y(s) &= \frac{2}{s} - \frac{1}{s+1}.
\end{aligned}$$

Hence, $y(t) = 2 - e^{-t}$. ◆

The **transfer function** $H(s)$ of a linear system is defined as the ratio of the Laplace transform of the output function $y(t)$ to the Laplace transform of the input function $g(t)$, under the assumption that all initial conditions are zero. That is, $H(s) = Y(s)/G(s)$. If the linear system is governed by the differential equation

$$ay'' + by' + cy = g(t), \qquad t > 0, \tag{14}$$

where a, b, and c are constants, we can compute the transfer function as follows. Take the Laplace transform of both sides of (14) to get

$$as^2Y(s) - asy(0) - ay'(0) + bsY(s) - by(0) + cY(s) = G(s).$$

Because the initial conditions are assumed to be zero, the equation reduces to

$$(as^2 + bs + c)Y(s) = G(s).$$

Thus the transfer function for equation (14) is

$$\textbf{(15)} \qquad H(s) = \frac{Y(s)}{G(s)} = \frac{1}{as^2 + bs + c} .$$

You may note the similarity of these calculations to those for finding the auxiliary equation for the homogeneous equation associated with (14) (recall Section 4.2, page 157). Indeed, the first step in inverting $Y(s) = G(s)/(as^2 + bs + c)$ would be to find the roots of the denominator $as^2 + bs + c$, which is identical to solving the characteristic equation for (14).

The function $h(t) \coloneqq \mathscr{L}^{-1}\{H\}(t)$ is called the **impulse response function** for the system because, physically speaking, it describes the solution when a mass–spring system is struck by a hammer (see Section 7.9). We can also characterize $h(t)$ as the unique solution to the homogeneous problem

$$\textbf{(16)} \qquad ah'' + bh' + ch = 0 ; \qquad h(0) = 0 , \qquad h'(0) = 1/a .$$

Indeed, observe that taking the Laplace transform of the equation in (16) gives

$$\textbf{(17)} \qquad a[s^2H(s) - sh(0) - h'(0)] + b[sH(s) - h(0)] + cH(s) = 0 .$$

Substituting in $h(0) = 0$ and $h'(0) = 1/a$ and solving for $H(s)$ yields

$$H(s) = \frac{1}{as^2 + bs + c} ,$$

which is the same as the formula for the transfer function given in equation (15).

One nice feature of the impulse response function h is that it can help us describe the solution to the *general* initial value problem

$$\textbf{(18)} \qquad ay'' + by' + cy = g(t) ; \qquad y(0) = y_0 , \qquad y'(0) = y_1 .$$

From the discussion of equation (14), we can see that the convolution $h * g$ is the solution to (18) in the special case when the initial conditions are zero (i.e., $y_0 = y_1 = 0$). To deal with nonzero initial conditions, let y_k denote the solution to the corresponding *homogeneous* initial value problem; that is, y_k solves

$$\textbf{(19)} \qquad ay'' + by' + cy = 0 ; \qquad y(0) = y_0 , \qquad y'(0) = y_1 .$$

Then, the desired solution to the general initial value problem (18) must be $h * g + y_k$. Indeed, it follows from the superposition principle (see Theorem 3 in Section 4.5) that since $h * g$ is a solution to equation (14) and y_k is a solution to the corresponding homogeneous equation, then $h * g + y_k$ is a solution to equation (14). Moreover, since $h * g$ has initial conditions zero,

$$(h * g)(0) + y_k(0) = 0 + y_0 = y_0 ,$$
$$(h * g)'(0) + y_k'(0) = 0 + y_1 = y_1 .$$

We summarize these observations in the following theorem.

Solution Using Impulse Response Function

Theorem 12. Let I be an interval containing the origin. The unique solution to the initial value problem

$$ay'' + by' + cy = g\,; \qquad y(0) = y_0\,, \qquad y'(0) = y_1\,,$$

where a, b, and c are constants and g is continuous on I, is given by

$$y(t) = (h * g)(t) + y_k(t) = \int_0^t h(t-v)g(v)dv + y_k(t)\,, \tag{20}$$

where h is the impulse response function for the system and y_k is the unique solution to (19).

Equation (20) is instructive in that it highlights how the value of y at time t depends on the initial conditions (through $y_k(t)$) and on the nonhomogeneity $g(t)$ (through the convolution integral). It even displays the *causal* nature of the dependence, in that the value of $g(v)$ cannot influence $y(t)$ until $t \geq v$.

A proof of Theorem 12 that does not involve Laplace transforms is outlined in Project E in Chapter 4.

In the next example, we use Theorem 12 to find a formula for the solution to an initial value problem.

Example 4 A linear system is governed by the differential equation

$$y'' + 2y' + 5y = g(t)\,; \qquad y(0) = 2\,, \qquad y'(0) = -2\,. \tag{21}$$

Find the transfer function for the system, the impulse response function, and a formula for the solution.

Solution According to formula (15), the transfer function for (21) is

$$H(s) = \frac{1}{as^2 + bs + c} = \frac{1}{s^2 + 2s + 5} = \frac{1}{(s+1)^2 + 2^2}\,.$$

The inverse Laplace transform of $H(s)$ is the impulse response function

$$\begin{aligned} h(t) &= \mathscr{L}^{-1}\{H\}(t) = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s+1)^2 + 2^2}\right\}(t) \\ &= \frac{1}{2}e^{-t}\sin 2t\,. \end{aligned}$$

To solve the initial value problem, we need the solution to the corresponding homogeneous problem. The auxiliary equation for the homogeneous equation is $r^2 + 2r + 5 = 0$, which has roots $r = -1 \pm 2i$. Thus a general solution is $C_1e^{-t}\cos 2t + C_2e^{-t}\sin 2t$. Choosing C_1 and C_2 so that the initial conditions in (21) are satisfied, we obtain $y_k(t) = 2e^{-t}\cos 2t$.

Hence, a formula for the solution to the initial value problem (21) is

$$(h * g)(t) + y_k(t) = \frac{1}{2}\int_0^t e^{-(t-v)}\sin[2(t-v)]g(v)\,dv + 2e^{-t}\cos 2t\,. \blacklozenge$$

7.8 EXERCISES

In Problems 1–4, use the convolution theorem to obtain a formula for the solution to the given initial value problem, where $g(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order.

1. $y'' - 2y' + y = g(t)\,; \quad y(0) = -1\,, \quad y'(0) = 1$

2. $y'' + 9y = g(t)\,; \quad y(0) = 1\,, \quad y'(0) = 0$

3. $y'' + 4y' + 5y = g(t)\,; \quad y(0) = 1\,, \quad y'(0) = 1$

4. $y'' + y = g(t)\,; \quad y(0) = 0\,, \quad y'(0) = 1$

In Problems 5–12, use the convolution theorem to find the inverse Laplace transform of the given function.

5. $\dfrac{1}{s(s^2+1)}$ **6.** $\dfrac{1}{(s+1)(s+2)}$

7. $\dfrac{14}{(s+2)(s-5)}$ **8.** $\dfrac{1}{(s^2+4)^2}$

9. $\dfrac{s}{(s^2+1)^2}$ **10.** $\dfrac{1}{s^3(s^2+1)}$

11. $\dfrac{s}{(s-1)(s+2)} \quad \left[\textit{Hint: } \dfrac{s}{s-1} = 1 + \dfrac{1}{s-1}.\right]$

12. $\dfrac{s+1}{(s^2+1)^2}$

13. Find the Laplace transform of

$$f(t) := \int_0^t (t-v)e^{3v}dv\,.$$

14. Find the Laplace transform of

$$f(t) := \int_0^t e^v \sin(t-v)\ dv\,.$$

In Problems 15–22, solve the given integral equation or integro-differential equation for $y(t)$.

15. $y(t) + 3\displaystyle\int_0^t y(v)\sin(t-v)\,dv = t$

16. $y(t) + \displaystyle\int_0^t e^{t-v}y(v)\,dv = \sin t$

17. $y(t) + \displaystyle\int_0^t (t-v)y(v)\,dv = 1$

18. $y(t) + \displaystyle\int_0^t (t-v)y(v)\,dv = t^2$

19. $y(t) + \displaystyle\int_0^t (t-v)^2y(v)\,dv = t^3 + 3$

20. $y'(t) + \displaystyle\int_0^t (t-v)y(v)\,dv = t\,, \quad y(0) = 0$

21. $y'(t) + y(t) - \displaystyle\int_0^t y(v)\sin(t-v)\,dv = -\sin t\,,$
$y(0) = 1$

22. $y'(t) - 2\displaystyle\int_0^t e^{t-v}y(v)\,dv = t\,, \quad y(0) = 2$

In Problems 23–28, a linear system is governed by the given initial value problem. Find the transfer function $H(s)$ for the system and the impulse response function $h(t)$ and give a formula for the solution to the initial value problem.

23. $y'' + 9y = g(t)\,; \quad y(0) = 2\,, \quad y'(0) = -3$

24. $y'' - 9y = g(t)\,; \quad y(0) = 2\,, \quad y'(0) = 0$

25. $y'' - y' - 6y = g(t)\,; \quad y(0) = 1\,, \quad y'(0) = 8$

26. $y'' + 2y' - 15y = g(t)\,; \quad y(0) = 0\,, \quad y'(0) = 8$

27. $y'' - 2y' + 5y = g(t)\,; \quad y(0) = 0\,, \quad y'(0) = 2$

28. $y'' - 4y' + 5y = g(t)\,; \quad y(0) = 0\,, \quad y'(0) = 1$

In Problems 29 and 30, the current $I(t)$ in an RLC circuit with voltage source $E(t)$ is governed by the initial value problem

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = e(t)\,,$$
$$I(0) = a\,, \qquad I'(0) = b\,,$$

where $e(t) = E'(t)$ (see Figure 7.25). For the given constants R, L, C, a, and b, find a formula for the solution $I(t)$ in terms of $e(t)$.

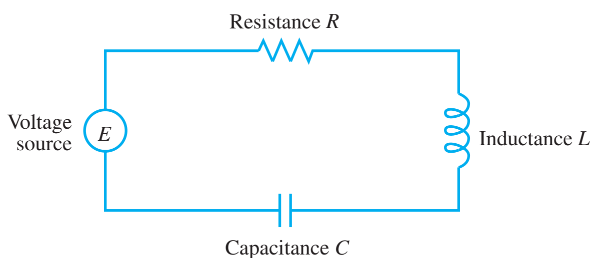

Figure 7.25 Schematic representation of an *RLC* series circuit

29. $R = 20\ \Omega,\ L = 5\ \text{H},\ C = 0.005\ \text{F},\ a = -1\ \text{A},$
$b = 8$ A/sec.

30. $R = 80\ \Omega$, $L = 10$ H, $C = 1/410$ F, $a = 2$ A, $b = -8$ A/sec.

31. Use the convolution theorem and Laplace transforms to compute $1 * 1 * 1$.

32. Use the convolution theorem and Laplace transforms to compute $1 * t * t^2$.

33. Prove property (5) in Theorem 10.

34. Prove property (6) in Theorem 10.

35. Use the convolution theorem to show that

$$\mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\}(t) = \int_0^t f(v)\,dv\,,$$

where $F(s) = \mathscr{L}\{f\}(s)$.

36. Using Theorem 5 in Section 7.3 and the convolution theorem, show that

$$\int_0^t \int_0^v f(z)dz\,dv = \mathscr{L}^{-1}\left\{\frac{F(s)}{s^2}\right\}(t)$$
$$= t\int_0^t f(v)dv - \int_0^t vf(v)dv\,,$$

where $F(s) = \mathscr{L}\{f\}(s)$.

37. Prove directly that if $h(t)$ is the impulse response function characterized by equation (16), then for any continuous $g(t)$, we have $(h * g)(0) = (h * g)'(0) = 0$. [*Hint:* Use Leibniz's rule, described in Project E of Chapter 4.]

7.9 Impulses and the Dirac Delta Function

In mechanical systems, electrical circuits, bending of beams, and other applications, one encounters functions that have a very large value over a very short interval. For example, the strike of a hammer exerts a relatively large force over a relatively short time, and a heavy weight concentrated at a spot on a suspended beam exerts a large force over a very small section of the beam. To deal with violent forces of short duration, physicists and engineers use the delta function introduced by Paul A. M. Dirac. Relaxing our standards of rigor for the moment, we present the following somewhat informal definition.

Dirac Delta Function

Definition 10. The **Dirac delta function** $\delta(t)$ is characterized by the following two properties:

$$\textbf{(1)} \qquad \delta(t) = \begin{cases} 0\,, & t \neq 0\,, \\ \text{“infinite,”} & t = 0\,, \end{cases}$$

and

$$\textbf{(2)} \qquad \int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

for any function $f(t)$ that is continuous on an open interval containing $t = 0$.

By shifting the argument of $\delta(t)$, we have $\delta(t-a) = 0$, $t \neq a$, and

$$\text{(3)} \qquad \int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

for any function $f(t)$ that is continuous on an interval containing $t = a$.

It is obvious that $\delta(t-a)$ is not a function in the usual sense; instead it is an example of what is called a **generalized function** or a **distribution.** Despite this shortcoming, the Dirac delta function was successfully used for several years to solve various physics and engineering problems before Laurent Schwartz mathematically justified its use!

A heuristic argument for the existence of the Dirac delta function can be made by considering the impulse of a force over a short interval. If a force $\mathscr{F}(t)$ is applied from time t_0 to time t_1, then the **impulse** due to $\mathscr{F}$ is the integral

$$\text{Impulse} := \int_{t_0}^{t_1} \mathscr{F}(t)\,dt\,.$$

By Newton's second law, we see that

$$\int_{t_0}^{t_1} \mathscr{F}(t)\,dt = \int_{t_0}^{t_1} m\frac{dv}{dt}dt = mv(t_1) - mv(t_0)\,, \tag{4}$$

where m denotes mass and v denotes velocity. Since mv represents the momentum, we can interpret equation (4) as saying: **The impulse equals the change in momentum.**

When a hammer strikes an object, it transfers momentum to the object. This change in momentum takes place over a very short period of time, say, $[t_0, t_1]$. If we let $\mathscr{F}_1(t)$ represent the force due to the hammer, then the *area* under the curve $\mathscr{F}_1(t)$ is the impulse or change in momentum (see Figure 7.26). If, as is illustrated in Figure 7.27, the same change in momentum takes place over shorter and shorter time intervals—say, $[t_0, t_2]$ or $[t_0, t_3]$—then the average force must get greater and greater in order for the impulses (the areas under the curves $\mathscr{F}_n$) to remain the same. In fact, if the forces $\mathscr{F}_n$ having the same impulse act, respectively, over the intervals $[t_0, t_n]$, where $t_n \to t_0$ as $n \to \infty$, then $\mathscr{F}_n$ approaches a function that is zero for $t \neq t_0$ but has an infinite value for $t = t_0$. Moreover, the areas under the $\mathscr{F}_n$'s have a common value. Normalizing this value to be 1 gives

$$\int_{-\infty}^{\infty} \mathscr{F}_n(t)\,dt = 1 \qquad \text{for all } n\,.$$

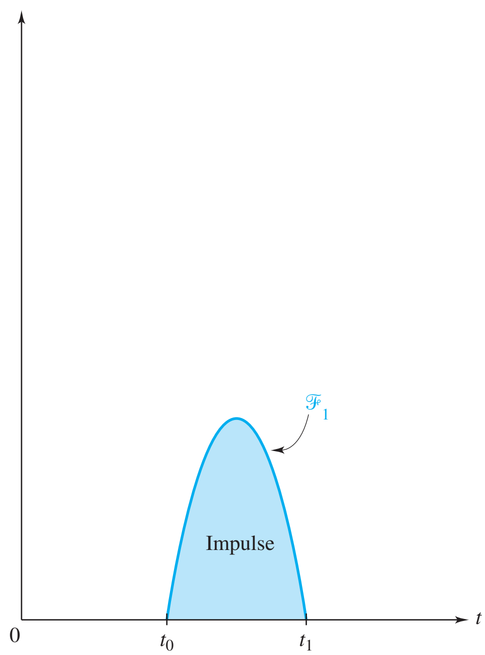

Figure 7.26 Force due to a blow from a hammer

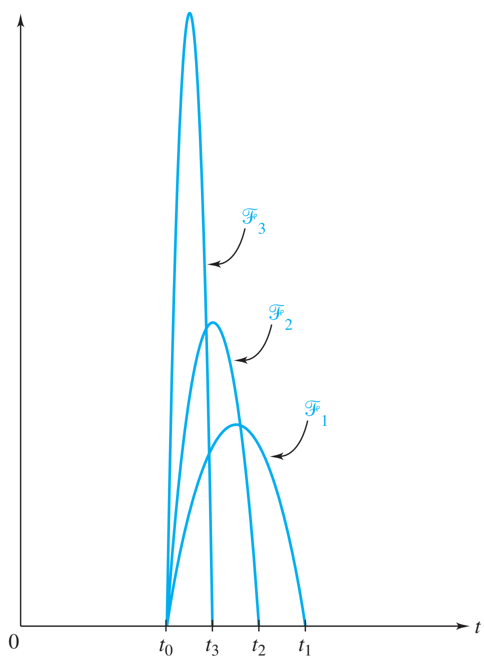

Figure 7.27 Forces with the same impulse

When $t_0 = 0$, we derive from the limiting properties of the $\mathcal{F}_n$'s a "function" δ that satisfies property (1) and the integral condition

(5) $$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 .$$

Notice that (5) is a special case of property (2) that is obtained by taking $f(t) \equiv 1$. It is interesting to note that (1) and (5) actually imply the general property (2) (see Problem 33).

The Laplace transform of the Dirac delta function can be quickly derived from property (3). Since $\delta(t-a) = 0$ for $t \neq a$, then setting $f(t) = e^{-st}$ in (3), we find for $a \geq 0$

$$\int_{0}^{\infty} e^{-st}\delta(t-a)\, dt = \int_{-\infty}^{\infty} e^{-st}\delta(t-a)\, dt = e^{-as} .$$

Thus, for $a \geq 0$,

(6) $$\boldsymbol{\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}} .$$

An interesting connection exists between the unit step function and the Dirac delta function. Observe that as a consequence of equation (5) and the fact that $\delta(x-a)$ is zero for $x < a$ and for $x > a$, we have

(7) $$\begin{aligned} \int_{-\infty}^{t} \delta(x-a)\, dx &= \begin{cases} 0, & t < a, \\ 1, & t > a \end{cases} \\ &= u(t-a) . \end{aligned}$$

If we formally differentiate both sides of (7) with respect to t (in the spirit of the fundamental theorem of calculus), we find

$$\delta(t-a) = u'(t-a) .$$

Thus it appears that the Dirac delta function is the derivative of the unit step function. That is, in fact, the case if we consider "differentiation" in a more general sense.[†]

The Dirac delta function is used in modeling mechanical vibration problems involving an impulse. For example, a mass–spring system at rest that is struck by a hammer exerting an impulse on the mass might be governed by the *symbolic* initial value problem

(8) $$x'' + x = \delta(t) ; \qquad x(0) = 0 , \qquad x'(0) = 0 ,$$

where $x(t)$ denotes the displacement from equilibrium at time t. We refer to this as a symbolic problem because while the left-hand side of equation (8) represents an ordinary function, the right-hand side does *not*. Consequently, it is not clear what we mean by a solution to problem (8). Because $\delta(t)$ is zero everywhere except at $t = 0$, one might be tempted to treat (8) as a homogeneous equation with zero initial conditions. But the solution to the latter is zero everywhere, which certainly does not describe the motion of the spring after the mass is struck by the hammer.

To define what is meant by a solution to (8), recall that $\delta(t)$ is depicted as the limit of forces $\mathcal{F}_n(t)$ having unit impulse and acting over shorter and shorter intervals. If we let $y_n(t)$ be the solution to the initial value problem

(9) $$y_n'' + y_n = \mathcal{F}_n(t) ; \qquad y_n(0) = 0 , \qquad y_n'(0) = 0 ,$$

where δ is replaced by $\mathcal{F}_n$, then we can think of the solution $x(t)$ to (8) as the limit (as $n \to \infty$) of the solutions $y_n(t)$.

[†]See *Distributions, Complex Variables, and Fourier Transforms,* by H. J. Bremermann (Addison-Wesley, Reading, MA, 1965).

For example, let

$$\mathscr{F}_n(t) := n - nu(t - 1/n) = \begin{cases} n\,, & 0 < t < 1/n\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Taking the Laplace transform of equation (9), we find

$$(s^2 + 1)Y_n(s) = \frac{n}{s}(1 - e^{-s/n})\,,$$

and so

$$Y_n(s) = \frac{n}{s(s^2+1)} - e^{-s/n}\frac{n}{s(s^2+1)}\,.$$

Now

$$\frac{n}{s(s^2+1)} = \frac{n}{s} - \frac{ns}{s^2+1} = \mathscr{L}\{n - n\cos t\}(s)\,.$$

Hence,

(10)
$$\begin{aligned} y_n(t) &= n - n\cos t - [n - n\cos(t - 1/n)]\,u(t - 1/n) \\ &= \begin{cases} n - n\cos t\,, & 0 < t < 1/n\,, \\ n\cos(t - 1/n) - n\cos t\,, & 1/n < t\,. \end{cases} \end{aligned}$$

Fix $t > 0$. Then for n large enough, we have $1/n < t$. Thus,

$$\begin{aligned} \lim_{n\to\infty} y_n(t) &= \lim_{n\to\infty}[n\cos(t - 1/n) - n\cos t] \\ &= -\lim_{n\to\infty}\frac{\cos(t - 1/n) - \cos t}{-1/n} \\ &= -\lim_{h\to\infty}\frac{\cos(t + h) - \cos t}{h} \qquad (\text{where } h = -1/n)\,, \\ &= -\frac{d}{dt}(\cos t) = \sin t\,. \end{aligned}$$

Also, for $t = 0$, we have $\lim_{n\to\infty} y_n(0) = 0 = \sin 0$. Therefore,

$$\lim_{n\to\infty} y_n(t) = \sin t\,.$$

Hence, the solution to the symbolic initial value problem (8) is $x(t) = \sin t$.

Fortunately, we do not have to go through the tedious process of solving for each y_n in order to find the solution x of the symbolic problem. It turns out that the Laplace transform method when applied directly to (8) yields the derived solution $x(t)$. Indeed, simply taking the Laplace transform of both sides of (8), we obtain from (6) (with $a = 0$)

$$\begin{aligned} (s^2 + 1)X(s) &= 1\,, \\ X(s) &= \frac{1}{s^2+1}\,, \end{aligned}$$

which gives

$$x(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) = \sin t\,.$$

A peculiarity of using the Dirac delta function is that the solution $x(t) = \sin t$ of the symbolic initial value problem (8) does not satisfy both initial conditions; that is, $x'(0) = 1 \neq 0$. This reflects the fact that the impulse $\delta(t)$ is applied at $t = 0$. Thus, the momentum x' (observe that in equation (8) the mass equals one) jumps abruptly from $x'(0) = 0$ to $x'(0^+) = 1$.

In the next example, the Dirac delta function is used in modeling a mechanical vibration problem.

Example 1 A mass attached to a spring is released from rest 1 m below the equilibrium position for the mass–spring system and begins to vibrate. After π seconds, the mass is struck by a hammer exerting an impulse on the mass. The system is governed by the symbolic initial value problem

$$\frac{d^2x}{dt^2} + 9x = 3\delta(t-\pi)\,; \qquad x(0) = 1\,, \qquad \frac{dx}{dt}(0) = 0\,, \tag{11}$$

where $x(t)$ denotes the displacement from equilibrium at time t. Determine $x(t)$.

Solution Let $X(s) = \mathscr{L}\{x\}(s)$. Since

$$\mathscr{L}\{x''\}(s) = s^2X(s) - s \qquad \text{and} \qquad \mathscr{L}\{\delta(t-\pi)\}(s) = e^{-\pi s}\,,$$

taking the Laplace transform of both sides of (11) and solving for $X(s)$ yields

$$\begin{aligned} s^2X(s) - s + 9X(s) &= 3e^{-\pi s} \\ X(s) &= \frac{s}{s^2+9} + e^{-\pi s}\frac{3}{s^2+9} \\ &= \mathscr{L}\{\cos 3t\}(s) + e^{-\pi s}\mathscr{L}\{\sin 3t\}(s)\,. \end{aligned}$$

Using the translation property (cf. page 386) to determine the inverse Laplace transform of $X(s)$, we find

$$\begin{aligned} x(t) &= \cos 3t + [\sin 3(t-\pi)]u(t-\pi) \\ &= \begin{cases} \cos 3t\,, & t < \pi\,, \\ \cos 3t - \sin 3t\,, & \pi < t \end{cases} \\ &= \begin{cases} \cos 3t\,, & t < \pi\,, \\ \sqrt{2}\cos\left(3t + \dfrac{\pi}{4}\right), & \pi < t\,. \end{cases} \end{aligned}$$

The graph of $x(t)$ is given in color in Figure 7.28. For comparison, the dashed curve depicts the displacement of an undisturbed vibrating spring. Note that the impulse effectively adds 3 units to the momentum at time $t = \pi$. ◆

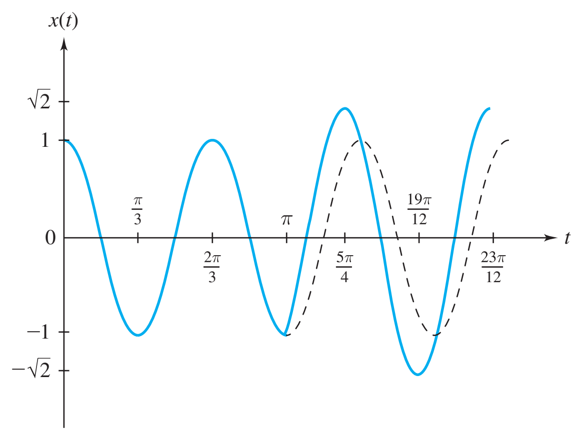

Figure 7.28 Displacement of a vibrating spring that is struck by a hammer at $t = \pi$

In Section 7.8 we defined the **impulse response function** for

$$ay'' + by' + cy = g(t) \tag{12}$$

as the function $h(t) \coloneqq \mathscr{L}^{-1}\{H\}(t)$, where $H(s)$ is the **transfer function.** Recall that $H(s)$ is the ratio

$$H(s) \coloneqq \frac{Y(s)}{G(s)},$$

where $Y(s)$ is the Laplace transform of the solution to (12) with zero initial conditions and $G(s)$ is the Laplace transform of $g(t)$. It is important to note that $H(s)$, and hence $h(t)$, does not depend on the choice of the function $g(t)$ in (12) [see equation (15) in Section 7.8, page 402]. However, it is useful to think of the impulse response function as the solution of the symbolic initial value problem

$$ay'' + by' + cy = \delta(t)\,; \qquad y(0) = 0\,, \qquad y'(0) = 0\,. \tag{13}$$

Indeed, with $g(t) = \delta(t)$, we have $G(s) = 1$, and hence $H(s) = Y(s)$. Consequently, $h(t) = y(t)$. So we see that the function $h(t)$ is the response to the impulse $\delta(t)$ for a mechanical system governed by the symbolic initial value problem (13).

7.9 EXERCISES

In Problems 1–6, evaluate the given integral.

1. $\displaystyle\int_{-\infty}^{\infty} (t^2 - 1)\delta(t)\,dt$

2. $\displaystyle\int_{-\infty}^{\infty} e^{3t}\delta(t)\,dt$

3. $\displaystyle\int_{-\infty}^{\infty} (\sin 3t)\delta\left(t - \frac{\pi}{2}\right)dt$

4. $\displaystyle\int_{-\infty}^{\infty} e^{-2t}\delta(t+1)\,dt$

5. $\displaystyle\int_{0}^{\infty} e^{-2t}\delta(t-1)\,dt$

6. $\displaystyle\int_{-1}^{1} (\cos 2t)\delta(t)\,dt$

In Problems 7–12, determine the Laplace transform of the given generalized function.

7. $\delta(t-1) - \delta(t-3)$ **8.** $3\delta(t-1)$

9. $t\delta(t-1)$ **10.** $t^3\delta(t-3)$

11. $\delta(t-\pi)\sin t$ **12.** $e^t\delta(t-3)$

In Problems 13–20, solve the given symbolic initial value problem.

13. $w'' + w = \delta(t-\pi)\,; \quad w(0) = 0\,, \quad w'(0) = 0$

14. $y'' + 2y' + 2y = \delta(t-\pi)\,; \quad y(0) = 1\,, \quad y'(0) = 1$

15. $y'' + 2y' - 3y = \delta(t-1) - \delta(t-2)\,;$
$y(0) = 2\,, \quad y'(0) = -2$

16. $y'' - 2y' - 3y = 2\delta(t-1) - \delta(t-3)\,;$
$y(0) = 2\,, \quad y'(0) = 2$

17. $y'' - y = 4\delta(t-2) + t^2\,;$
$y(0) = 0\,, \quad y'(0) = 2$

18. $y'' - y' - 2y = 3\delta(t-1) + e^t\,;$
$y(0) = 0\,, \quad y'(0) = 3$

19. $w'' + 6w' + 5w = e^t\delta(t-1)\,;$
$w(0) = 0\,, \quad w'(0) = 4$

20. $y'' + 5y' + 6y = e^{-t}\delta(t-2)\,;$
$y(0) = 2\,, \quad y'(0) = -5$

In Problems 21–24, solve the given symbolic initial value problem and sketch a graph of the solution.

21. $y'' + y = \delta(t-2\pi)\,;$
$y(0) = 0\,, \quad y'(0) = 1$

22. $y'' + y = \delta(t-\pi/2)\,;$
$y(0) = 0\,, \quad y'(0) = 1$

23. $y'' + y = -\delta(t-\pi) + \delta(t-2\pi)\,;$
$y(0) = 0\,, \quad y'(0) = 1$

24. $y'' + y = \delta(t-\pi) - \delta(t-2\pi)\,;$
$y(0) = 0\,, \quad y'(0) = 1$

In Problems 25–28, find the impulse response function $h(t)$ by using the fact that $h(t)$ is the solution to the symbolic initial value problem with $g(t) = \delta(t)$ and zero initial conditions.

25. $y'' + 4y' + 8y = g(t)$ **26.** $y'' - 6y' + 13y = g(t)$

27. $y'' - 2y' + 5y = g(t)$ **28.** $y'' - y = g(t)$

29. A mass attached to a spring is released from rest 1 m below the equilibrium position for the mass–spring system and begins to vibrate. After $\pi/2$ sec, the mass is struck by a hammer exerting an impulse on the mass. The system is governed by the symbolic initial value problem

$$\frac{d^2x}{dt^2} + 9x = -3\delta\left(t - \frac{\pi}{2}\right);$$

$$x(0) = 1, \quad \frac{dx}{dt}(0) = 0,$$

where $x(t)$ denotes the displacement from equilibrium at time t. What happens to the mass after it is struck?

30. You have probably heard that soldiers are told not to march in cadence when crossing a bridge. By solving the symbolic initial value problem

$$y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi); \quad y(0) = 0, \quad y'(0) = 0,$$

explain why soldiers are so instructed. [*Hint:* See Section 4.10.]

31. A linear system is said to be **stable** if its impulse response function $h(t)$ remains bounded as $t \to +\infty$. If the linear system is governed by

$$ay'' + by' + cy = g(t),$$

where b and c are not both zero, show that the system is stable if and only if the real parts of the roots to

$$ar^2 + br + c = 0$$

are less than or equal to zero.

32. A linear system is said to be **asymptotically stable** if its impulse response function satisfies $h(t) \to 0$ as $t \to +\infty$. If the linear system is governed by

$$ay'' + by' + cy = g(t),$$

show that the system is asymptotically stable if and only if the real parts of the roots to

$$ar^2 + br + c = 0$$

are strictly less than zero.

33. The Dirac delta function may also be characterized by the properties

$$\delta(t) = \begin{cases} 0, & t \neq 0, \\ \text{“infinite,”} & t = 0, \end{cases}$$

and $$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

Formally using the mean value theorem for definite integrals, verify that if $f(t)$ is continuous, then the above properties imply

$$\int_{-\infty}^{\infty} f(t)\delta(t)\, dt = f(0).$$

34. Formally using integration by parts, show that

$$\int_{-\infty}^{\infty} f(t)\delta'(t)\, dt = -f'(0).$$

Also show that, in general,

$$\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t)\, dt = (-1)^n f^{(n)}(0).$$

35. Figure 7.29 shows a beam of length 2λ that is imbedded in a support on the left side and free on the right. The vertical deflection of the beam a distance x from the support is denoted by $y(x)$. If the beam has a concentrated load L acting on it in the center of the beam, then the deflection must satisfy the symbolic boundary value problem

$$EIy^{(4)}(x) = L\delta(x - \lambda);$$
$$y(0) = y'(0) = y''(2\lambda) = y'''(2\lambda) = 0,$$

where E, the modulus of elasticity, and I, the moment of inertia, are constants. Find a formula for the displacement $y(x)$ in terms of the constants λ, L, E, and I. [*Hint:* Let $y''(0) = A$ and $y'''(0) = B$. First solve the fourth-order symbolic initial value problem and then use the conditions $y''(2\lambda) = y'''(2\lambda) = 0$ to determine A and B.]

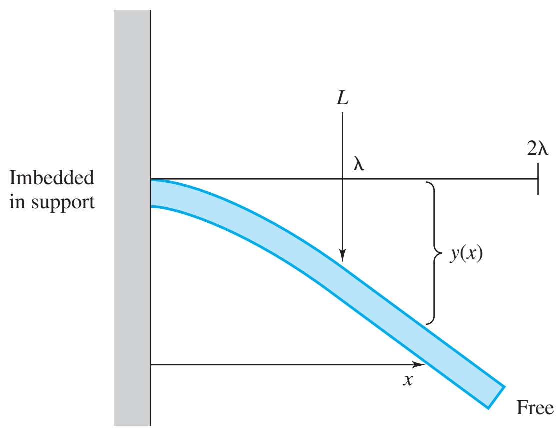

Figure 7.29 Beam imbedded in a support under a concentrated load at $x = \lambda$

7.10 Solving Linear Systems with Laplace Transforms

We can use the Laplace transform to reduce certain *systems* of linear differential equations with initial conditions to a system of linear algebraic equations, where again the unknowns are the transforms of the functions that make up the solution. Solving for these unknowns and taking their inverse Laplace transforms, we can then obtain the solution to the initial value problem for the system.

Example 1 Solve the initial value problem

$$\begin{aligned} x'(t) - 2y(t) &= 4t\,; \qquad & x(0) &= 4\,, \\ y'(t) + 2y(t) - 4x(t) &= -4t - 2\,; \qquad & y(0) &= -5\,. \end{aligned} \tag{1}$$

Solution Taking the Laplace transform of both sides of the differential equations gives

$$\begin{aligned} \mathscr{L}\{x'\}(s) - 2\mathscr{L}\{y\}(s) &= \frac{4}{s^2}\,, \\ \mathscr{L}\{y'\}(s) + 2\mathscr{L}\{y\}(s) - 4\mathscr{L}\{x\}(s) &= -\frac{4}{s^2} - \frac{2}{s}\,. \end{aligned} \tag{2}$$

Let $X(s) \coloneqq \mathscr{L}\{x\}(s)$ and $Y(s) \coloneqq \mathscr{L}\{y\}(s)$. Then, by Theorem 4 on page 362,

$$\begin{aligned} \mathscr{L}\{x'\}(s) &= sX(s) - x(0) = sX(s) - 4\,, \\ \mathscr{L}\{y'\}(s) &= sY(s) - y(0) = sY(s) + 5\,. \end{aligned}$$

Substituting these expressions into system (2) and simplifying, we find

$$\begin{aligned} sX(s) - 2Y(s) &= \frac{4s^2 + 4}{s^2}\,, \\ -4X(s) + (s+2)Y(s) &= -\frac{5s^2 + 2s + 4}{s^2}\,. \end{aligned} \tag{3}$$

To eliminate $Y(s)$ from the system, we multiply the first equation by $(s+2)$ and the second by 2 and then add to obtain

$$[s(s+2) - 8]X(s) = \frac{(s+2)(4s^2+4)}{s^2} - \frac{10s^2 + 4s + 8}{s^2}\,.$$

This simplifies to

$$X(s) = \frac{4s - 2}{(s+4)(s-2)}\,.$$

To compute the inverse transform, we first write $X(s)$ in the partial fraction form

$$X(s) = \frac{3}{s+4} + \frac{1}{s-2}\,.$$

Hence, from the Laplace transform table on the inside back cover, we find that

$$x(t) = 3e^{-4t} + e^{2t}\,. \tag{4}$$

To determine $y(t)$, we could solve system (3) for $Y(s)$ and then compute its inverse Laplace transform. However, it is easier just to solve the first equation in system (1) for $y(t)$ in terms of $x(t)$. Thus,

$$y(t) = \frac{1}{2}x'(t) - 2t\,.$$

Substituting $x(t)$ from equation (4), we find that

$$y(t) = -6e^{-4t} + e^{2t} - 2t\,. \tag{5}$$

The solution to the initial value problem (1) consists of the pair of functions $x(t)$, $y(t)$ given by equations (4) and (5). ◆

7.10 EXERCISES

In Problems 1–19, use the method of Laplace transforms to solve the given initial value problem. Here x', y', etc., denotes differentiation with respect to t; so does the symbol D.

1. $x' = 3x - 2y\,;\quad x(0) = 1\,,$
$y' = 3y - 2x\,;\quad y(0) = 1$

2. $x' = x - y\,;\quad x(0) = -1\,,$
$y' = 2x + 4y\,;\quad y(0) = 0$

3. $z' + w' = z - w\,;\quad z(0) = 1\,,$
$z' - w' = z - w\,;\quad w(0) = 0$

4. $x' - 3x + 2y = \sin t\,;\quad x(0) = 0\,,$
$4x - y' - y = \cos t\,;\quad y(0) = 0$

5. $x' = y + \sin t\,;\quad x(0) = 2\,,$
$y' = x + 2\cos t\,;\quad y(0) = 0$

6. $x' - x - y = 1\,;\quad x(0) = 0\,,$
$-x + y' - y = 0\,;\quad y(0) = -5/2$

7. $(D-4)[x] + 6y = 9e^{-3t}\,;\quad x(0) = -9\,,$
$x - (D-1)[y] = 5e^{-3t}\,;\quad y(0) = 4$

8. $D[x] + y = 0\,;\quad x(0) = 7/4\,,$
$4x + D[y] = 3\,;\quad y(0) = 4$

9. $x'' + 2y' = -x\,;\quad x(0) = 2\,,\quad x'(0) = -7\,,$
$-3x'' + 2y'' = 3x - 4y\,;\quad y(0) = 4\,,\quad y'(0) = -9$

10. $x'' + y = 1\,;\quad x(0) = 1\,,\quad x'(0) = 1\,,$
$x + y'' = -1\,;\quad y(0) = 1\,,\quad y'(0) = -1$

11. $x' + y = 1 - u(t-2)\,;\quad x(0) = 0\,,$
$x + y' = 0\,;\quad y(0) = 0$

12. $x' + y = x\,;\quad x(0) = 0\,,\quad y(0) = 1\,,$
$2x' + y'' = u(t-3)\,;\quad y'(0) = -1$

13. $x' - y' = (\sin t)u(t-\pi)\,;\quad x(0) = 1\,,$
$x + y' = 0\,;\quad y(0) = 1$

14. $x'' = y + u(t-1)\,;\quad x(0) = 1\,,\quad x'(0) = 0\,,$
$y'' = x + 1 - u(t-1)\,;\quad y(0) = 0\,,\quad y'(0) = 0$

15. $x' - 2y = 2\,;\quad x(1) = 1\,,$
$x' + x - y' = t^2 + 2t - 1\,;\quad y(1) = 0$

16. $x' - 2x + y' = -(\cos t + 4\sin t)\,;\quad x(\pi) = 0\,,$
$2x + y' + y = \sin t + 3\cos t\,;\quad y(\pi) = 3$

17. $x' + x - y' = 2(t-2)e^{t-2}\,;\quad x(2) = 0\,,$
$x'' - x' - 2y = -e^{t-2}\,;\quad x'(2) = 1\,,\quad y(2) = 1$

18. $x' - 2y = 0\,;\quad x(0) = 0\,,$
$x' - z' = 0\,;\quad y(0) = 0\,,$
$x + y' - z = 3\,;\quad z(0) = -2$

19. $x' = 3x + y - 2z\,;\quad x(0) = -6\,,$
$y' = -x + 2y + z\,;\quad y(0) = 2\,,$
$z' = 4x + y - 3z\,;\quad z(0) = -12$

20. Use the method of Laplace transforms to solve
$x'' + y' = 2\,;\quad x(0) = 3\,,\quad x'(0) = 0\,,$
$4x + y' = 6\,;\quad y(1) = 4\,.$
[*Hint:* Let $y(0) = c$ and then solve for c.]

21. For the interconnected tanks problem of Section 5.1, page 241, suppose that the input to tank A is now controlled by a valve which for the first 5 min delivers 6 L/min of pure water, but thereafter delivers 6 L/min of brine at a concentration of 0.02 kg/L. Assuming that all other data remain the same (see Figure 5.1, page 241), determine the mass of salt in each tank for $t > 0$ if $x_0 = 0$ and $y_0 = 0.04$.

22. Recompute the coupled mass–spring oscillator motion in Problem 1, Exercises 5.6 (page 287), using Laplace transforms.

In Problems 23 and 24, find a system of differential equations and initial conditions for the currents in the networks given by the schematic diagrams; the initial currents are all assumed to be zero. Solve for the currents in each branch of the network. (See Section 5.7 for a discussion of electrical networks.)

23.

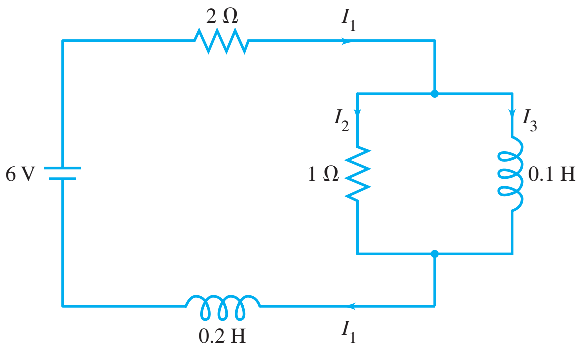

Figure 7.30 *RL* network for Problem 23

24.

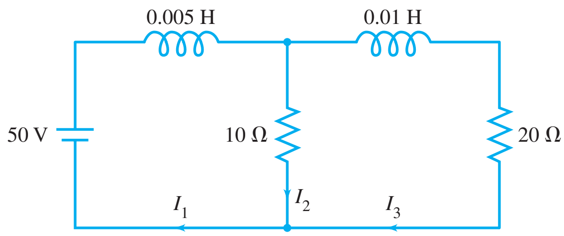

Figure 7.31 *RL* network for Problem 24

Chapter 7 Summary

The use of the Laplace transform helps to simplify the process of solving initial value problems for certain differential and integral equations, especially when a forcing function with jump discontinuities is involved. The Laplace transform $\mathscr{L}\{f\}$ of a function $f(t)$ is defined by

$$\mathscr{L}\{f\}(s) := \int_0^{\infty} e^{-st} f(t)\, dt$$

for all values of s for which the improper integral exists. If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α (that is, $|f(t)|$ grows no faster than a constant times $e^{\alpha t}$ as $t \to \infty$), then $\mathscr{L}\{f\}(s)$ exists for all $s > \alpha$.

The Laplace transform can be interpreted as an integral operator that maps a function $f(t)$ to a function $F(s)$. The transforms of commonly occurring functions appear in Table 7.1, page 356, and on the inside back cover of this book. The use of these tables is enhanced by several important properties of the operator $\mathscr{L}$.

Linearity: $\mathscr{L}\{af + bg\} = a\mathscr{L}\{f\} + b\mathscr{L}\{g\}$.

Translation in *s*: $\mathscr{L}\{e^{at}f(t)\}(s) = F(s-a)$, where $F = \mathscr{L}\{f\}$.

Translation in *t*: $\mathscr{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathscr{L}\{g(t+a)\}(s)$, where $u(t-a)$ is the step function that equals 1 for $t > a$ and 0 for $t < a$. If $f(t)$ is continuous and $f(0) = 0$, then

$$\mathscr{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a)\,,$$

where $f = \mathscr{L}^{-1}\{F\}$.

Convolution Property: $\mathscr{L}\{f * g\} = \mathscr{L}\{f\}\mathscr{L}\{g\}$, where $f * g$ denotes the convolution function

$$(f * g)(t) := \int_0^t f(t-v)g(v)\, dv\,.$$

CHAPTER 8

Series Solutions of Differential Equations

8.1 Introduction: The Taylor Polynomial Approximation

Probably the best tool for numerically approximating a function $f(x)$ near a particular point x_0 is the *Taylor polynomial.* The formula for the Taylor polynomial of degree n centered at x_0, approximating a function $f(x)$ possessing n derivatives at x_0, is given by

$$
\begin{aligned}
p_n(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\
&\quad + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\
&= \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j\,.
\end{aligned} \tag{1}
$$

This polynomial matches the value of f and the values of its derivatives, up to the order of the polynomial, at the point x_0:

$$
\begin{aligned}
p_n(x_0) &= f(x_0)\,, \\
p_n'(x_0) &= f'(x_0)\,, \\
p_n''(x_0) &= f''(x_0)\,, \\
&\;\;\vdots \\
p_n^{(n)}(x_0) &= f^{(n)}(x_0)\,.
\end{aligned}
$$

For example, the first four Taylor polynomials for e^x, expanded around $x_0 = 0$, are

$$
\begin{aligned}
p_0(x) &= 1\,, \\
p_1(x) &= 1 + x\,, \\
p_2(x) &= 1 + x + \frac{x^2}{2}\,, \\
p_3(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}\,.
\end{aligned} \tag{2}
$$

Their efficacy in approximating the exponential function is demonstrated in Figure 8.1, page 422.

The Taylor polynomial of degree n differs from the polynomial of the next lower degree only in the addition of a single term:

$$
p_n(x) = p_{n-1}(x) + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n\,,
$$

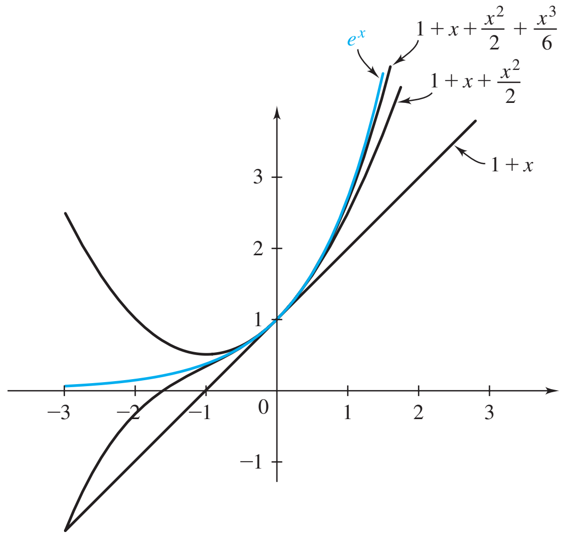

Figure 8.1 Graphs of Taylor polynomials for e^x

so a listing like (2) is clearly redundant—one can read off $p_0(x)$, $p_1(x)$, and $p_2(x)$ from the formula for $p_3(x)$. In fact, if f is infinitely differentiable, $p_n(x)$ is just the $(n+1)$st partial sum of the **Taylor series**†

$$\textbf{(3)} \qquad \sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j .$$

Example 1 Determine the fourth-degree Taylor polynomials matching the functions e^x, $\cos x$, and $\sin x$ at $x_0 = 2$.

Solution For $f(x) = e^x$ we have $f^{(j)}(2) = e^2$ for each $j = 0, 1, \ldots$, so from (1) we obtain

$$e^x \approx e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \frac{e^2}{4!}(x-2)^4 .$$

For $f(x) = \cos x$, we have $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$, so that

$$\cos x \approx \cos 2 - (\sin 2)(x-2) - \frac{\cos 2}{2!}(x-2)^2 + \frac{\sin 2}{3!}(x-2)^3 + \frac{\cos 2}{4!}(x-2)^4 .$$

In a similar fashion we find

$$\sin x \approx \sin 2 + (\cos 2)(x-2) - \frac{\sin 2}{2!}(x-2)^2 - \frac{\cos 2}{3!}(x-2)^3$$
$$+ \frac{\sin 2}{4!}(x-2)^4 . \quad ◆$$

†Truncated Taylor series were introduced in Section 3.7 (page 132) as a tool for constructing recursive formulas for approximate solutions of differential equations.

To relate this approximation scheme to our theme (the solution of differential equations), we alter our point of view; we regard a differential equation *not* as a "condition to be satisfied," but as a prescription for constructing the Taylor polynomials for its solutions. Besides providing a very general method for computing accurate approximate solutions to the equation near any particular "starting" point, this interpretation also provides insight into the role of the initial conditions. The following example illustrates the method.

Example 2 Find the first few Taylor polynomials approximating the solution around $x_0 = 0$ of the initial value problem

$$y'' = 3y' + x^2 y\,; \qquad y(0) = 10\,, \qquad y'(0) = 5\,.$$

Solution To construct

$$p_n(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots + \frac{y^{(n)}(0)}{n!}x^n\,,$$

we need the values of $y(0), y'(0), y''(0), y'''(0)$, etc. The first two are provided by the given initial conditions. The value of $y''(0)$ can be deduced from the differential equation itself and the values of the lower derivatives:

$$y''(0) = 3y'(0) + 0^2 y(0) = 3\cdot 5 + 0\cdot 10 = 15\,.$$

Now since $y'' = 3y' + x^2y$ holds for some interval around $x_0 = 0$, we can differentiate both sides to derive

$$\begin{aligned}
y''' &= 3y'' + 2xy + x^2y'\,,\\
y^{(4)} &= 3y''' + 2y + 2xy' + 2xy' + x^2y'' = 3y''' + 2y + 4xy' + x^2y''\,,\\
y^{(5)} &= 3y^{(4)} + 2y' + 4y' + 4xy'' + 2xy'' + x^2y''' = 3y^{(4)} + 6y' + 6xy'' + x^2y'''\,.
\end{aligned}$$

Thus on substituting $x = 0$ we deduce, in turn, that

$$\begin{aligned}
y'''(0) &= 3\cdot 15 + 2\cdot 0\cdot 10 + 0^2\cdot 5 = 45\,,\\
y^{(4)}(0) &= 3\cdot 45 + 2\cdot 10 + 4\cdot 0\cdot 5 + 0^2\cdot 15 = 155\,,\\
y^{(5)}(0) &= 3\cdot 155 + 6\cdot 5 + 6\cdot 0\cdot 15 + 0^2\cdot 45 = 495\,.
\end{aligned}$$

Consequently the Taylor polynomial of degree 5 for the solution is given by

$$\begin{aligned}
p_5(x) &= 10 + 5x + \frac{15}{2!}x^2 + \frac{45}{3!}x^3 + \frac{155}{4!}x^4 + \frac{495}{5!}x^5\\
&= 10 + 5x + \frac{15}{2}x^2 + \frac{15}{2}x^3 + \frac{155}{24}x^4 + \frac{33}{8}x^5\,. \quad \blacklozenge
\end{aligned}$$

It is of interest to note that if the original equation in Example 2 were replaced by $y'' = 3y' + x^{1/3}y$, the third derivative would look like $y''' = 3y'' + y/(3x^{2/3}) + x^{1/3}y'$, and $y'''(0)$ would not exist. Only Taylor polynomials of degree 0 through 2 can be constructed for the solution to this problem.

The next example demonstrates the application of the Taylor polynomial method to a *nonlinear* equation.

Example 3 Determine the Taylor polynomial of degree 3 for the solution to the initial value problem

$$\text{(4)} \qquad y' = \frac{1}{x+y+1}\,, \qquad y(0) = 0\,.$$

Solution Using $y(0) = 0$, we substitute $x = 0$ and $y = 0$ into equation (4) and find that $y'(0) = 1$. To determine $y''(0)$, we differentiate both sides of the equation in (4) with respect to x, thereby getting an expression for $y''(x)$ in terms of x, $y(x)$, and $y'(x)$. That is,

$$y''(x) = (-1)[x + y(x) + 1]^{-2}[1 + y'(x)] . \tag{5}$$

Substituting $x = 0$, $y(0) = 0$, and $y'(0) = 1$ in (5), we obtain

$$y''(0) = (-1)(1)^{-2}(1 + 1) = -2 .$$

Similarly, differentiating (5) and substituting, we obtain

$$y'''(x) = 2[x + y(x) + 1]^{-3}[1 + y'(x)]^2 - [x + y(x) + 1]^{-2}y''(x) ,$$
$$y'''(0) = 2(1)^{-3}(1 + 1)^2 - (1)^{-2}(-2) = 10 .$$

Thus, the Taylor polynomial of degree 3 is

$$p_3(x) = 0 + x - x^2 + \frac{10}{6}x^3 = x - x^2 + \frac{5}{3}x^3 . \quad \blacklozenge$$

In a theoretical sense we can estimate the accuracy to which a Taylor polynomial $p_n(x)$ approximates its target function $f(x)$ for x near x_0. Indeed, if we let $\varepsilon_n(x)$ measure the accuracy of the approximation,

$$\varepsilon_n(x) \coloneqq f(x) - p_n(x) ,$$

then calculus provides us with several formulas for estimating ε_n. The most transparent is due to Lagrange: *if the $(n + 1)$st derivative of f exists and is continuous on an interval containing x_0 and x, then*

$$\boldsymbol{\varepsilon_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}} , \tag{6}$$

where ξ, although unknown, is guaranteed to lie between x_0 and x.[†]

Figure 8.1 on page 422 and equation (6) suggest that one might control the error in the Taylor polynomial approximation by increasing the degree n of the polynomial (i.e., taking more terms), thereby increasing the factor $(n + 1)!$ in the denominator. This possibility is limited, of course, by the number of times f can be differentiated. In Example 2, for instance, the solution did not have a fifth derivative at $x_0 = 0$ ($f^{(5)}(0)$ is "infinite"). Thus, we could not construct $p_5(x)$, nor could we conclude anything about the accuracy of $p_4(x)$ from the Lagrange formula.

However, for Example 3 we could, in theory, compute *every* derivative of the solution $y(x)$ at $x_0 = 0$, and speculate on the *convergence* of the Taylor series

$$\sum_{j=0}^{\infty} \frac{y^{(j)}(x_0)}{j!}(x - x_0)^j = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{y^{(j)}(x_0)}{j!}(x - x_0)^j$$

to the solution $y(x)$. Now for nonlinear equations such as (4), the factor $f^{(n+1)}(\xi)$ in the Lagrange error formula may grow too rapidly with n, and the convergence can be thwarted. But if the differential equation is linear and its coefficients and nonhomogeneous term enjoy a feature known as *analyticity,* our wish is granted; the error does indeed diminish to zero as the degree n goes to infinity, and the sequence of Taylor polynomials can be guaranteed to converge to the actual solution on a certain (known) interval. For instance, the exponential,

---

[†]Equation (6) is proved by invoking the mean value theorem; see, e.g., *Principles of Mathematical Analysis,* 3rd ed., by Walter Rudin (McGraw-Hill, New York, 1976).

sine, and cosine functions in Example 1 all satisfy linear differential equations with constant coefficients, and their Taylor series converge to the corresponding function values. (Indeed, *all* their derivatives are bounded on any interval of finite length, so the Lagrange formula for their approximation errors approaches zero as n increases, for each value of x.) This topic is the theme for the early sections of this chapter.

In Sections 8.5 and 8.6 we'll see that solutions to second-order linear equations can exhibit very wild behavior near points x_0 where the coefficient of y'' is zero; so wild, in fact, that no Euler or Runge–Kutta algorithm could hope to keep up with them. But a clever modification of the Taylor polynomial method, due to Frobenius, provides very accurate approximations to the solutions in such regions. It is this latter feature, perhaps, that underscores the value of the Taylor methodology in the current practice of applied mathematics.

8.1 EXERCISES

In Problems 1–8, determine the first three nonzero terms in the Taylor polynomial approximations for the given initial value problem.

1. $y' = x^2 + y^2\,; \quad y(0) = 1$

2. $y' = y^2\,; \quad y(0) = 2$

3. $y' = \sin y + e^x\,; \quad y(0) = 0$

4. $y' = \sin(x + y)\,; \quad y(0) = 0$

5. $x'' + tx = 0\,; \quad x(0) = 1\,, \quad x'(0) = 0$

6. $y'' + y = 0\,; \quad y(0) = 0\,, \quad y'(0) = 1$

7. $y''(\theta) + y(\theta)^3 = \sin\theta\,;$
$y(0) = 0\,, \quad y'(0) = 0$

8. $y'' + \sin y = 0\,; \quad y(0) = 1\,, \quad y'(0) = 0$

9. **(a)** Construct the Taylor polynomial $p_3(x)$ of degree 3 for the function $f(x) = \ln x$ around $x = 1$.
(b) Using the error formula (6), show that

$$|\ln(1.5) - p_3(1.5)| \le \frac{(0.5)^4}{4} = 0.015625\,.$$

(c) Compare the estimate in part (b) with the actual error by calculating $|\ln(1.5) - p_3(1.5)|$.
(d) Sketch the graphs of $\ln x$ and $p_3(x)$ (on the same axes) for $0 < x < 2$.

10. **(a)** Construct the Taylor polynomial $p_3(x)$ of degree 3 for the function $f(x) = 1/(2 - x)$ around $x = 0$.
(b) Using the error formula (6), show that

$$\left|f\left(\frac{1}{2}\right) - p_3\left(\frac{1}{2}\right)\right| = \left|\frac{2}{3} - p_3\left(\frac{1}{2}\right)\right| \le \frac{2}{3^5}\,.$$

(c) Compare the estimate in part (b) with the actual error

$$\left|\frac{2}{3} - p_3\left(\frac{1}{2}\right)\right|\,.$$

(d) Sketch the graphs of $1/(2 - x)$ and $p_3(x)$ (on the same axes) for $-2 < x < 2$.

11. Argue that if $y = \phi(x)$ is a solution to the differential equation $y'' + p(x)y' + q(x)y = g(x)$ on the interval (a, b), where p, q, and g are each twice-differentiable, then the fourth derivative of $\phi(x)$ exists on (a, b).

12. Argue that if $y = \phi(x)$ is a solution to the differential equation $y'' + p(x)y' + q(x)y = g(x)$ on the interval (a, b), where p, q, and g possess derivatives of all orders, then ϕ has derivatives of all orders on (a, b).

13. **Duffing's Equation.** In the study of a nonlinear spring with periodic forcing, the following equation arises:

$$y'' + ky + ry^3 = A\cos\omega t\,.$$

Let $k = r = A = 1$ and $\omega = 10$. Find the first three nonzero terms in the Taylor polynomial approximations to the solution with initial values $y(0) = 0$, $y'(0) = 1$.

14. **Soft versus Hard Springs.** For Duffing's equation given in Problem 13, the behavior of the solutions changes as r changes sign. When $r > 0$, the restoring force $ky + ry^3$ becomes stronger than for the linear spring $(r = 0)$. Such a spring is called **hard.** When $r < 0$, the restoring force becomes weaker than the linear spring and the spring is called **soft.** Pendulums act like soft springs.
(a) Redo Problem 13 with $r = -1$. Notice that for the initial conditions $y(0) = 0, y'(0) = 1$, the soft and hard springs appear to respond in the same way for t small.
(b) Keeping $k = A = 1$ and $\omega = 10$, change the initial conditions to $y(0) = 1$ and $y'(0) = 0$. Now redo Problem 13 with $r = \pm 1$.
(c) Based on the results of part (b), is there a difference between the behavior of soft and hard springs for t small? Describe.

15. The solution to the initial value problem

$$xy''(x) + 2y'(x) + xy(x) = 0\,;$$
$$y(0) = 1\,, \qquad y'(0) = 0$$

has derivatives of all orders at $x = 0$ (although this is far from obvious). Use L'Hôpital's rule to compute the Taylor polynomial of degree 2 approximating this solution.

16. van der Pol Equation. In the study of the vacuum tube, the following equation is encountered:

$$y'' + (0.1)(y^2 - 1)y' + y = 0\,.$$

Find the Taylor polynomial of degree 4 approximating the solution with the initial values $y(0) = 1$, $y'(0) = 0$.

8.2 Power Series and Analytic Functions

The differential equations studied in earlier sections often possessed solutions $y(x)$ that could be written in terms of elementary functions such as polynomials, exponentials, sines, and cosines. However, many important equations arise whose solutions cannot be so expressed. In the previous chapters, when we encountered such an equation we either settled for expressing the solution as an integral (see Exercises 2.2, Problem 27, page 46) or as a numerical approximation (Sections 3.6, 3.7, and 5.3). However, the Taylor polynomial approximation scheme of the preceding section suggests another possibility. Suppose the differential equation (and initial conditions) permit the computation of *every* derivative $y^{(n)}$ at the expansion point x_0. Are there any conditions that would guarantee that the sequence of Taylor polynomials would *converge* to the solution $y(x)$ as the degree of the polynomials tends to infinity:

$$\lim_{n\to\infty}\sum_{j=0}^{n}\frac{y^{(j)}(x_0)}{j!}(x-x_0)^j = \sum_{j=0}^{\infty}\frac{y^{(j)}(x_0)}{j!}(x-x_0)^j = y(x)\ ?$$

In other words, when can we be sure that a solution to a differential equation is represented by its *Taylor series*? As we'll see, the answer is quite favorable, and it enables a powerful new technique for solving equations.

The most efficient way to begin an exploration of this issue is by investigating the algebraic and convergence properties of generic expressions that include Taylor series—"long polynomials" so to speak, or more conventionally, *power series.*

Power Series

A **power series** about the point x_0 is an expression of the form

$$\textbf{(1)}\qquad \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots\,,$$

where x is a variable and the a_n's are constants. We say that (1) **converges** at the point $x = c$ if the infinite series (of real numbers) $\sum_{n=0}^{\infty} a_n(c-x_0)^n$ converges; that is, the limit of the partial sums,

$$\lim_{N\to\infty}\sum_{n=0}^{N} a_n(c-x_0)^n\,,$$

exists (as a finite number). If this limit does not exist, the power series is said to **diverge** at $x = c$. Observe that (1) converges at $x = x_0$, since

$$\sum_{n=0}^{\infty} a_n(x_0-x_0)^n = a_0 + 0 + 0 + \cdots = a_0\,.$$

But what about convergence for other values of x? As stated in the following Theorem 1, a power series of the form (1) converges for all values of x in some "interval" centered at x_0 and diverges for x outside this interval. Moreover, at the interior points of this interval, the power series **converges absolutely** in the sense that $\sum_{n=0}^{\infty}|a_n(x-x_0)^n|$ converges. [Recall that absolute convergence of a series implies (ordinary) convergence of the series.]

Radius of Convergence

Theorem 1. For each power series of the form (1), there is a number ρ $(0 \le \rho \le \infty)$, called the **radius of convergence** of the power series, such that (1) converges absolutely for $|x-x_0| < \rho$ and diverges for $|x-x_0| > \rho$. (See Figure 8.2.)

If the series (1) converges for all values of x, then $\rho = \infty$. When the series (1) converges only at x_0, then $\rho = 0$.

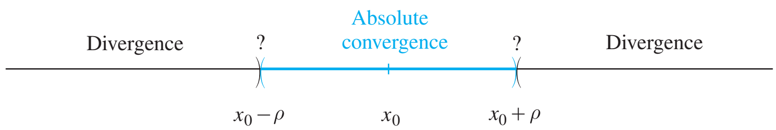

Figure 8.2 Interval of convergence

Notice that Theorem 1 settles the question of convergence except at the endpoints $x_0 \pm \rho$. Thus, these two points require separate analysis. To determine the radius of convergence ρ, one method that is often easy to apply is the ratio test.

Ratio Test for Power Series

Theorem 2. If, for n large, the coefficients a_n are nonzero and satisfy

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = L \qquad (0 \le L \le \infty)\,,$$

then the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is $\rho = L$.

Remark. We caution that if the ratio $|a_n/a_{n+1}|$ does not have a limit, then methods other than the ratio test (e.g., root test) must be used to determine ρ. In particular, if infinitely many of the a_n's are zero, then the ratio test cannot be directly applied. (However, Problem 7 demonstrates how to apply the result for series containing only "even-order" or "odd-order" terms.)

Example 1 Determine the convergence set of

$$\sum_{n=0}^{\infty}\frac{(-2)^n}{n+1}(x-3)^n\,. \tag{2}$$

Solution Since $a_n = (-2)^n/(n+1)$, we have

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-2)^n(n+2)}{(-2)^{n+1}(n+1)}\right|$$
$$= \lim_{n\to\infty}\frac{n+2}{2(n+1)} = \frac{1}{2} = L\,.$$

By the ratio test, the radius of convergence is $\rho = 1/2$. Hence, the series (2) converges absolutely for $|x-3| < 1/2$ and diverges when $|x-3| > 1/2$. It remains only to determine what happens when $|x-3| = 1/2$, that is, when $x = 5/2$ and $x = 7/2$.

Set $x = 5/2$, and the series (2) becomes the harmonic series $\Sigma_{n=0}^{\infty}(n+1)^{-1}$, which is known to diverge. When $x = 7/2$, the series (2) becomes an **alternating** harmonic series, which is known to converge. Thus, the power series converges for each x in the half-open interval $(5/2, 7/2]$; outside this interval it diverges. ◆

For each value of x for which the power series $\Sigma_{n=0}^{\infty} a_n(x-x_0)^n$ converges, we get a number that is the sum of the series. It is appropriate to denote this sum by $f(x)$, since its value depends on the choice of x. Thus, we write

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

for all numbers x in the convergence interval. For example, the **geometric series** $\Sigma_{n=0}^{\infty} x^n$ has the radius of convergence $\rho = 1$ and the sum function $f(x) = 1/(1-x)$; that is,

$$\textbf{(3)} \qquad \frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad -1 < x < 1\,.$$

In this chapter we'll frequently appeal to the following basic property of power series.

Power Series Vanishing on an Interval

Theorem 3. If $\Sigma_{n=0}^{\infty} a_n(x-x_0)^n = 0$ for all x in some open interval, then each coefficient a_n equals zero.

Given two power series

$$\textbf{(4)} \qquad f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n\,, \qquad g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n\,,$$

with nonzero radii of convergence, we want to find power series representations for the sum, product, and quotient of the functions $f(x)$ and $g(x)$. The sum is simply obtained by termwise addition:

$$f(x) + g(x) = \sum_{n=0}^{\infty}(a_n+b_n)(x-x_0)^n$$

for all x in the common interval of convergence of the power series in (4). The power series representation for the product $f(x)g(x)$ is a bit more complicated. To provide motivation for the formula, we treat the power series for $f(x)$ and $g(x)$ as "long polynomials," apply the distributive law, and group the terms in powers of $(x - x_0)$:

$$\begin{aligned}&[a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots] \cdot [b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \cdots] \\ &\quad = a_0b_0 + (a_0b_1 + a_1b_0)(x - x_0) + (a_0b_2 + a_1b_1 + a_2b_0)(x - x_0)^2 + \cdots .\end{aligned}$$

The general formula for the product is

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n , \tag{5}$$

where

$$c_n := \sum_{k=0}^{n} a_k b_{n-k} . \tag{6}$$

The power series in (5) is called the **Cauchy product,** and it will converge for all x in the common *open* interval of convergence for the power series of f and g.[†]

The quotient $f(x)/g(x)$ will also have a power series expansion about x_0, provided $g(x_0) \neq 0$. However, the radius of convergence for this quotient series may be smaller than that for $f(x)$ or $g(x)$. Unfortunately, there is no nice formula for obtaining the coefficients in the power series for $f(x)/g(x)$. However, we can use the Cauchy product to divide power series indirectly (see Problem 15 on page 434). The quotient series can also be obtained by formally carrying out polynomial long division (see Problem 16).

The next theorem explains, in part, why power series are so useful.

Differentiation and Integration of Power Series

Theorem 4. If the series $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ has a positive radius of convergence ρ, then f is differentiable in the interval $|x - x_0| < \rho$ and termwise differentiation gives the power series for the derivative:

$$f'(x) = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1} \qquad \text{for} \qquad |x - x_0| < \rho .$$

Furthermore, termwise integration gives the power series for the integral of f:

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1} + C \quad \text{for} \quad |x - x_0| < \rho .$$

Example 2 Starting with the geometric series (3) for $1/(1 - x)$, find a power series for each of the following functions:

(a) $\dfrac{1}{1 + x^2}$. **(b)** $\dfrac{1}{(1 - x)^2}$. **(c)** $\arctan x$.

Solution **(a)** Replacing x by $-x^2$ in (3) immediately gives

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots . \tag{7}$$

[†]Actually, it may happen that the radius of convergence of the power series for $f(x)g(x)$ or $f(x) + g(x)$ is larger than that for the power series of f or g.

(b) Notice that $1/(1-x)^2$ is the derivative of the function $f(x) = 1/(1-x)$. Hence, on differentiating (3) term by term, we obtain

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots . \tag{8}$$

(c) Since

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt ,$$

we can integrate the series in (7) termwise to obtain the series for arctan x. Thus,

$$\int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \{1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \cdots\}\,dt$$

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots . \tag{9}$$ ◆

It is important to keep in mind that since the geometric series (3) has the (open) interval of convergence $(-1, 1)$, the representations (7), (8), and (9) are at least valid in this interval. [Actually, the series (9) for arctan x converges for all $|x| \le 1$.]

Shifting the Summation Index

The index of summation in a power series is a dummy index just like the variable of integration in a definite integral. Hence,

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{k=0}^{\infty} a_k (x-x_0)^k = \sum_{i=0}^{\infty} a_i (x-x_0)^i .$$

Just as there are times when we want to change the variable of integration, there are situations (and we will encounter many in this chapter) when it is desirable to change or shift the index of summation. This is particularly important when one has to combine two different power series.

Example 3 Express the series

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

as a series where the generic term is x^k instead of x^{n-2}.

Solution Writing out a few terms of the series, we have

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} &= 2\cdot 1a_2x^{2-2} + 3\cdot 2a_3x^{3-2} + 4\cdot 3a_4x^{4-2} + 5\cdot 4a_5x^{5-2} + 6\cdot 5a_6x^{6-2} + \cdots \\ &= 2\cdot 1a_2x^0 + 3\cdot 2a_3x^1 + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 + 6\cdot 5a_6x^4 + \cdots .\end{aligned}$$

Relabeling the terms by the exponents of x, we get

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k$$

In effect we have implemented the change of variables $k = n-2$, or $n = k+2$; thus when $n = 2$, then $k = 0$. ◆

We employ this technique for changing indices in the next two examples.

Example 4 Show that

$$x^3 \sum_{n=0}^{\infty} n^2(n-2)a_n x^n = \sum_{n=3}^{\infty} (n-3)^2(n-5)a_{n-3}x^n .$$

Solution We start by taking the x^3 inside the summation on the left-hand side:

$$x^3 \sum_{n=0}^{\infty} n^2(n-2)a_n x^n = \sum_{n=0}^{\infty} n^2(n-2)a_n x^{n+3} .$$

To rewrite this with generic term x^k, we set $k = n + 3$. Thus $n = k - 3$, and $n = 0$ corresponds to $k = 3$. Straightforward substitution thus yields

$$\sum_{n=0}^{\infty} n^2(n-2)a_n x^{n+3} = \sum_{k=3}^{\infty} (k-3)^2(k-5)a_{k-3}x^k .$$

By replacing k by n, we obtain the desired form. ◆

Example 5 Show that the identity

$$\sum_{n=1}^{\infty} na_{n-1}x^{n-1} + \sum_{n=2}^{\infty} b_n x^{n+1} = 0$$

implies that $a_0 = a_1 = a_2 = 0$ and $a_n = -b_{n-1}/(n+1)$ for $n \geq 3$.

Solution First, we rewrite both series in terms of x^k. For the first series, we set $k = n - 1$, and hence $n = k + 1$, to write

$$\sum_{n=1}^{\infty} na_{n-1}x^{n-1} = \sum_{k=0}^{\infty} (k+1)a_k x^k .$$

Then with $k = n + 1$, $n = k - 1$, the second series becomes

$$\sum_{n=2}^{\infty} b_n x^{n+1} = \sum_{k=3}^{\infty} b_{k-1}x^k .$$

The identity thus states

$$\sum_{k=0}^{\infty} (k+1)a_k x^k + \sum_{k=3}^{\infty} b_{k-1}x^k = 0 ,$$

and so we have a power series that sums to zero; consequently, by Theorem 3, each of its coefficients equals zero. For $k = 3, 4, \ldots$, both series contribute to the coefficient of x^k, and thus we confirm that

$$(k+1)a_k + b_{k-1} = 0$$

or $a_k = -b_{k-1}/(k+1)$ for $k \geq 3$. For $k = 0$, 1, or 2, only the first series contributes, and we find, in turn,

$$\begin{aligned}(0+1)a_0 &= 0 , \\ (1+1)a_1 &= 0 , \\ (2+1)a_2 &= 0 .\end{aligned}$$

Hence, $a_0 = a_1 = a_2 = 0$. ◆

Analytic Functions

Not all functions are expressible as power series. Those distinguished functions that can be so represented are called **analytic.**

Analytic Function

Definition 1. A function f is said to be **analytic at x_0** if, in an open interval about x_0, this function is the sum of a power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ that has a positive radius of convergence.

For example, a polynomial function $b_0 + b_1x + \cdots + b_nx^n$ is analytic at every x_0, since we can always rewrite it in the form $a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n$. A rational function $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials without a common factor, is an analytic function except at those x_0 for which $Q(x_0) = 0$. As you may recall from calculus, the elementary functions e^x, $\sin x$, and $\cos x$ are analytic for all x, while $\ln x$ is analytic for $x > 0$. Some familiar representations are

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,, \tag{10}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\,, \tag{11}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\,, \tag{12}$$

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n\,, \tag{13}$$

where (10), (11), and (12) are valid for all x, whereas (13) is valid for x in the half-open interval $(0, 2]$.

From Theorem 4 on the differentiation of power series, we see that a function f analytic at x_0 is differentiable in a neighborhood of x_0. Moreover, because f' has a power series representation in this neighborhood, it too is analytic at x_0. Repeating this argument, we see that f'', $f^{(3)}$, etc., exist and are analytic at x_0. Consequently, if a function does not have derivatives of all orders at x_0, then it cannot be analytic at x_0. The function $f(x) = |x - 1|$ is not analytic at $x_0 = 1$ because $f'(1)$ does not exist; and $f(x) = x^{7/3}$ is not analytic at $x_0 = 0$ because $f'''(0)$ does not exist.

Now we can deduce something very specific about the possible power series that can represent an analytic function. If $f(x)$ is analytic at x_0, then (by definition) it is the sum of *some* power series that converges in a neighborhood of x_0:

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n\,.$$

By the reasoning in the previous paragraph, the derivatives of f have convergent power series representations

$$
\begin{aligned}
f'(x) &= \sum_{n=0}^{\infty} na_n(x-x_0)^{n-1} = 0 + \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}\,, \\
f''(x) &= \sum_{n=0}^{\infty} n(n-1)a_n(x-x_0)^{n-2} = 0 + 0 + \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}\,, \\
&\ \vdots \\
f^{(j)}(x) &= \sum_{n=0}^{\infty} n(n-1)\cdots\big(n-[j-1]\big)a_n(x-x_0)^{n-j}\,, \\
&= \sum_{n=j}^{\infty} n(n-1)\cdots\big(n-[j-1]\big)a_n(x-x_0)^{n-j}\,, \\
&\ \vdots
\end{aligned}
$$

But if we evaluate these series at $x = x_0$, we learn that

$$
\begin{aligned}
f(x_0) &= a_0\,, \\
f'(x_0) &= 1 \cdot a_1\,, \\
f''(x_0) &= 2 \cdot 1 \cdot a_2\,, \\
&\ \vdots \\
f^{(j)}(x_0) &= j! \cdot a_j\,, \\
&\ \vdots
\end{aligned}
$$

that is, $a_j = f^{(j)}(x_0)/j!$ and the power series *must coincide* with the Taylor series[†]

$$
\sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j
$$

about x_0. *Any power series—regardless of how it is derived—that converges in some neighborhood of x_0 to a function has to be the Taylor series of that function.* For example, the expansion for $\arctan x$ given in (9) of Example 2 must be its Taylor expansion.

With these facts in mind we are ready to turn to the study of the effectiveness of power series techniques for solving differential equations. In the next sections, you will find it helpful to keep in mind that if f and g are analytic at x_0, then so are $f + g$, cf, fg, and f/g if $g(x_0) \neq 0$. These facts follow from the algebraic properties of power series discussed earlier.

8.2 EXERCISES

In Problems 1–6, determine the convergence set of the given power series.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1}(x-1)^n$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}x^n$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n}(x+2)^n$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2+2n}(x-3)^n$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^3}(x-2)^n$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{(n+2)!}{n!}(x+2)^n$

[†]When the expansion point x_0 is zero, the Taylor series is also known as the **Maclaurin series.**

7. Sometimes the ratio test (Theorem 2) can be applied to a power series containing an infinite number of zero coefficients, provided the zero pattern is regular. Use Theorem 2 to show, for example, that the series

$$a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \cdots = \sum_{k=0}^{\infty} a_{2k}x^{2k}$$

has a radius of convergence $\rho = \sqrt{L}$, if

$$\lim_{n\to\infty} \left| \frac{a_{2k}}{a_{2k+2}} \right| = L\,,$$

and that

$$a_1x + a_3x^3 + a_5x^5 + a_7x^7 + \cdots$$
$$= \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1}$$

has a radius of convergence $\rho = \sqrt{M}$, if

$$\lim_{k\to\infty} \left| \frac{a_{2k+1}}{a_{2k+3}} \right| = M\,.$$

[*Hint:* Let $z = x^2$.]

8. Determine the convergence set of the given power series.

(a) $\sum_{k=0}^{\infty} 2^{2k}x^{2k}$ **(b)** $\sum_{k=0}^{\infty} 2^{2k+1}x^{2k+1}$

(c) $\sin x$ [equation (11)] **(d)** $\cos x$ [equation (12)]

(e) $(\sin x)/x = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n+1)!$

(f) $\sum_{k=0}^{\infty} 2^{2k}x^{4k}$

In Problems 9 and 10, find the power series expansion $\sum_{n=0}^{\infty} a_nx^n$ for $f(x) + g(x)$, given the expansions for $f(x)$ and $g(x)$.

9. $f(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}x^n\,,\qquad g(x) = \sum_{n=1}^{\infty} 2^{-n}x^{n-1}$

10. $f(x) = \sum_{n=3}^{\infty} \frac{2^n}{n!}(x-1)^{n-2}\,,\quad g(x) = \sum_{n=1}^{\infty} \frac{n^2}{2^n}(x-1)^{n-1}$

In Problems 11–14, find the first three nonzero terms in the power series expansion for the product $f(x)g(x)$.

11. $f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n\,,$

$g(x) = \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$

12. $f(x) = \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}\,,$

$g(x) = \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$

13. $f(x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n\,,$

$g(x) = (1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$

14. $f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n\,,$

$g(x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n$

15. Find the first few terms of the power series for the quotient

$$q(x) = \left(\sum_{n=0}^{\infty} \frac{1}{2^n}x^n\right) \Big/ \left(\sum_{n=0}^{\infty} \frac{1}{n!}x^n\right)$$

by completing the following:

(a) Let $q(x) = \sum_{n=0}^{\infty} a_nx^n$, where the coefficients a_n are to be determined. Argue that $\sum_{n=0}^{\infty} x^n/2^n$ is the Cauchy product of $q(x)$ and $\sum_{n=0}^{\infty} x^n/n!$.

(b) Use formula (6) of the Cauchy product (page 429) to deduce the equations

$$\frac{1}{2^0} = a_0\,,\quad \frac{1}{2} = a_0 + a_1\,,\quad \frac{1}{2^2} = \frac{a_0}{2} + a_1 + a_2\,,$$
$$\frac{1}{2^3} = \frac{a_0}{6} + \frac{a_1}{2} + a_2 + a_3\,,\ \ldots\,.$$

(c) Solve the equations in part (b) to determine the constants a_0, a_1, a_2, a_3.

16. To find the first few terms in the power series for the quotient $q(x)$ in Problem 15, treat the power series in the numerator and denominator as "long polynomials" and carry out long division. That is, perform

$$1 + x + \frac{1}{2}x^2 + \cdots \overline{\big)\, 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \cdots}\,.$$

In Problems 17–20, find a power series expansion for $f'(x)$, given the expansion for $f(x)$.

17. $f(x) = (1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$

18. $f(x) = \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$

19. $f(x) = \sum_{k=0}^{\infty} a_kx^{2k}$ **20.** $f(x) = \sum_{n=1}^{\infty} na_nx^{n-1}$

In Problems 21 and 22, find a power series expansion for $g(x) := \int_0^x f(t)\,dt$, given the expansion for $f(x)$.

21. $f(x) = (1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$

22. $f(x) = \frac{\sin x}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k}$

In Problems 23–26, express the given power series as a series with generic term x^k.

23. $\sum_{n=1}^{\infty} n a_n x^{n-1}$

24. $\sum_{n=2}^{\infty} n(n-1) a_n x^{n+2}$

25. $\sum_{n=0}^{\infty} a_n x^{n+1}$

26. $\sum_{n=1}^{\infty} \frac{a_n}{n+3} x^{n+3}$

27. Show that

$$x^2 \sum_{n=0}^{\infty} n(n+1) a_n x^n = \sum_{n=2}^{\infty} (n-2)(n-1) a_{n-2} x^n .$$

28. Show that

$$2 \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} n b_n x^{n-1} = b_1 + \sum_{n=1}^{\infty} [2a_{n-1} + (n+1) b_{n+1}] x^n .$$

In Problems 29–34, determine the Taylor series about the point x_0 for the given functions and values of x_0.

29. $f(x) = \cos x$, $\quad x_0 = \pi$

30. $f(x) = x^{-1}$, $\quad x_0 = 1$

31. $f(x) = \dfrac{1+x}{1-x}$, $\quad x_0 = 0$

32. $f(x) = \ln(1+x)$, $\quad x_0 = 0$

33. $f(x) = x^3 + 3x - 4$, $\quad x_0 = 1$

34. $f(x) = \sqrt{x}$, $\quad x_0 = 1$

35. The Taylor series for $f(x) = \ln x$ about $x_0 = 1$ given in equation (13) can also be obtained as follows:

(a) Starting with the expansion $1/(1-s) = \Sigma_{n=0}^{\infty} s^n$ and observing that
$$\frac{1}{x} = \frac{1}{1+(x-1)} ,$$
obtain the Taylor series for $1/x$ about $x_0 = 1$.

(b) Since $\ln x = \int_1^x 1/t\, dt$, use the result of part (a) and termwise integration to obtain the Taylor series for $f(x) = \ln x$ about $x_0 = 1$.

36. Let $f(x)$ and $g(x)$ be analytic at x_0. Determine whether the following statements are always true or sometimes false:

(a) $3f(x) + g(x)$ is analytic at x_0

(b) $f(x)/g(x)$ is analytic at x_0

(c) $f'(x)$ is analytic at x_0

(d) $[f(x)]^3 - \int_{x_0}^{x} g(t)\, dt$ is analytic at x_0

37. Let

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 , \\ 0 , & x = 0 . \end{cases}$$

Show that $f^{(n)}(0) = 0$ for $n = 0, 1, 2, \ldots$ and hence that the Maclaurin series for $f(x)$ is $0 + 0 + 0 + \cdots$, which converges for all x but is equal to $f(x)$ only when $x = 0$. This is an example of a function possessing derivatives of all orders (at $x_0 = 0$), whose Taylor series converges, but the Taylor series (about $x_0 = 0$) does not converge to the original function! Consequently, this function is not analytic at $x = 0$.

38. Compute the Taylor series for $f(x) = \ln(1+x^2)$ about $x_0 = 0$. [*Hint:* Multiply the series for $(1+x^2)^{-1}$ by $2x$ and integrate.]

8.3 Power Series Solutions to Linear Differential Equations

In this section we demonstrate a method for obtaining a power series solution to a linear differential equation with polynomial coefficients. This method is easier to use than the Taylor series method discussed in Section 8.1 and sometimes gives a nice expression for the general term in the power series expansion. Knowing the form of the general term also allows us to test for the radius of convergence of the power series.

We begin by writing the linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

in the standard form

$$y'' + p(x)y' + q(x)y = 0 , \tag{2}$$

where $p(x) := a_1(x)/a_2(x)$ and $q(x) := a_0(x)/a_2(x)$.

Ordinary and Singular Points

Definition 2. A point x_0 is called an **ordinary point** of equation (2) if both p and q are analytic at x_0. If x_0 is not an ordinary point, it is called a **singular point** of the equation.

Example 1 Determine all the singular points of

$$xy'' + x(1-x)^{-1}y' + (\sin x)y = 0 .$$

Solution Dividing the equation by x, we find that

$$p(x) = \frac{x}{x(1-x)}, \qquad q(x) = \frac{\sin x}{x} .$$

The singular points are those points where $p(x)$ *or* $q(x)$ fails to be analytic. Observe that $p(x)$ and $q(x)$ are the ratios of functions that are everywhere analytic. Hence, $p(x)$ and $q(x)$ are analytic except, *perhaps,* when their denominators are zero. For $p(x)$ this occurs at $x = 0$ and $x = 1$. But since we can cancel an x in the numerator and denominator of $p(x)$, that is,

$$p(x) = \frac{x}{x(1-x)} = \frac{1}{1-x},$$

we see that $p(x)$ is actually analytic at $x = 0$.† Therefore, $p(x)$ is analytic except at $x = 1$. For $q(x)$, the denominator is zero at $x = 0$. Just as with $p(x)$, this zero is removable since $q(x)$ has the power series expansion

$$q(x) = \frac{\sin x}{x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots .$$

Thus, $q(x)$ is everywhere analytic. Consequently, the only singular point of the given equation is $x = 1$. ◆

At an ordinary point x_0 of equation (1) (or (2)), the coefficient functions $p(x)$ and $q(x)$ are analytic. Hence, we might expect that the solutions to these equations inherit this property. From the discussion in Section 6.1 on linear equations, the continuity of p and q in a neighborhood of x_0 is sufficient to imply that equation (2) has two linearly independent solutions defined in that neighborhood. But analytic functions are not merely continuous—they possess derivatives of all orders in a neighborhood of x_0. Thus we can differentiate equation (2) to show that $y^{(3)}$ exists and, by a "bootstrap" argument, prove that solutions to (2) must likewise possess derivatives of all orders. Although we cannot conclude by this reasoning that the solutions enjoy the stronger property of analyticity, this is nonetheless the case (see Theorem 5 in Section 8.4, page 445). Hence, in a neighborhood of an ordinary point x_0, the solutions to (1) (or (2)) can be expressed as a power series about x_0.

To illustrate the power series method about an ordinary point, let's look at a simple *first-order* linear differential equation.

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†Such points are called **removable singularities.** In this chapter we assume in such cases that the function has been defined (or redefined) so that it is analytic at the point.

Example 2 Find a power series solution about $x = 0$ to

$$y' + 2xy = 0 \,. \tag{3}$$

Solution The coefficient of y is the polynomial $2x$, which is analytic everywhere, so $x = 0$ is an ordinary point[†] of equation (3). Thus, we expect to find a power series solution of the form

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n \,. \tag{4}$$

Our task is to determine the coefficients a_n.

For this purpose we need the expansion for $y'(x)$ that is given by termwise differentiation of (4):

$$y'(x) = 0 + a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=0}^{\infty} n a_n x^{n-1} \,.$$

We now substitute the series expansions for y and y' into (3) and obtain

$$\sum_{n=0}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n = 0 \,,$$

which simplifies to

$$\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0 \,. \tag{5}$$

To add the two power series in (5), we add the coefficients of like powers of x. If we write out the first few terms of these summations and add, we get

$$\begin{aligned} (a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots) + (2a_0 x + 2a_1 x^2 + 2a_2 x^3 + \cdots) &= 0 \,, \\ a_1 + (2a_2 + 2a_0)x + (3a_3 + 2a_1)x^2 + (4a_4 + 2a_2)x^3 + \cdots &= 0 \,. \end{aligned} \tag{6}$$

For the power series on the left-hand side of equation (6) to be identically zero, we must have all the coefficients equal to zero. Thus,

$$\begin{aligned} a_1 &= 0 \,, & 2a_2 + 2a_0 &= 0 \,, \\ 3a_3 + 2a_1 &= 0 \,, & 4a_4 + 2a_2 &= 0 \,, & \text{etc.} \end{aligned}$$

Solving the preceding system, we find

$$a_1 = 0 \,, \qquad a_2 = -a_0 \,, \qquad a_3 = -\frac{2}{3} a_1 = 0 \,,$$

$$a_4 = -\frac{1}{2} a_2 = -\frac{1}{2}(-a_0) = \frac{1}{2} a_0 \,.$$

Hence, the power series for the solution takes the form

$$y(x) = a_0 - a_0 x^2 + \frac{1}{2} a_0 x^4 + \cdots \,. \tag{7}$$

Although the first few terms displayed in (7) are useful, it is sometimes advantageous to have a formula for the *general term* in the power series expansion for the solution. To achieve this goal, let's return to equation (5). This time, instead of just writing out a few terms, let's

[†]By an ordinary point of a first-order equation $y' + q(x)y = 0$, we mean a point where $q(x)$ is analytic.

shift the indices in the two power series so that they sum over the same powers of x, say, x^k. To do this we shift the index in the first summation in (5) by letting $k = n - 1$. Then $n = k + 1$ and $k = 0$ when $n = 1$. Hence, the first summation in (5) becomes

$$\textbf{(8)} \qquad \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k .$$

In the second summation of (5), we put $k = n + 1$ so that $n = k - 1$ and $k = 1$ when $n = 0$. This gives

$$\textbf{(9)} \qquad \sum_{n=0}^{\infty} 2 a_n x^{n+1} = \sum_{k=1}^{\infty} 2 a_{k-1} x^k .$$

Substituting (8) and (9) into (5) yields

$$\textbf{(10)} \qquad \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k + \sum_{k=1}^{\infty} 2 a_{k-1} x^k = 0 .$$

Since the first summation in (10) begins at $k = 0$ and the second at $k = 1$, we break up the first into

$$\sum_{k=0}^{\infty} (k+1) a_{k+1} x^k = a_1 + \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k .$$

Then (10) becomes

$$\textbf{(11)} \qquad a_1 + \sum_{k=1}^{\infty} [(k+1) a_{k+1} + 2 a_{k-1}] x^k = 0 .$$

When we set all the coefficients in (11) equal to zero, we find

$$a_1 = 0 ,$$

and, for all $k \geq 1$,

$$\textbf{(12)} \qquad (k+1) a_{k+1} + 2 a_{k-1} = 0 .$$

Equation (12) is a **recurrence relation** that we can use to determine the coefficient a_{k+1} in terms of a_{k-1}; that is,

$$a_{k+1} = -\frac{2}{k+1} a_{k-1} .$$

Setting $k = 1, 2, \ldots, 8$ and using the fact that $a_1 = 0$, we find

$$\begin{aligned}
a_2 &= -\frac{2}{2} a_0 = -a_0 && (k = 1) , & a_3 &= -\frac{2}{3} a_1 = 0 && (k = 2) , \\
a_4 &= -\frac{2}{4} a_2 = \frac{1}{2} a_0 && (k = 3) , & a_5 &= -\frac{2}{5} a_3 = 0 && (k = 4) , \\
a_6 &= -\frac{2}{6} a_4 = -\frac{1}{3!} a_0 && (k = 5) , & a_7 &= -\frac{2}{7} a_5 = 0 && (k = 6) , \\
a_8 &= -\frac{2}{8} a_6 = \frac{1}{4!} a_0 && (k = 7) , & a_9 &= -\frac{2}{9} a_7 = 0 && (k = 8) .
\end{aligned}$$

After a moment's reflection, we realize that

$$\begin{aligned}
a_{2n} &= \frac{(-1)^n}{n!} a_0 , \qquad n = 1, 2, \ldots , \\
a_{2n+1} &= 0 , \qquad n = 0, 1, 2, \ldots .
\end{aligned}$$

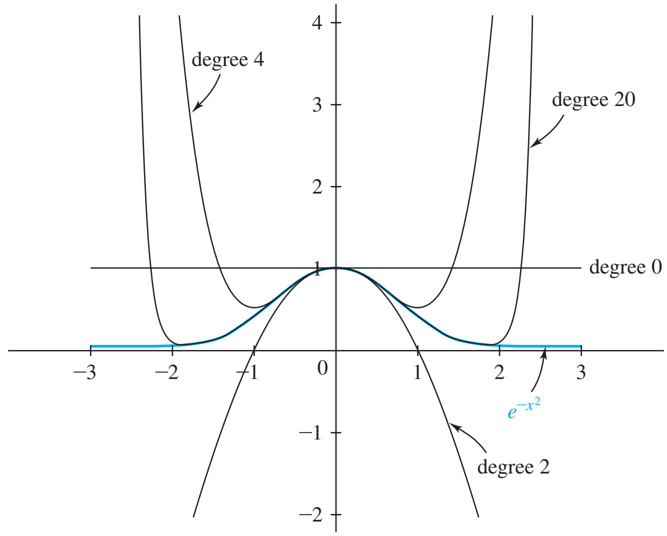

Figure 8.3 Partial sum approximations to e^{-x^2}

Substituting back into the expression (4), we obtain the power series solution

$$y(x) = a_0 - a_0x^2 + \frac{1}{2!}a_0x^4 + \cdots = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^{2n} . \tag{13}$$

Since a_0 is left undetermined, it serves as an arbitrary constant, and hence (13) gives a general solution to equation (3). ◆

Applying the ratio test as described in Problem 7, Exercises 8.2 (page 434), we can verify that the power series in (13) has radius of convergence $\rho = \infty$. Moreover, (13) is reminiscent of the expansion for the exponential function; you can check that it converges to

$$y(x) = a_0 e^{-x^2} .$$

This general solution to the simple equation (3) can also be obtained by the method of separation of variables.

The convergence of the partial sums of (13), with $a_0 = 1$, to the actual solution e^{-x^2} is depicted in Figure 8.3. Notice that taking more terms results in better approximations around $x = 0$. Note also, however, that every partial sum is a *polynomial* and hence must diverge at $x = \pm\infty$, while e^{-x^2} converges to zero, of course. Thus, each partial sum approximation eventually deteriorates for large $|x|$. This is a typical feature of power series approximations.

In the next example we use the power series method to obtain a general solution to a linear second-order differential equation.

Example 3 Find a general solution to

$$2y'' + xy' + y = 0 \tag{14}$$

in the form of a power series about the ordinary point $x = 0$.

Solution Writing

$$y(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_nx^n , \tag{15}$$

we differentiate termwise to obtain

$$y'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty} na_nx^{n-1},$$

$$y''(x) = 2a_2 + 6a_3x + 12a_4x^2 + \cdots = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}.$$

Substituting these power series into equation (14), we find

(16) $$\sum_{n=2}^{\infty} 2n(n-1)a_nx^{n-2} + \sum_{n=1}^{\infty} na_nx^n + \sum_{n=0}^{\infty} a_nx^n = 0.$$

To simplify the addition of the three summations in (16), let's shift the indices so that the general term in each is a constant times x^k. For the first summation, we substitute $k = n - 2$ and get

$$\sum_{n=2}^{\infty} 2n(n-1)a_nx^{n-2} = \sum_{k=0}^{\infty} 2(k+2)(k+1)a_{k+2}x^k.$$

In the second and third summations, we simply substitute k for n. With these changes of indices, equation (16) becomes

$$\sum_{k=0}^{\infty} 2(k+2)(k+1)a_{k+2}x^k + \sum_{k=1}^{\infty} ka_kx^k + \sum_{k=0}^{\infty} a_kx^k = 0.$$

Next, we separate the x^0 terms from the others and then combine the like powers of x in the three summations to get

$$4a_2 + a_0 + \sum_{k=1}^{\infty} [2(k+2)(k+1)a_{k+2} + ka_k + a_k]x^k = 0.$$

Setting the coefficients of this power series equal to zero yields

(17) $$4a_2 + a_0 = 0$$

and the recurrence relation

(18) $$2(k+2)(k+1)a_{k+2} + (k+1)a_k = 0, \qquad k \geq 1.$$

We can now use (17) and (18) to determine all the coefficients a_k of the solution in terms of a_0 and a_1. Solving (18) for a_{k+2} gives

(19) $$a_{k+2} = \frac{-1}{2(k+2)}a_k, \qquad k \geq 1.$$

Thus,

$$a_2 = \frac{-1}{2^2}a_0,$$

$$a_3 = \frac{-1}{2 \cdot 3}a_1 \qquad (k=1),$$

$$a_4 = \frac{-1}{2 \cdot 4}a_2 = \frac{1}{2^2 \cdot 2 \cdot 4}a_0 \qquad (k=2),$$

$$a_5 = \frac{-1}{2 \cdot 5}a_3 = \frac{1}{2^2 \cdot 3 \cdot 5}a_1 \qquad (k=3),$$

$$a_6 = \frac{-1}{2 \cdot 6}a_4 = \frac{-1}{2^3 \cdot 2 \cdot 4 \cdot 6}a_0 = \frac{-1}{2^6 \cdot 3!}a_0 \qquad (k=4),$$

$$a_7 = \frac{-1}{2 \cdot 7}a_5 = \frac{-1}{2^3 \cdot 3 \cdot 5 \cdot 7}a_1 \qquad (k=5),$$

$$a_8 = \frac{-1}{2 \cdot 8}a_6 = \frac{1}{2^4 \cdot 2 \cdot 4 \cdot 6 \cdot 8}a_0 = \frac{1}{2^8 \cdot 4!}a_0 \qquad (k=6).$$

The pattern for the coefficients is now apparent. Since a_0 and a_1 are not restricted, we find

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!} a_0 , \qquad n \geq 1 ,$$

and

$$a_{2n+1} = \frac{(-1)^n}{2^n [1 \cdot 3 \cdot 5 \cdots (2n+1)]} a_1 , \qquad n \geq 1.$$

From this, two linearly independent solutions emerge; namely,

(20) $$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!} x^{2n} \qquad (\text{take } a_0 = 1, a_1 = 0) ,$$

(21) $$y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n [1 \cdot 3 \cdot 5 \cdots (2n+1)]} x^{2n+1} \qquad (\text{take } a_0 = 0, a_1 = 1) .$$

Hence, a general solution to (14) is $a_0 y_1(x) + a_1 y_2(x)$. Approximations to the solutions $y_1(x), y_2(x)$ are depicted in Figure 8.4. ◆

The method illustrated in Example 3 can also be used to solve initial value problems. Suppose we are given the values of $y(0)$ and $y'(0)$; then, from equation (15), we see that $a_0 = y(0)$ and $a_1 = y'(0)$. Knowing these two coefficients leads to a unique power series solution for the initial value problem.

The recurrence relation (18) in Example 3 involved just two of the coefficients, a_{k+2} and a_k, and we were fortunate in being able to deduce from this relation the general form for the coefficient a_n. However, many cases arise that lead to more complicated two-term or even to many-term recurrence relations. When this occurs, it may be impossible to determine the general form for the coefficients a_n. In the next example, we consider an equation that gives rise to a three-term recurrence relation.

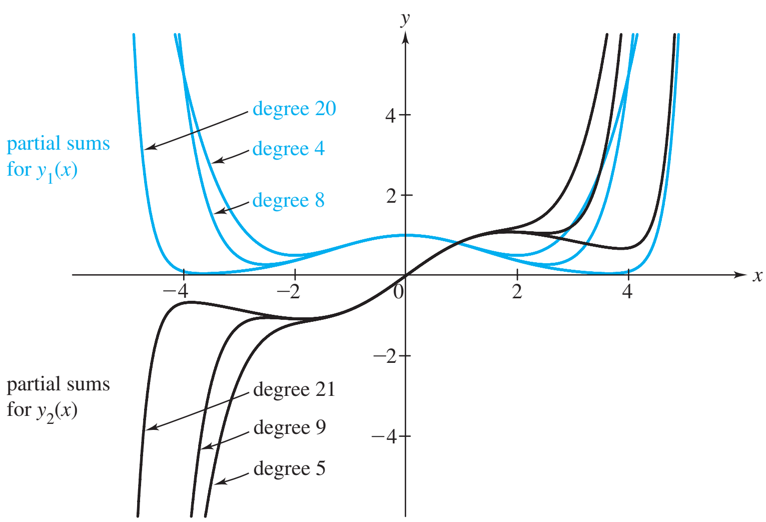

Figure 8.4 Partial sum approximations to solutions for Example 3

Example 4 Find the first few terms in a power series expansion about $x = 0$ for a general solution to

$$(1+x^2)y'' - y' + y = 0\,. \tag{22}$$

Solution Since $p(x) = -(1+x^2)^{-1}$ and $q(x) = (1+x^2)^{-1}$ are analytic at $x = 0$, then $x = 0$ is an ordinary point for equation (22). Hence, we can express its general solution in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n\,.$$

Substituting this expansion into (22) yields

$$(1+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0\,,$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0\,. \tag{23}$$

To sum over like powers x^k, we put $k = n-2$ in the first summation of (23), $k = n-1$ in the third, and $k = n$ in the second and fourth. This gives

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k + \sum_{k=2}^{\infty} k(k-1)a_k x^k - \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^k = 0\,.$$

Separating the terms corresponding to $k = 0$ and $k = 1$ and combining the rest under one summation, we have

$$(2a_2 - a_1 + a_0) + (6a_3 - 2a_2 + a_1)x + \sum_{k=2}^{\infty}\left[(k+2)(k+1)a_{k+2} - (k+1)a_{k+1} + (k(k-1)+1)a_k\right]x^k = 0\,.$$

Setting the coefficients equal to zero gives

$$2a_2 - a_1 + a_0 = 0\,, \tag{24}$$

$$6a_3 - 2a_2 + a_1 = 0\,, \tag{25}$$

and the recurrence relation

$$(k+2)(k+1)a_{k+2} - (k+1)a_{k+1} + (k^2 - k + 1)a_k = 0\,, \qquad k \ge 2\,. \tag{26}$$

We can solve (24) for a_2 in terms of a_0 and a_1:

$$a_2 = \frac{a_1 - a_0}{2}\,.$$

Now that we have a_2, we can use (25) to express a_3 in terms of a_0 and a_1:

$$a_3 = \frac{2a_2 - a_1}{6} = \frac{(a_1 - a_0) - a_1}{6} = \frac{-a_0}{6}\,.$$

Solving the recurrence relation (26) for a_{k+2}, we obtain

$$a_{k+2} = \frac{(k+1)a_{k+1} - (k^2 - k + 1)a_k}{(k+2)(k+1)}\,, \qquad k \ge 2\,. \tag{27}$$

For $k = 2, 3$, and 4, this gives

$$a_4 = \frac{3a_3 - 3a_2}{4 \cdot 3} = \frac{a_3 - a_2}{4}$$

$$= \frac{\dfrac{-a_0}{6} - \left(\dfrac{a_1 - a_0}{2}\right)}{4} = \frac{2a_0 - 3a_1}{24} \qquad (k = 2)\,,$$

$$a_5 = \frac{4a_4 - 7a_3}{5 \cdot 4} = \frac{3a_0 - a_1}{40} \qquad (k = 3)\,,$$

$$a_6 = \frac{5a_5 - 13a_4}{6 \cdot 5} = \frac{36a_1 - 17a_0}{720} \qquad (k = 4)\,.$$

We can now express a general solution in terms up to order 6, using a_0 and a_1 as the arbitrary constants. Thus,

$$\text{(28)} \qquad y(x) = a_0 + a_1 x + \left(\frac{a_1 - a_0}{2}\right)x^2 - \frac{a_0}{6}x^3$$
$$+ \left(\frac{2a_0 - 3a_1}{24}\right)x^4 + \left(\frac{3a_0 - a_1}{40}\right)x^5 + \left(\frac{36a_1 - 17a_0}{720}\right)x^6 + \cdots$$
$$= a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 - \frac{17}{720}x^6 + \cdots\right)$$
$$+ a_1\left(x + \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{40}x^5 + \frac{1}{20}x^6 + \cdots\right). \quad ◆$$

Given specific values for a_0 and a_1, will the partial sums of the power series representation (28) yield useful approximations to the solution when $x = 0.5$? What about when $x = 2.3$ or $x = 7.8$? The answers to these questions certainly depend on the radius of convergence of the power series in (28). But since we were not able to determine a general form for the coefficients a_n in this example, we cannot use the ratio test (or other methods such as the root test, integral test, or comparison test) to compute the radius ρ. In the next section we remedy this situation by giving a simple procedure that determines a lower bound for the radius of convergence of power series solutions.

8.3 EXERCISES

In Problems 1–10, determine all the singular points of the given differential equation.

1. $(x + 1)y'' - x^2y' + 3y = 0$

2. $x^2y'' + 3y' - xy = 0$

3. $(\theta^2 - 2)y'' + 2y' + (\sin\theta)y = 0$

4. $(x^2 + x)y'' + 3y' - 6xy = 0$

5. $(t^2 - t - 2)x'' + (t + 1)x' - (t - 2)x = 0$

6. $(x^2 - 1)y'' + (1 - x)y' + (x^2 - 2x + 1)y = 0$

7. $(\sin x)y'' + (\cos x)y = 0$

8. $e^xy'' - (x^2 - 1)y' + 2xy = 0$

9. $(\sin\theta)y'' - (\ln\theta)y = 0$

10. $[\ln(x - 1)]y'' + (\sin 2x)y' - e^xy = 0$

In Problems 11–18, find at least the first four nonzero terms in a power series expansion about $x = 0$ for a general solution to the given differential equation.

11. $y' + (x + 2)y = 0$

12. $y' - y = 0$

13. $z'' - x^2z = 0$

14. $(x^2 + 1)y'' + y = 0$

15. $y'' + (x - 1)y' + y = 0$

16. $y'' - 2y' + y = 0$

17. $w'' - x^2w' + w = 0$

18. $(2x - 3)y'' - xy' + y = 0$

In Problems 19–24, find a power series expansion about $x = 0$ for a general solution to the given differential equation. Your answer should include a general formula for the coefficients.

19. $y' - 2xy = 0$

20. $y'' + y = 0$

21. $y'' - xy' + 4y = 0$

22. $y'' - xy = 0$

23. $z'' - x^2z' - xz = 0$

24. $(x^2 + 1)y'' - xy' + y = 0$

In Problems 25–28, find at least the first four nonzero terms in a power series expansion about $x = 0$ for the solution to the given initial value problem.

25. $w'' + 3xw' - w = 0\,;$
$w(0) = 2\,, \quad w'(0) = 0$

26. $(x^2 - x + 1)y'' - y' - y = 0\,;$
$y(0) = 0\,, \quad y'(0) = 1$

27. $(x + 1)y'' - y = 0\,;$
$y(0) = 0\,, \quad y'(0) = 1$

28. $y'' + (x - 2)y' - y = 0\,;$
$y(0) = -1\,, \quad y'(0) = 0$

In Problems 29–31, use the first few terms of the power series expansion to find a cubic polynomial approximation for the solution to the given initial value problem. Graph the linear, quadratic, and cubic polynomial approximations for $-5 \le x \le 5$.

29. $y'' + y' - xy = 0\,;$
$y(0) = 1\,, \quad y'(0) = -2$

30. $y'' - 4xy' + 5y = 0\,;$
$y(0) = -1\,, \quad y'(0) = 1$

31. $(x^2 + 2)y'' + 2xy' + 3y = 0\,;$
$y(0) = 1\,, \qquad y'(0) = 2$

32. Consider the initial value problem

$$y'' - 2xy' - 2y = 0\,;$$
$$y(0) = a_0\,, \qquad y'(0) = a_1\,,$$

where a_0 and a_1 are constants.

(a) Show that if $a_0 = 0$, then the solution will be an odd function [that is, $y(-x) = -y(x)$ for all x]. What happens when $a_1 = 0$?

(b) Show that if a_0 and a_1 are positive, then the solution is increasing on $(0, \infty)$.

(c) Show that if a_0 is negative and a_1 is positive, then the solution is increasing on $(-\infty, 0)$.

(d) What conditions on a_0 and a_1 would guarantee that the solution is increasing on $(-\infty, \infty)$?

33. Use the ratio test to show that the radius of convergence of the series in equation (13) is infinite. [*Hint:* See Problem 7, Exercises 8.2, page 434.]

34. Emden's Equation. A classical nonlinear equation that occurs in the study of the thermal behavior of a spherical cloud is **Emden's equation**

$$y'' + \frac{2}{x}y' + y^n = 0\,,$$

with initial conditions $y(0) = 1, y'(0) = 0$. Even though $x = 0$ is *not* an ordinary point for this equation (which is nonlinear for $n \neq 1$), it turns out that there does exist a solution analytic at $x = 0$. Assuming that n is a positive integer, show that the first few terms in a power series solution are

$$y = 1 - \frac{x^2}{3!} + n\frac{x^4}{5!} + \cdots\ .$$

[*Hint:* Substitute $y = 1 + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots$ into the equation and carefully compute the first few terms in the expansion for y^n.]

35. Variable Resistor. In Section 5.7, we showed that the charge q on the capacitor in a simple *RLC* circuit is governed by the equation

$$Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = E(t)\,,$$

where L is the inductance, R the resistance, C the capacitance, and E the voltage source. Since the resistance of a resistor increases with temperature, let's assume that the resistor is heated so that the resistance at time t is $R(t) = 1 + t/10\ \Omega$ (see Figure 8.5). If $L = 0.1$ H, $C = 2$ F, $E(t) \equiv 0$, $q(0) = 10$ C, and $q'(0) = 0$ A, find at least the first four nonzero terms in a power series expansion about $t = 0$ for the charge on the capacitor.

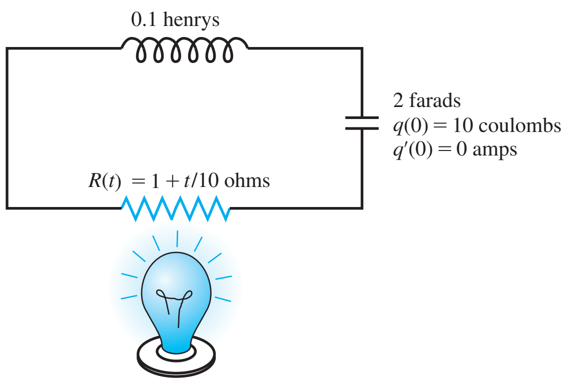

Figure 8.5 An *RLC* circuit whose resistor is being heated

36. Variable Spring Constant. As a spring is heated, its spring "constant" decreases. Suppose the spring is heated so that the spring "constant" at time t is $k(t) = 6 - t$ N/m (see Figure 8.6). If the unforced mass–spring system has mass $m = 2$ kg and a damping constant $b = 1$ N-sec/m with initial conditions $x(0) = 3$ m and $x'(0) = 0$ m/sec, then the displacement $x(t)$ is governed by the initial value problem

$$2x''(t) + x'(t) + (6 - t)x(t) = 0 ;$$
$$x(0) = 3 , \qquad x'(0) = 0 .$$

Find at least the first four nonzero terms in a power series expansion about $t = 0$ for the displacement.

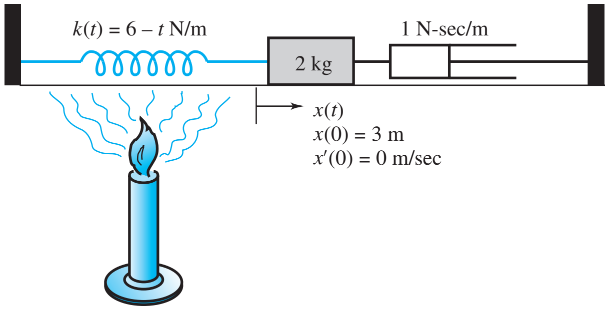

Figure 8.6 A mass–spring system whose spring is being heated

8.4 Equations with Analytic Coefficients

In Section 8.3 we introduced a method for obtaining a power series solution about an ordinary point. In this section we continue the discussion of this procedure. We begin by stating a basic existence theorem for the equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 , \tag{1}$$

which justifies the power series method.

Existence of Analytic Solutions

Theorem 5. Suppose x_0 is an ordinary point for equation (1). Then (1) has two linearly independent analytic solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n . \tag{2}$$

Moreover, the radius of convergence of any power series solution of the form given by (2) is at least as large as the distance from x_0 to the nearest singular point (real or complex-valued) of equation (1).

The key element in the proof of Theorem 5 is the construction of a convergent geometric series that dominates the series expansion (2) of a solution to equation (1). The convergence of the series in (2) then follows by the comparison test. The details of the proof can be found in more advanced books on differential equations.†

As we saw in Section 8.3, the power series method gives us a general solution in the same form as (2), with a_0 and a_1 as arbitrary constants. The two linearly independent solutions referred to in Theorem 5 can be obtained by taking $a_0 = 1$, $a_1 = 0$ for the first and $a_0 = 0$, $a_1 = 1$ for the second. Thus, we can extend Theorem 5 by saying that *equation* (1) *has a general solution of the form* (2) *with* a_0 *and* a_1 *as the arbitrary constants.*

†See, for example, *Ordinary Differential Equations,* 4th ed., by G. Birkhoff and G.-C. Rota (Wiley, New York, 1989).

The second part of Theorem 5 gives a simple way of determining a minimum value for the radius of convergence of the power series. We need only find the singular points of equation (1) and then determine the distance between the ordinary point x_0 and the nearest singular point.

Example 1 Find a minimum value for the radius of convergence of a power series solution about $x = 0$ to

$$2y'' + xy' + y = 0 . \tag{3}$$

Solution For this equation, $p(x) = x/2$ and $q(x) = 1/2$. Both of these functions are analytic for all real or complex values of x. Since equation (3) has no singular points, the distance between the ordinary point $x = 0$ and the nearest singular point is infinite. Hence, the radius of convergence is infinite. ◆

The next example helps to answer the questions posed at the end of the last section.

Example 2 Find a minimum value for the radius of convergence of a power series solution about $x = 0$ to

$$(1 + x^2)y'' - y' + y = 0 . \tag{4}$$

Solution Here $p(x) = -1/(1 + x^2)$, $q(x) = 1/(1 + x^2)$, and so the singular points of equation (4) occur when $1 + x^2 = 0$; that is, when $x = \pm\sqrt{-1} = \pm i$. Since the only singular points of equation (4) are the complex numbers $\pm i$, we see that $x = 0$ is an ordinary point. Moreover, the distance† from 0 to either $\pm i$ is 1. Thus, the radius of convergence of a power series solution about $x = 0$ is at least 1. ◆

In equation (28) of Section 8.3, page 443, we found the first few terms of a power series solution to equation (4). Because we now know that this series has radius of convergence at least 1, the partial sums of this series will converge to the solution for $|x| < 1$. However, when $|x| \geq 1$, we have no basis on which to decide whether we can use the series to approximate the solution.

Power series expansions about $x_0 = 0$ are somewhat easier to manipulate than expansions about nonzero points. As the next example shows, a simple shift in variable enables us always to expand about the origin.

Example 3 Find the first few terms in a power series expansion about $x = 1$ for a general solution to

$$2y'' + xy' + y = 0 . \tag{5}$$

Also determine the radius of convergence of the series.

Solution As seen in Example 1, there are no singular points for equation (5). Thus, $x = 1$ is an ordinary point, and, as a consequence of Theorem 5, equation (5) has a general solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n(x - 1)^n . \tag{6}$$

Moreover, the radius of convergence of the series in (6) must be infinite.

---

†Recall that the distance between the two complex numbers $z = a + bi$ and $w = c + di$ is given by $\sqrt{(a - c)^2 + (b - d)^2}$.

We can simplify the computation of the coefficients a_n by shifting the center of the expansion (6) from $x_0 = 1$ to $t_0 = 0$. This is accomplished by the substitution $x = t + 1$. Setting $Y(t) = y(t+1)$, we find via the chain rule

$$\frac{dy}{dx} = \frac{dY}{dt}, \qquad \frac{d^2y}{dx^2} = \frac{d^2Y}{dt^2},$$

and, hence, equation (5) is changed into

$$2\frac{d^2Y}{dt^2} + (t+1)\frac{dY}{dt} + Y = 0 . \tag{7}$$

We now seek a general solution of the form

$$Y(t) = \sum_{n=0}^{\infty} a_n t^n , \tag{8}$$

where the a_n's in equations (6) and (8) are the same. Proceeding as usual, we substitute the power series for $Y(t)$ into (7), derive a recurrence relation for the coefficients, and ultimately find that

$$Y(t) = a_0\left\{1 - \frac{1}{4}t^2 + \frac{1}{24}t^3 + \cdots\right\} + a_1\left\{t - \frac{1}{4}t^2 - \frac{1}{8}t^3 + \cdots\right\}$$

(the details are left as an exercise). Thus, restoring $t = x - 1$ we have

$$\begin{aligned} y(x) = a_0&\left\{1 - \frac{1}{4}(x-1)^2 + \frac{1}{24}(x-1)^3 + \cdots\right\} \\ &+ a_1\left\{(x-1) - \frac{1}{4}(x-1)^2 - \frac{1}{8}(x-1)^3 + \cdots\right\} . \end{aligned} \tag{9}$$ ◆

When the coefficients of a linear equation are not polynomials in x, but are analytic functions, we can still find analytic solutions by essentially the same method.

Example 4 Find a power series expansion for the solution to

$$y''(x) + e^x y'(x) + (1+x^2)y(x) = 0 ; \qquad y(0) = 1 , \qquad y'(0) = 0 . \tag{10}$$

Solution Here $p(x) = e^x$ and $q(x) = 1 + x^2$, and both are analytic for all x. Thus, by Theorem 5, the initial value problem (10) has a power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{11}$$

that converges for all x $(\rho = \infty)$. To find the first few terms of this series, we first expand $p(x) = e^x$ in its Maclaurin series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

Substituting the expansions for $y(x)$, $y'(x)$, $y''(x)$, and e^x into (10) gives

$$\begin{aligned} &\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\sum_{n=1}^{\infty} n a_n x^{n-1} \\ &\quad + (1+x^2)\sum_{n=0}^{\infty} a_n x^n = 0 . \end{aligned} \tag{12}$$

Because of the computational difficulties due to the appearance of the product of the power series for e^x and $y'(x)$, we concern ourselves with just those terms up to order 4. Writing out (12) and keeping track of all such terms, we find

(13)
$$
\begin{aligned}
&(2a_2 + 6a_3x + \mathbf{12a_4x^2} + 20a_5x^3 + 30a_6x^4 + \cdots) \\
&+ (a_1 + 2a_2x + \mathbf{3a_3x^2} + 4a_4x^3 + 5a_5x^4 + \cdots) &&\longleftarrow 1\cdot\Sigma\, na_nx^{n-1} \\
&+ (a_1x + \mathbf{2a_2x^2} + 3a_3x^3 + 4a_4x^4 + \cdots) &&\longleftarrow x\cdot\Sigma\, na_nx^{n-1} \\
&+ \left(\mathbf{\frac{1}{2}a_1x^2} + a_2x^3 + \frac{3}{2}a_3x^4 + \cdots\right) &&\longleftarrow \frac{x^2}{2}\cdot\Sigma\, na_nx^{n-1} \\
&+ \left(\frac{1}{6}a_1x^3 + \frac{1}{3}a_2x^4 + \cdots\right) &&\qquad\vdots \\
&+ \left(\frac{1}{24}a_1x^4 + \cdots\right) \\
&+ (a_0 + a_1x + \mathbf{a_2x^2} + a_3x^3 + a_4x^4 + \cdots) &&\longleftarrow 1\cdot\Sigma\, a_nx^n \\
&+ (\mathbf{a_0x^2} + a_1x^3 + a_2x^4 + \cdots) = 0\,. &&\longleftarrow x^2\cdot\Sigma\, a_nx^n
\end{aligned}
$$

Grouping the like powers of x in equation (13) (for example, the x^2 terms are shown in color) and then setting the coefficients equal to zero yields the system of equations

$$
\begin{aligned}
2a_2 + a_1 + a_0 &= 0 && (x^0 \text{ term})\,, \\
6a_3 + 2a_2 + 2a_1 &= 0 && (x^1 \text{ term})\,, \\
\mathbf{12a_4 + 3a_3 + 3a_2 + \frac{1}{2}a_1 + a_0} &\mathbf{= 0} && (\mathbf{x^2 \textbf{ term}})\,, \\
20a_5 + 4a_4 + 4a_3 + a_2 + \frac{7}{6}a_1 &= 0 && (x^3 \text{ term})\,, \\
30a_6 + 5a_5 + 5a_4 + \frac{3}{2}a_3 + \frac{4}{3}a_2 + \frac{1}{24}a_1 &= 0 && (x^4 \text{ term})\,.
\end{aligned}
$$

The initial conditions in (10) imply that $y(0) = a_0 = 1$ and $y'(0) = a_1 = 0$. Using these values for a_0 and a_1, we can solve the above system first for a_2, then a_3, and so on:

$$
\begin{aligned}
2a_2 + 0 + 1 &= 0 \Rightarrow a_2 = -\frac{1}{2}\,, \\
6a_3 - 1 + 0 &= 0 \Rightarrow a_3 = \frac{1}{6}\,, \\
12a_4 + \frac{1}{2} - \frac{3}{2} + 0 + 1 &= 0 \Rightarrow a_4 = 0\,, \\
20a_5 + 0 + \frac{2}{3} - \frac{1}{2} + 0 &= 0 \Rightarrow a_5 = -\frac{1}{120}\,, \\
30a_6 - \frac{1}{24} + 0 + \frac{1}{4} - \frac{2}{3} + 0 &= 0 \Rightarrow a_6 = \frac{11}{720}\,.
\end{aligned}
$$

Thus, the solution to the initial value problem in (10) is

(14)
$$
y(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{11}{720}x^6 + \cdots\,. \quad ◆
$$

Thus far we have used the power series method only for homogeneous equations. But the same method applies, with obvious modifications, to nonhomogeneous equations of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x) , \tag{15}$$

provided the forcing term $g(x)$ and the coefficient functions are analytic at x_0. For example, to find a power series about $x = 0$ for a general solution to

$$y''(x) - xy'(x) - y(x) = \sin x , \tag{16}$$

we use the substitution $y(x) = \Sigma\, a_n x^n$ to obtain a power series expansion for the left-hand side of (16). We then equate the coefficients of this series with the corresponding coefficients of the Maclaurin expansion for sin x:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} .$$

Carrying out the details (see Problem 20 on page 450), we ultimately find that an expansion for a general solution to (16) is

$$y(x) = a_0 y_1(x) + a_1 y_2(x) + y_p(x) , \tag{17}$$

where

$$y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \cdots , \tag{18}$$

$$y_2(x) = x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7 + \cdots \tag{19}$$

are the solutions to the homogeneous equation associated with equation (16) and

$$y_p(x) = \frac{1}{6}x^3 + \frac{1}{40}x^5 + \frac{19}{5040}x^7 + \cdots \tag{20}$$

is a particular solution to equation (16).

8.4 EXERCISES

In Problems 1–6, find a minimum value for the radius of convergence of a power series solution about x_0.

1. $(x+1)y'' - 3xy' + 2y = 0 ; \quad x_0 = 1$
2. $y'' - xy' - 3y = 0 ; \quad x_0 = 2$
3. $(1 + x + x^2)y'' - 3y = 0 ; \quad x_0 = 1$
4. $(x^2 - 5x + 6)y'' - 3xy' - y = 0 ; \quad x_0 = 0$
5. $y'' - (\tan x)y' + y = 0 ; \quad x_0 = 0$
6. $(1 + x^3)y'' - xy' + 3x^2 y = 0 ; \quad x_0 = 1$

In Problems 7–12, find at least the first four nonzero terms in a power series expansion about x_0 for a general solution to the given differential equation with the given value for x_0.

7. $y' + 2(x-1)y = 0 ; \quad x_0 = 1$
8. $y' - 2xy = 0 ; \quad x_0 = -1$
9. $(x^2 - 2x)y'' + 2y = 0 ; \quad x_0 = 1$
10. $x^2 y'' - xy' + 2y = 0 ; \quad x_0 = 2$
11. $x^2 y'' - y' + y = 0 ; \quad x_0 = 2$
12. $y'' + (3x - 1)y' - y = 0 ; \quad x_0 = -1$

In Problems 13–19, find at least the first four nonzero terms in a power series expansion of the solution to the given initial value problem.

13. $x' + (\sin t)x = 0 ; \quad x(0) = 1$
14. $y' - e^x y = 0 ; \quad y(0) = 1$
15. $(x^2 + 1)y'' - e^x y' + y = 0 ; \quad y(0) = 1 , \quad y'(0) = 1$
16. $y'' + ty' + e^t y = 0 ; \quad y(0) = 0 , \quad y'(0) = -1$
17. $y'' - (\sin x)y = 0 ; \quad y(\pi) = 1 , \quad y'(\pi) = 0$

18. $y'' - (\cos x)y' - y = 0\,;$
$y(\pi/2) = 1\,, \quad y'(\pi/2) = 1$

19. $y'' - e^{2x}y' + (\cos x)y = 0\,;$
$y(0) = -1\,, \quad y'(0) = 1$

20. To derive the general solution given by equations (17)–(20) for the nonhomogeneous equation (16), complete the following steps:

(a) Substitute $y(x) = \Sigma_{n=0}^{\infty} a_n x^n$ and the Maclaurin series for $\sin x$ into equation (16) to obtain

$$(2a_2 - a_0) + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} - (k+1)a_k]x^k$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\,.$$

(b) Equate the coefficients of like powers of x on both sides of the equation in part (a) and thereby deduce the equations

$$a_2 = \frac{a_0}{2}\,, \quad a_3 = \frac{1}{6} + \frac{a_1}{3}\,, \quad a_4 = \frac{a_0}{8}\,,$$
$$a_5 = \frac{1}{40} + \frac{a_1}{15}\,, \quad a_6 = \frac{a_0}{48}\,,$$
$$a_7 = \frac{19}{5040} + \frac{a_1}{105}\,.$$

(c) Show that the relations in part (b) yield the general solution to (16) given in equations (17)–(20).

In Problems 21–28, use the procedure illustrated in Problem 20 to find at least the first four nonzero terms in a power series expansion about $x = 0$ of a general solution to the given differential equation.

21. $y' - xy = \sin x$

22. $w' + xw = e^x$

23. $z'' + xz' + z = x^2 + 2x + 1$

24. $y'' - 2xy' + 3y = x^2$

25. $(1 + x^2)y'' - xy' + y = e^{-x}$

26. $y'' - xy' + 2y = \cos x$

27. $(1 - x^2)y'' - y' + y = \tan x$

28. $y'' - (\sin x)y = \cos x$

29. The equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0,$$

where n is an unspecified parameter, is called **Legendre's equation.** This equation occurs in applications of differential equations to engineering systems in spherical coordinates.

(a) Find a power series expansion about $x = 0$ for a solution to Legendre's equation.

(b) Show that for n a nonnegative integer, there exists an nth-degree polynomial that is a solution to Legendre's equation. These polynomials, up to a constant multiple, are called **Legendre polynomials.**

(c) Determine the first three Legendre polynomials (up to a constant multiple).

30. Aging Spring. As a spring ages, its spring "constant" decreases in value. One such model for a mass–spring system with an aging spring is

$$mx''(t) + bx'(t) + ke^{-\eta t}x(t) = 0\,,$$

where m is the mass, b the damping constant, k and η positive constants, and $x(t)$ the displacement of the spring from its equilibrium position. Let $m = 1$ kg, $b = 2$ N-sec/m, $k = 1$ N/m, and $\eta = 1(\text{sec})^{-1}$. The system is set in motion by displacing the mass 1 m from its equilibrium position and then releasing it $(x(0) = 1, x'(0) = 0)$. Find at least the first four nonzero terms in a power series expansion about $t = 0$ for the displacement.

31. Aging Spring without Damping. In the mass–spring system for an aging spring discussed in Problem 30, assume that there is no damping (i.e., $b = 0$), $m = 1$, and $k = 1$. To see the effect of aging, consider η as a positive parameter.

(a) Redo Problem 30 with $b = 0$ and η arbitrary but fixed.

(b) Set $\eta = 0$ in the expansion obtained in part (a). Does this expansion agree with the expansion for the solution to the problem with $\eta = 0$? [*Hint:* When $\eta = 0$, the solution is $x(t) = \cos t$.]

8.5 Cauchy–Euler (Equidimensional) Equations

In the previous sections we considered methods for obtaining power series solutions about an ordinary point for a linear second-order equation. However, in certain cases we may want a series expansion about a *singular point* of the equation. To motivate a procedure for finding such expansions, we consider the class of **Cauchy–Euler equations.** In Section 4.7 we studied this topic briefly. Here we will use the operator approach to rederive and extend our conclusions.

A second-order homogeneous Cauchy–Euler equation has the form

$$\textbf{(1)} \qquad ax^2y''(x) + bxy'(x) + cy(x) = 0\,, \qquad x > 0\,,$$

8.6 Method of Frobenius

In the previous section we showed that a homogeneous Cauchy–Euler equation has a solution of the form $y(x) = x^r$, $x > 0$, where r is a certain constant. Cauchy–Euler equations have, of course, a very special form with only one singular point (at $x = 0$). In this section we show how the theory for Cauchy–Euler equations generalizes to other equations that have a special type of singularity.

To motivate the procedure, let's rewrite the Cauchy–Euler equation,

$$ax^2y''(x) + bxy'(x) + cy(x) = 0\,, \qquad x > 0\,, \tag{1}$$

in the standard form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0\,, \qquad x > 0\,, \tag{2}$$

where

$$p(x) = \frac{p_0}{x}\,, \qquad q(x) = \frac{q_0}{x^2}\,,$$

and p_0, q_0 are the constants b/a and c/a, respectively. When we substitute $w(r, x) = x^r$ for y into equation (2), we get

$$[r(r-1) + p_0 r + q_0]x^{r-2} = 0\,,$$

which yields the indicial equation

$$r(r-1) + p_0 r + q_0 = 0\,. \tag{3}$$

Thus, if r_1 is a root of (3), then $w(r_1, x) = x^{r_1}$ is a solution to equations (1) and (2).

Let's now assume, more generally, that (2) is an equation for which $xp(x)$ and $x^2q(x)$, instead of being constants, are *analytic functions*. That is, in some open interval about $x = 0$,

$$xp(x) = p_0 + p_1x + p_2x^2 + \cdots = \sum_{n=0}^{\infty} p_n x^n\,, \tag{4}$$

$$x^2q(x) = q_0 + q_1x + q_2x^2 + \cdots = \sum_{n=0}^{\infty} q_n x^n\,. \tag{5}$$

It follows from (4) and (5) that

$$\lim_{x\to 0} xp(x) = p_0 \qquad \text{and} \qquad \lim_{x\to 0} x^2q(x) = q_0\,, \tag{6}$$

and hence, for x near 0 we have $xp(x) \approx p_0$ and $x^2q(x) \approx q_0$. Therefore, it is reasonable to expect that the solutions to (2) will behave (for x near 0) like the solutions to the Cauchy–Euler equation

$$x^2y'' + p_0xy' + q_0y = 0\,.$$

When $p(x)$ and $q(x)$ satisfy (4) and (5), we say that the singular point at $x = 0$ is regular. More generally, we state the following.

Regular Singular Point

Definition 3. A singular point x_0 of

(7) $$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

is said to be a **regular singular point** if both $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0.† Otherwise x_0 is called an **irregular singular point.**

Example 1 Classify the singular points of the equation

(8) $$(x^2 - 1)^2 y''(x) + (x + 1)y'(x) - y(x) = 0 .$$

Solution Here

$$p(x) = \frac{x+1}{(x^2-1)^2} = \frac{1}{(x+1)(x-1)^2},$$

$$q(x) = \frac{-1}{(x^2-1)^2} = \frac{-1}{(x+1)^2(x-1)^2},$$

from which we see that ± 1 are the singular points of (8). For the singularity at 1, we have

$$(x-1)p(x) = \frac{1}{(x+1)(x-1)},$$

which is not analytic at $x = 1$. Therefore, $x = 1$ is an irregular singular point.

For the singularity at -1, we have

$$(x+1)p(x) = \frac{1}{(x-1)^2}, \qquad (x+1)^2 q(x) = \frac{-1}{(x-1)^2},$$

both of which are analytic at $x = -1$. Hence, $x = -1$ is a regular singular point. ◆

Let's assume that $x = 0$ is a regular singular point for equation (7) so that $p(x)$ and $q(x)$ satisfy (4) and (5); that is,

(9) $$p(x) = \sum_{n=0}^{\infty} p_n x^{n-1}, \qquad q(x) = \sum_{n=0}^{\infty} q_n x^{n-2} .$$

The idea of the mathematician Frobenius was that since Cauchy–Euler equations have solutions of the form x^r, then for the regular singular point $x = 0$, there should be solutions to (7) of the form x^r *times an analytic function*.‡ Hence we seek solutions to (7) of the form

(10) $$\boldsymbol{w(r,x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad x > 0 .}$$

In writing (10), we have assumed a_0 is the first nonzero coefficient, so we are left with determining r and the coefficients a_n, $n \geq 1$. Differentiating $w(r, x)$ with respect to x, we have

(11) $$w'(r,x) = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1},$$

(12) $$w''(r,x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} .$$

†In the terminology of complex variables, p has a pole of order at most 1, and q has a pole of order at most 2, at x_0.

‡*Historical Footnote:* George Frobenius (1848–1917) developed this method in 1873. He is also known for his research on group theory.

If we substitute the above expansions for $w(r,x)$, $w'(r,x)$, $w''(r,x)$, $p(x)$, and $q(x)$ into (7), we obtain

$$\textbf{(13)} \quad \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \left(\sum_{n=0}^{\infty} p_n x^{n-1}\right)\left(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} q_n x^{n-2}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0\,.$$

Now we use the Cauchy product to perform the series multiplications and then group like powers of x, starting with the lowest power, x^{r-2}. This gives

$$\textbf{(14)} \quad [r(r-1)+p_0 r+q_0]a_0 x^{r-2} + [(r+1)ra_1+(r+1)p_0 a_1+p_1 r a_0+q_0 a_1+q_1 a_0]x^{r-1}+\cdots = 0\,.$$

For the expansion on the left-hand side of equation (14) to sum to zero, each coefficient must be zero. Considering the first term, x^{r-2}, we find

$$\textbf{(15)} \quad [r(r-1)+p_0 r+q_0]a_0 = 0\,.$$

We have assumed that $a_0 \neq 0$, so the quantity in brackets must be zero. This gives the indicial equation; it is the same as the one we derived for Cauchy–Euler equations.

Indicial Equation

Definition 4. If x_0 is a regular singular point of $y''+py'+qy=0$, then the **indicial equation** for this point is

$$\textbf{(16)} \quad r(r-1)+p_0 r+q_0 = 0\,,$$

where

$$p_0 := \lim_{x\to x_0}(x-x_0)p(x)\,, \qquad q_0 := \lim_{x\to x_0}(x-x_0)^2 q(x)\,.$$

The roots of the indicial equation are called the **exponents (indices)** of the singularity x_0.

Example 2 Find the indicial equation and the exponents at the singularity $x=-1$ of

$$\textbf{(17)} \quad (x^2-1)^2 y''(x)+(x+1)y'(x)-y(x)=0\,.$$

Solution In Example 1 we showed that $x=-1$ is a regular singular point. Since $p(x) = (x+1)^{-1}(x-1)^{-2}$ and $q(x) = -(x+1)^{-2}(x-1)^{-2}$, we find

$$p_0 = \lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1}(x-1)^{-2} = \frac{1}{4}\,,$$

$$q_0 = \lim_{x\to -1}(x+1)^2 q(x) = \lim_{x\to -1}[-(x-1)^{-2}] = -\frac{1}{4}\,.$$

Substituting these values for p_0 and q_0 into (16), we obtain the indicial equation

$$\textbf{(18)} \quad r(r-1)+\frac{1}{4}r-\frac{1}{4}=0\,.$$

Multiplying by 4 and factoring gives $(4r+1)(r-1)=0$. Hence, $r=1,-1/4$ are the exponents. ◆

As we have seen, we can use the indicial equation to determine those values of r for which the coefficient of x^{r-2} in (14) is zero. If we set the coefficient of x^{r-1} in (14) equal to zero, we have

$$[(r+1)r + (r+1)p_0 + q_0]a_1 + (p_1 r + q_1)a_0 = 0 . \tag{19}$$

Since a_0 is arbitrary and we know the p_i's, q_i's, and r, we can solve equation (19) for a_1, provided the coefficient of a_1 in (19) is not zero. This will be the case if we take r to be the *larger* of the two roots of the indicial equation (see Problem 43, page 464).[†] Similarly, when we set the coefficient of x^r equal to zero, we can solve for a_2 in terms of the p_i's, q_i's, r, a_0, and a_1. Continuing in this manner, we can recursively solve for the a_n's. The procedure is illustrated in the following example.

Example 3 Find a series expansion about the regular singular point $x = 0$ for a solution to

$$(x+2)x^2 y''(x) - xy'(x) + (1+x)y(x) = 0 , \qquad x > 0 . \tag{20}$$

Solution Here $p(x) = -x^{-1}(x+2)^{-1}$ and $q(x) = x^{-2}(x+2)^{-1}(1+x)$, so

$$p_0 = \lim_{x \to 0} xp(x) = \lim_{x \to 0} [-(x+2)^{-1}] = -\frac{1}{2} ,$$

$$q_0 = \lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} (x+2)^{-1}(1+x) = \frac{1}{2} .$$

Since $x = 0$ is a regular singular point, we seek a solution to (20) of the form

$$w(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} . \tag{21}$$

By the previous discussion, r must satisfy the indicial equation (16). Substituting for p_0 and q_0 in (16), we obtain

$$r(r-1) - \frac{1}{2}r + \frac{1}{2} = 0 ,$$

which simplifies to $2r^2 - 3r + 1 = (2r-1)(r-1) = 0$. Thus, $r = 1$ and $r = 1/2$ are the roots of the indicial equation associated with $x = 0$.

Let's use the larger root $r = 1$ and solve for a_1, a_2, etc., to obtain the solution $w(1, x)$. We can simplify the computations by substituting $w(r, x)$ directly into equation (20), where the coefficients are polynomials in x, rather than dividing by $(x+2)x^2$ and having to work with the rational functions $p(x)$ and $q(x)$. Inserting $w(r, x)$ in (20) and recalling the formulas for $w'(r, x)$ and $w''(r, x)$ in (11) and (12) gives (with $r = 1$)

$$\begin{aligned}(x+2)x^2 \sum_{n=0}^{\infty} (n+1)na_n x^{n-1} - x \sum_{n=0}^{\infty} (n+1)a_n x^n \\ + (1+x) \sum_{n=0}^{\infty} a_n x^{n+1} = 0 ,\end{aligned} \tag{22}$$

which we can write as

$$\begin{aligned}\sum_{n=0}^{\infty} (n+1)na_n x^{n+2} + \sum_{n=0}^{\infty} 2(n+1)na_n x^{n+1} - \sum_{n=0}^{\infty} (n+1)a_n x^{n+1} \\ + \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0 .\end{aligned} \tag{23}$$

[†]"Larger" in the sense of Problem 43.

Next we shift the indices so that each summation in (23) is in powers x^k. With $k = n + 2$ in the first and last summations and $k = n + 1$ in the rest, (23) becomes

(24) $$\sum_{k=2}^{\infty}[(k-1)(k-2)+1]a_{k-2}x^k + \sum_{k=1}^{\infty}[2k(k-1)-k+1]a_{k-1}x^k = 0 .$$

Separating off the $k = 1$ term and combining the rest under one summation yields

(25) $$[2(1)(0)-1+1]a_0x + \sum_{k=2}^{\infty}[(k^2-3k+3)a_{k-2} + (2k-1)(k-1)a_{k-1}]x^k = 0 .$$

Notice that the coefficient of x in (25) is zero. This is because $r = 1$ is a root of the indicial equation, which is the equation we obtained by setting the coefficient of the lowest power of x equal to zero.

We can now determine the a_k's in terms of a_0 by setting the coefficients of x^k in equation (25) equal to zero for $k = 2, 3$, etc. This gives the recurrence relation

(26) $$(k^2-3k+3)a_{k-2} + (2k-1)(k-1)a_{k-1} = 0 ,$$

or, equivalently,

(27) $$a_{k-1} = -\frac{k^2-3k+3}{(2k-1)(k-1)}a_{k-2} , \qquad k \geq 2 \ .$$

Setting $k = 2$, 3, and 4 in (27), we find

$$a_1 = -\frac{1}{3}a_0 \qquad (k = 2) ,$$

$$a_2 = -\frac{3}{10}a_1 = \frac{1}{10}a_0 \qquad (k = 3) ,$$

$$a_3 = -\frac{1}{3}a_2 = -\frac{1}{30}a_0 \qquad (k = 4) .$$

Substituting these values for r, a_1, a_2, and a_3 into (21) gives

(28) $$w(1, x) = a_0x^1\left(1 - \frac{1}{3}x + \frac{1}{10}x^2 - \frac{1}{30}x^3 + \cdots\right),$$

where a_0 is arbitrary. In particular, for $a_0 = 1$, we get the solution

$$y_1(x) = x - \frac{1}{3}x^2 + \frac{1}{10}x^3 - \frac{1}{30}x^4 + \cdots \qquad (x > 0) .$$

See Figure 8.8 on page 459. ◆

To find a second linearly independent solution to equation (20), we could try setting $r = 1/2$ and solving for $a_1, a_2, \ldots$ to obtain a solution $w(1/2, x)$ (see Problem 44, page 464). In this particular case, the approach would work. However, if we encounter an indicial equation that has a repeated root, then the method of Frobenius would yield just one solution (apart from constant multiples). To find the desired second solution, we must use another technique, such as the reduction of order procedure discussed in Section 4.7 or Exercises 6.1, Problem 31, page 327. We tackle the problem of finding a second linearly independent solution in the next section.

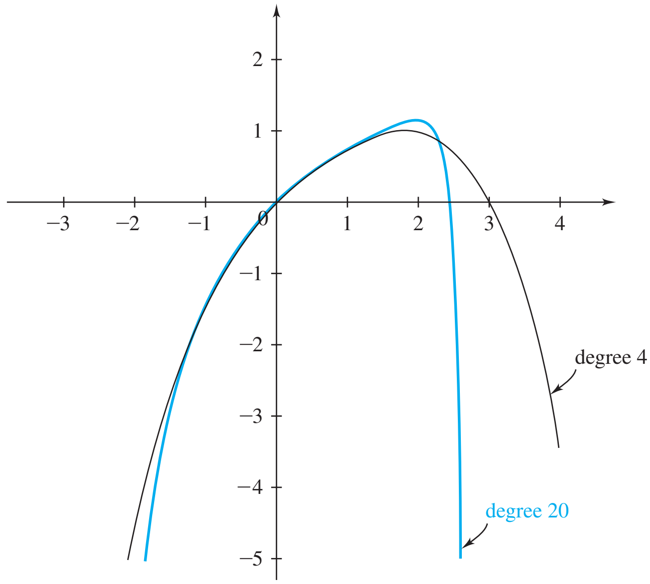

Figure 8.8 Partial sums approximating the solution $y_1(x)$ of Example 3

The method of Frobenius can be summarized as follows.

Method of Frobenius

To derive a series solution about the singular point x_0 of

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0\,, \qquad x > x_0: \tag{29}$$

(a) Set $p(x) := a_1(x)/a_2(x)$, $q(x) := a_0(x)/a_2(x)$. If both $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0, then x_0 is a regular singular point and the remaining steps apply.

(b) Let

$$w(r, x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}\,, \tag{30}$$

and, using termwise differentiation, substitute $w(r, x)$ into equation (29) to obtain an equation of the form

$$A_0 (x - x_0)^{r+J} + A_1 (x - x_0)^{r+J+1} + \cdots = 0\,.$$

(c) Set the coefficients $A_0, A_1, A_2, \ldots$ equal to zero. [Notice that the equation $A_0 = 0$ is just a constant multiple of the indicial equation $r(r - 1) + p_0 r + q_0 = 0$.]

(d) Use the system of equations

$$A_0 = 0\,, \qquad A_1 = 0\,, \qquad \ldots, \qquad A_k = 0$$

to find a recurrence relation involving a_k and $a_0, a_1, \ldots, a_{k-1}$.

(e) Take $r = r_1$, the larger root of the indicial equation, and use the relation obtained in step (d) to determine $a_1, a_2, \ldots$ recursively in terms of a_0 and r_1.

(f) A series expansion of a solution to (29) is

$$w(r_1, x) = (x - x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n\,, \qquad x > x_0\,, \tag{31}$$

where a_0 is arbitrary and the a_n's are defined in terms of a_0 and r_1.

One important question that remains concerns the radius of convergence of the power series that appears in (31). The following theorem contains an answer.†

Frobenius's Theorem

Theorem 6. If x_0 is a regular singular point of equation (29), then there exists at least one series solution of the form (30), where $r = r_1$ is the larger root of the associated indicial equation. Moreover, this series converges for all x such that $0 < x - x_0 < R$, where R is the distance from x_0 to the nearest other singular point (real or complex) of (29).

For simplicity, in the examples that follow we consider only series expansions about the regular singular point $x = 0$ and only those equations for which the associated indicial equation has real roots.

The following three examples not only illustrate the method of Frobenius but also are important models to which we refer in later sections.

Example 4 Find a series solution about the regular singular point $x = 0$ of

$$x^2y''(x) - xy'(x) + (1 - x)y(x) = 0\,, \qquad x > 0\,. \tag{32}$$

Solution Here $p(x) = -x^{-1}$ and $q(x) = (1 - x)x^{-2}$. It is easy to check that $x = 0$ is a regular singular point of (32), so we compute

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0}(-1) = -1\,,$$
$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0}(1 - x) = 1\,.$$

Then the indicial equation is

$$r(r-1) - r + 1 = r^2 - 2r + 1 = (r-1)^2 = 0\,,$$

which has the roots $r_1 = r_2 = 1$.

Next we substitute

$$w(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{33}$$

into (32) and obtain

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} + (1-x)\sum_{n=0}^{\infty} a_n x^{n+r} = 0\,, \tag{34}$$

which we write as

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0\,. \tag{35}$$

†For a proof of this theorem, see *Ordinary Differential Equations,* by E. L. Ince (Dover Publications, New York, 1956), Chapter 16.

Shifting the indices so that each summation in (35) is in powers x^{k+r}, we take $k = n + 1$ in the last summation and $k = n$ in the rest. This gives

$$\sum_{k=0}^{\infty}[(k+r)(k+r-1)-(k+r)+1]a_k x^{k+r} - \sum_{k=1}^{\infty} a_{k-1}x^{k+r} = 0 . \tag{36}$$

Singling out the term corresponding to $k = 0$ and combining the rest under one summation yields

$$\begin{aligned}&[r(r-1)-r+1]a_0 x^r \\ &\quad + \sum_{k=1}^{\infty}\{[(k+r)(k+r-1)-(k+r)+1]a_k - a_{k-1}\}x^{k+r} = 0 .\end{aligned} \tag{37}$$

When we set the coefficients equal to zero, we recover the indicial equation

$$[r(r-1)-r+1]a_0 = 0 , \tag{38}$$

and obtain, for $k \geq 1$, the recurrence relation

$$[(k+r)^2 - 2(k+r) + 1]a_k - a_{k-1} = 0 , \tag{39}$$

which reduces to

$$(k+r-1)^2 a_k - a_{k-1} = 0 . \tag{40}$$

Relation (40) can be used to solve for a_k in terms of a_{k-1}:

$$a_k = \frac{1}{(k+r-1)^2}a_{k-1} , \qquad k \geq 1 . \tag{41}$$

Setting $r = r_1 = 1$ in (38) gives (as expected) $0 \cdot a_0 = 0$, and in (41) gives

$$\boldsymbol{a_k = \frac{1}{k^2}a_{k-1} , \qquad k \geq 1 .} \tag{42}$$

For $k = 1$, 2, and 3, we now find

$$\begin{aligned}
a_1 &= \frac{1}{1^2}a_0 = a_0 && (k = 1) ,\\
a_2 &= \frac{1}{2^2}a_1 = \frac{1}{(2 \cdot 1)^2}a_0 = \frac{1}{4}a_0 && (k = 2) ,\\
a_3 &= \frac{1}{3^2}a_2 = \frac{1}{(3 \cdot 2 \cdot 1)^2}a_0 = \frac{1}{36}a_0 && (k = 3) .
\end{aligned}$$

In general, we have

$$a_k = \frac{1}{(k!)^2}a_0 . \tag{43}$$

Hence, equation (32) has a series solution given by

$$\begin{aligned} w(1, x) &= a_0 x\left\{1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \cdots\right\} \\ &= a_0 x \sum_{k=0}^{\infty}\frac{1}{(k!)^2}x^k , \qquad x > 0 . \ \blacklozenge \end{aligned} \tag{44}$$

Since $x = 0$ is the only singular point for equation (32), it follows from Frobenius's theorem or directly by the ratio test that the series solution (44) converges for all $x > 0$.

In the next two examples, we only outline the method; we leave it to you to furnish the intermediate steps.

Example 5 Find a series solution about the regular singular point $x = 0$ of

$$xy''(x) + 4y'(x) - xy(x) = 0\,, \qquad x > 0\,. \tag{45}$$

Solution Since $p(x) = 4/x$ and $q(x) = -1$, we see that $x = 0$ is indeed a regular singular point and

$$p_0 = \lim_{x \to 0} xp(x) = 4\,, \qquad q_0 = \lim_{x \to 0} x^2 q(x) = 0\,.$$

The indicial equation is

$$r(r-1) + 4r = r^2 + 3r = r(r+3) = 0\,,$$

with roots $r_1 = 0$ and $r_2 = -3$.

Now we substitute

$$w(r,x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{46}$$

into (45). After a little algebra and a shift in indices, we get

$$\begin{aligned} &[r(r-1) + 4r]a_0 x^{r-1} + [(r+1)r + 4(r+1)]a_1 x^r \\ &\quad + \sum_{k=1}^{\infty} [(k+r+1)(k+r+4)a_{k+1} - a_{k-1}]x^{k+r} = 0\,. \end{aligned} \tag{47}$$

Next we set the coefficients equal to zero and find

$$[r(r-1) + 4r]a_0 = 0\,, \tag{48}$$

$$[(r+1)r + 4(r+1)]a_1 = 0\,, \tag{49}$$

and, for $k \geq 1$, the recurrence relation

$$(k+r+1)(k+r+4)a_{k+1} - a_{k-1} = 0\,. \tag{50}$$

For $r = r_1 = 0$, equation (48) becomes $0 \cdot a_0 = 0$ and (49) becomes $4 \cdot a_1 = 0$. Hence, although a_0 is arbitrary, a_1 must be zero. Setting $r = r_1 = 0$ in (50), we find

$$\boldsymbol{a_{k+1} = \frac{1}{(k+1)(k+4)} a_{k-1}\,, \qquad k \geq 1\,,} \tag{51}$$

from which it follows (after a few experimental computations) that $a_{2k+1} = 0$ for $k = 0, 1, \ldots,$ and

$$\begin{aligned} a_{2k} &= \frac{1}{[2 \cdot 4 \cdots (2k)][5 \cdot 7 \cdots (2k+3)]} a_0 \\ &= \frac{1}{2^k k! [5 \cdot 7 \cdots (2k+3)]} a_0\,, \qquad k \geq 1\,. \end{aligned} \tag{52}$$

Hence equation (45) has the power series solution

$$w(0,x) = a_0 \left\{ 1 + \sum_{k=1}^{\infty} \frac{1}{2^k k! [5 \cdot 7 \cdots (2k+3)]} x^{2k} \right\}, \qquad x > 0\,. \ \blacklozenge \tag{53}$$

If in Example 5 we had worked with the root $r = r_2 = -3$, then we would actually have obtained *two* linearly independent solutions (see Problem 45).

Example 6 Find a series solution about the regular singular point of $x = 0$ of

$$xy''(x) + 3y'(x) - xy(x) = 0, \qquad x > 0. \tag{54}$$

Solution Since $p(x) = 3/x$ and $q(x) = -1$, we see that $x = 0$ is a regular singular point. Moreover,

$$p_0 = \lim_{x \to 0} xp(x) = 3, \qquad q_0 = \lim_{x \to 0} x^2 q(x) = 0.$$

So the indicial equation is

$$r(r-1) + 3r = r^2 + 2r = r(r+2) = 0, \tag{55}$$

with roots $r_1 = 0$ and $r_2 = -2$.

Substituting

$$w(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{56}$$

into (54) ultimately gives

$$\begin{aligned}[r(r-1) + 3r]a_0 x^{r-1} &+ [(r+1)r + 3(r+1)]a_1 x^r \\ &+ \sum_{k=1}^{\infty} [(k+r+1)(k+r+3)a_{k+1} - a_{k-1}]x^{k+r} = 0.\end{aligned} \tag{57}$$

Setting the coefficients equal to zero, we have

$$[r(r-1) + 3r]a_0 = 0, \tag{58}$$

$$[(r+1)r + 3(r+1)]a_1 = 0, \tag{59}$$

and, for $k \geq 1$, the recurrence relation

$$\boldsymbol{(k + r + 1)(k + r + 3)a_{k+1} - a_{k-1} = 0.} \tag{60}$$

With $r = r_1 = 0$, these equations lead to the following formulas: $a_{2k+1} = 0$, $k = 0, 1, \ldots,$ and

$$a_{2k} = \frac{1}{[2 \cdot 4 \cdots (2k)][4 \cdot 6 \cdots (2k+2)]} a_0 = \frac{1}{2^{2k} k!(k+1)!} a_0, \qquad k \geq 0. \tag{61}$$

Hence equation (54) has the power series solution

$$w(0, x) = a_0 \sum_{k=0}^{\infty} \frac{1}{2^{2k} k!(k+1)!} x^{2k}, \qquad x > 0. \; \blacklozenge \tag{62}$$

Unlike in Example 5, if we work with the second root $r = r_2 = -2$ in Example 6, then we do *not* obtain a second linearly independent solution (see Problem 46).

In the preceding examples we were able to use the method of Frobenius to find a series solution valid to the right ($x > 0$) of the regular singular point $x = 0$. For $x < 0$, we can use the change of variables $x = -t$ and then solve the resulting equation for $t > 0$.

The method of Frobenius also applies to higher-order linear equations (see Problems 35–38).

8.6 EXERCISES

In Problems 1–10, classify each singular point (real or complex) of the given equation as regular or irregular.

1. $(x^2-1)y''+xy'+3y=0$
2. $x^2y''+8xy'-3xy=0$
3. $(x^2+1)z''+7x^2z'-3xz=0$
4. $x^2y''-5xy'+7y=0$
5. $(x^2-1)^2y''-(x-1)y'+3y=0$
6. $(x^2-4)y''+(x+2)y'+3y=0$
7. $(t^2-t-2)^2x''+(t^2-4)x'-tx=0$
8. $(x^2-x)y''+xy'+7y=0$
9. $(x^2+2x-8)^2y''+(3x+12)y'-x^2y=0$
10. $x^3(x-1)y''+(x^2-3x)(\sin x)y'-xy=0$

In Problems 11–18, find the indicial equation and the exponents for the specified singularity of the given differential equations.

11. $x^2y''-2xy'-10y=0\,,\quad \text{at}\quad x=0$
12. $x^2y''+4xy'+2y=0\,,\quad \text{at}\quad x=0$
13. $(x^2-x-2)^2z''+(x^2-4)z'-6xz=0\,,\quad \text{at}\quad x=2$
14. $(x^2-4)y''+(x+2)y'+3y=0\,,\quad \text{at}\quad x=-2$
15. $\theta^3y''+\theta(\sin\theta)y'-(\tan\theta)y=0\,,\quad \text{at}\quad \theta=0$
16. $(x^2-1)y''-(x-1)y'-3y=0\,,\quad \text{at}\quad x=1$
17. $(x-1)^2y''+(x^2-1)y'-12y=0\,,\quad \text{at}\quad x=1$
18. $4x(\sin x)y''-3y=0\,,\quad \text{at}\quad x=0$

In Problems 19–24, use the method of Frobenius to find at least the first four nonzero terms in the series expansion about $x=0$ for a solution to the given equation for $x>0$.

19. $9x^2y''+9x^2y'+2y=0$
20. $2x(x-1)y''+3(x-1)y'-y=0$
21. $x^2y''+xy'+x^2y=0$
22. $xy''+y'-4y=0$
23. $x^2z''+(x^2+x)z'-z=0$
24. $3xy''+(2-x)y'-y=0$

In Problems 25–30, use the method of Frobenius to find a general formula for the coefficient a_n in a series expansion about $x=0$ for a solution to the given equation for $x>0$.

25. $4x^2y''+2x^2y'-(x+3)y=0$
26. $x^2y''+(x^2-x)y'+y=0$
27. $xw''-w'-xw=0$
28. $3x^2y''+8xy'+(x-2)y=0$
29. $xy''+(x-1)y'-2y=0$
30. $x(x+1)y''+(x+5)y'-4y=0$

In Problems 31–34, first determine a recurrence formula for the coefficients in the (Frobenius) series expansion of the solution about $x=0$. Use this recurrence formula to determine if there exists a solution to the differential equation that is decreasing for $x>0$.

31. $xy''+(1-x)y'-y=0$
32. $x^2y''-x(1+x)y'+y=0$
33. $3xy''+2(1-x)y'-4y=0$
34. $xy''+(x+2)y'-y=0$

In Problems 35–38, use the method of Frobenius to find at least the first four nonzero terms in the series expansion about $x=0$ for a solution to the given linear third-order equation for $x>0$.

35. $6x^3y'''+13x^2y''+(x+x^2)y'+xy=0$
36. $6x^3y'''+11x^2y''-2xy'-(x-2)y=0$
37. $6x^3y'''+13x^2y''-(x^2+3x)y'-xy=0$
38. $6x^3y'''+(13x^2-x^3)y''+xy'-xy=0$

In Problems 39 and 40, try to use the method of Frobenius to find a series expansion about the irregular singular point $x=0$ for a solution to the given differential equation. If the method works, give at least the first four nonzero terms in the expansion. If the method does not work, explain what went wrong.

39. $x^2y''+(3x-1)y'+y=0$
40. $x^2y''+y'-2y=0$

*In certain applications, it is desirable to have **an expansion about the point at infinity**. To obtain such an expansion, we use the change of variables $z=1/x$ and expand about $z=0$. In Problems 41 and 42, show that infinity is a regular singular point of the given differential equation by showing that $z=0$ is a regular singular point for the transformed equation in z. Also find at least the first four nonzero terms in the series expansion about infinity of a solution to the original equation in x.*

41. $x^3y''-x^2y'-y=0$
42. $18(x-4)^2(x-6)y''+9x(x-4)y'-32y=0$

43. Show that if r_1 and r_2 are roots of the indicial equation (16) on page 456, with r_1 the larger root ($\operatorname{Re} r_1\ge \operatorname{Re} r_2$), then the coefficient of a_1 in equation (19) on page 457 is not zero when $r=r_1$.
44. To obtain a second linearly independent solution to equation (20):
    (a) Substitute $w(r,x)$ given in (21) into (20) and conclude that the coefficients a_k, $k\ge 1$, must satisfy the recurrence relation
$$(k+r-1)(2k+2r-1)a_k$$
$$+[(k+r-1)(k+r-2)+1]a_{k-1}=0\,.$$

(b) Use the recurrence relation with $r = 1/2$ to derive the second series solution

$$w\left(\frac{1}{2}, x\right) = a_0\left(x^{1/2} - \frac{3}{4}x^{3/2} + \frac{7}{32}x^{5/2} - \frac{133}{1920}x^{7/2} + \cdots\right).$$

(c) Use the recurrence relation with $r = 1$ to obtain $w(1, x)$ in (28) on page 458.

45. In Example 5, show that if we choose $r = r_2 = -3$, then we obtain *two* linearly independent solutions to equation (45). [*Hint:* a_0 and a_3 are arbitrary constants.]

46. In Example 6, page 463, show that if we choose $r = r_2 = -2$, then we obtain a solution that is a constant multiple of the solution given in (62). [*Hint:* Show that a_0 and a_1 must be zero while a_2 is arbitrary.]

47. In applying the method of Frobenius, the following recurrence relation arose: $a_{k+1} = 15^7 a_k/(k+1)^9$, $k = 0, 1, 2, \ldots$.

(a) Show that the coefficients are given by the formula $a_k = 15^{7k} a_0/(k!)^9$, $k = 0, 1, 2, \ldots$.

(b) Use the formula obtained in part (a) with $a_0 = 1$ to compute a_5, a_{10}, a_{15}, a_{20}, and a_{25} on your computer or calculator. What goes wrong?

(c) Now use the recurrence relation to compute a_k for $k = 1, 2, 3, \ldots, 25$, assuming $a_0 = 1$.

(d) What advantage does the recurrence relation have over the formula?

8.7 Finding a Second Linearly Independent Solution

In the previous section we showed that if $x = 0$ is a regular singular point of

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0\,, \qquad x > 0\,, \tag{1}$$

then the method of Frobenius can be used to find a series solution valid for x near zero. The first step in the method is to find the roots r_1 and r_2 $(\operatorname{Re} r_1 \geq \operatorname{Re} r_2)$ of the associated indicial equation

$$r(r-1) + p_0 r + q_0 = 0\,. \tag{2}$$

Then, utilizing the larger root r_1, equation (1) has a series solution of the form

$$w(r_1, x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r_1}\,, \tag{3}$$

where $a_0 \neq 0$. To find a second linearly independent solution, our first inclination is to set $r = r_2$ and seek a solution of the form

$$w(r_2, x) = x^{r_2}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r_2}\,. \tag{4}$$

We'll see that this procedure works, *provided* $r_1 - r_2$ *is not an integer.* However, when $r_1 - r_2$ *is* an integer, the Frobenius method with $r = r_2$ may just lead to the same solution that we obtained using the root r_1. (This is obviously true when $r_1 = r_2$.)

Example 1 Find the first few terms in the series expansion about the regular singular point $x = 0$ for a general solution to

$$(x+2)x^2 y''(x) - xy'(x) + (1+x)y(x) = 0\,, \qquad x > 0\,. \tag{5}$$

Solution In Example 3 of Section 8.6, we used the method of Frobenius to find a series solution for (5). In the process we determined that $p_0 = -1/2$, $q_0 = 1/2$ and that the indicial equation

CHAPTER

9

Matrix Methods for Linear Systems

9.1 Introduction

In this chapter we return to the analysis of systems of differential equations. When the equations in the system are *linear*, matrix algebra provides a compact notation for expressing the system. In fact, the notation itself suggests new and elegant ways of characterizing the solution properties, as well as novel, efficient techniques for explicitly obtaining solutions.

In Chapter 5 we analyzed physical situations wherein two fluid tanks containing brine solutions were interconnected and pumped so as ultimately to deplete the salt content in each tank. By accounting for the various influxes and outfluxes of brine, a system of differential equations for the salt contents $(x(t)$ and $y(t))$ of each tank was derived; a typical model is

(1)
$$\begin{aligned} dx/dt &= -4x + 2y\ , \\ dy/dt &= 4x - 4y\ . \end{aligned}$$

Express this system in matrix notation as a single equation.

The right-hand side of the first member of (1) possesses a mathematical structure that is familiar from vector calculus; namely, it is the **dot product**† of two vectors:

(2)
$$-4x + 2y = [-4 \quad 2] \cdot [x \quad y]\ .$$

Similarly, the second right-hand side in (1) is the dot product

$$4x - 4y = [4 \quad -4] \cdot [x \quad y]\ .$$

The frequent occurrence in mathematics of arrays of dot products, such as evidenced in the system (1), led to the development of **matrix algebra**, a mathematical discipline whose basic operation—the matrix product—is the arrangement of a set of dot products according to the following plan:

$$\begin{bmatrix} -4 & 2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} [-4 \quad 2] \cdot [x \quad y] \\ [4 \quad -4] \cdot [x \quad y] \end{bmatrix} = \begin{bmatrix} -4x + 2y \\ 4x - 4y \end{bmatrix}.$$

In general, the product of a **matrix**—i.e., an m by n rectangular array of numbers—and a *column vector* is defined to be the collection of dot products of the *rows* of the matrix with the

†Recall that the dot product of two vectors **u** and **v** equals the length of **u** times the length of **v** times the cosine of the angle between **u** and **v**. However, it is more conveniently computed from the *components* of **u** and **v** by the "inner product" indicated in equation (2).

vector, arranged as a column vector:

$$\begin{bmatrix} \textbf{row \# 1} \\ \textbf{row \# 2} \\ \vdots \\ \textbf{row \# m} \end{bmatrix} \begin{bmatrix} \\ \mathbf{v} \\ \\ \end{bmatrix} = \begin{bmatrix} [\textbf{row \# 1}] \cdot \mathbf{v} \\ [\textbf{row \# 2}] \cdot \mathbf{v} \\ \vdots \\ [\textbf{row \# m}] \cdot \mathbf{v} \end{bmatrix},$$

where the vector $\mathbf{v}$ has n components; the dot product of two n-dimensional vectors is computed in the obvious way:

$$[a_1 \quad a_2 \ \cdots \ a_n] \cdot [x_1 \quad x_2 \ \cdots \ x_n] = a_1x_1 + a_2x_2 + \cdots + a_nx_n .$$

Using the notation for the matrix product, we can write the system (1) for the interconnected tanks as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The following example demonstrates a four-dimensional implementation of this notation. Note that the coefficients in the linear system need not be constants.

Example 1 Express the system

$$\begin{aligned} x_1' &= 2x_1 + t^2x_2 + (4t + e^t)x_4 , \\ x_2' &= (\sin t)x_2 + (\cos t)x_3 , \\ x_3' &= x_1 + x_2 + x_3 + x_4 , \\ x_4' &= 0 \end{aligned} \tag{3}$$

as a matrix equation.

Solution We express the right-hand side of the first member of (3) as the dot product

$$2x_1 + t^2x_2 + (4t + e^t)x_4 = [2 \quad t^2 \quad 0 \quad (4t + e^t)] \cdot [x_1 \quad x_2 \quad x_3 \quad x_4] .$$

The other dot products are similarly identified, and the matrix form is given by

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 2 & t^2 & 0 & (4t + e^t) \\ 0 & \sin t & \cos t & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \quad \blacklozenge$$

In general, if a system of differential equations is expressed as

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n \\ &\vdots \\ x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n , \end{aligned}$$

it is said to be a linear homogeneous system in **normal form**.† The matrix formulation of such

†The normal form was defined for general systems in Section 5.3, page 256.

a system is then

$$\mathbf{x}' = \mathbf{A}\mathbf{x}\,,$$

where $\mathbf{A}$ is the **coefficient matrix**

$$\mathbf{A} = \mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

and $\mathbf{x}$ is the solution vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Note that we have used $\mathbf{x}'$ to denote the vector of derivatives

$$\mathbf{x}' = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}.$$

Example 2 Express the differential equation for the undamped, unforced mass–spring oscillator (recall Section 4.1, page 152)

(4) $$my'' + ky = 0$$

as an equivalent system of first-order equations in normal form, expressed in matrix notation.

Solution We have to express the *second* derivative, y'', as a *first* derivative in order to formulate (4) as a first-order system. This is easy; the acceleration y'' is the derivative of the *velocity* $v = y'$, so (4) becomes

(5) $$mv' + ky = 0\,.$$

The first-order system is then assembled by identifying v with y', and appending it to (5):

$$\begin{aligned} y' &= v \\ mv' &= -ky\,. \end{aligned}$$

To put this system in normal form and express it as a matrix equation, we need to divide the second equation by the mass m:

$$\begin{bmatrix} y \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}. \quad \blacklozenge$$

In general, the customary way to write an nth-order linear homogeneous differential equation

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = 0$$

as an equivalent system in normal form is to define the first $(n-1)$ derivatives of y (including y, the zeroth derivative, itself) to be new unknowns:

$$\begin{aligned} x_1(t) &= y(t) , \\ x_2(t) &= y'(t) , \\ &\vdots \\ x_n(t) &= y^{(n-1)}(t) . \end{aligned}$$

Then the system consists of the identification of $x_j(t)$ as the derivative of $x_{j-1}(t)$, together with the original differential equation expressed in these variables (and divided by $a_n(t)$):

$$\begin{aligned} x_1' &= x_2 , \\ x_2' &= x_3 , \\ &\vdots \\ x_{n-1}' &= x_n , \\ x_n' &= -\frac{a_0(t)}{a_n(t)}x_1 - \frac{a_1(t)}{a_n(t)}x_2 - \cdots - \frac{a_{n-1}(t)}{a_n(t)}x_n . \end{aligned}$$

For systems of two or more higher-order differential equations, the same procedure is applied to each unknown function in turn; an example will make this clear.

Example 3 The coupled mass–spring oscillator depicted in Figure 5.26 on page 283 was shown to be governed by the system

$$\begin{aligned} 2\frac{d^2x}{dt^2} + 6x - 2y &= 0 , \\ \frac{d^2y}{dt^2} + 2y - 2x &= 0 . \end{aligned} \tag{6}$$

Write (6) in matrix notation.

Solution We introduce notation for the lower-order derivatives:

$$x_1 = x , \qquad x_2 = x' , \qquad x_3 = y , \qquad x_4 = y' . \tag{7}$$

In these variables, the system (6) states

$$\begin{aligned} 2x_2' + 6x_1 - 2x_3 &= 0 , \\ x_4' + 2x_3 - 2x_1 &= 0 . \end{aligned} \tag{8}$$

The normal form is then

$$\begin{aligned} x_1' &= x_2 , \\ x_2' &= -3x_1 + x_3 , \\ x_3' &= x_4 , \\ x_4' &= 2x_1 - 2x_3 \end{aligned}$$

or in matrix notation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} . \quad ◆$$

9.1 EXERCISES

In Problems 1–6, express the given system of differential equations in matrix notation.

1. $x' = 7x + 2y\,,$
$y' = 3x - 2y$

2. $x' = y\,,$
$y' = -x$

3. $x' = x + y + z\,,$
$y' = 2z - x\,,$
$z' = 4y$

4. $x_1' = x_1 - x_2 + x_3 - x_4\,,$
$x_2' = x_1 + x_4\,,$
$x_3' = \sqrt{\pi}x_1 - x_3\,,$
$x_4' = 0$

5. $x' = (\sin t)x + e^t y\,,$
$y' = (\cos t)x + (a + bt^3)y$

6. $x_1' = (\cos 2t)x_1\,,$
$x_2' = (\sin 2t)x_2\,,$
$x_3' = x_1 - x_2$

In Problems 7–10, express the given higher-order differential equation as a matrix system in normal form.

7. The damped mass–spring oscillator equation $my'' + by' + ky = 0$

8. Legendre's equation $(1 - t^2)y'' - 2ty' + 2y = 0$

9. The Airy equation $y'' - ty = 0$

10. Bessel's equation $y'' + \dfrac{1}{t}y' + \left(1 - \dfrac{n^2}{t^2}\right)y = 0$

In Problems 11–13, express the given system of higher-order differential equations as a matrix system in normal form.

11. $x'' + 3x + 2y = 0\,,$
$y'' - 2x = 0$

12. $x'' + 3x' - y' + 2y = 0\,,$
$y'' + x' + 3y' + y = 0$

13. $x'' - 3x' + t^2 y - (\cos t)x = 0\,,$
$y''' + y'' - tx' + y' + e^t x = 0$

9.2 Review 1: Linear Algebraic Equations

Here and in the next section we review some basic facts concerning linear algebraic systems and matrix algebra that will be useful in solving linear systems of differential equations in normal form. Readers competent in these areas may proceed to Section 9.4.

A set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\,, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\,, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

(where the a_{ij}'s and b_i's are given constants) is called a *linear system of n algebraic equations* in the n unknowns $x_1, x_2, \ldots, x_n$. The procedure for solving the system using elimination methods is well known. Herein we describe a particularly convenient implementation of the method called the *Gauss–Jordan elimination algorithm.*[†] The basic idea of this formulation is to use the first equation to eliminate x_1 in all the other equations; then use the second equation to eliminate x_2 in all the others; and so on. If all goes well, the resulting system will be "uncoupled," and the values of the unknowns $x_1, x_2, \ldots, x_n$ will be apparent. A short example will make this clear.

[†]The Gauss–Jordan algorithm is neither the fastest nor the most accurate computer algorithm for solving a linear system of algebraic equations, but for solutions executed by hand it has many pedagogical advantages. Usually it is much faster than *Cramer's rule*, described in Appendix D.

Example 1 Solve the system

$$\begin{aligned} 2x_1 + \ \ 6x_2 + \ \ 8x_3 &= 16\,, \\ 4x_1 + 15x_2 + 19x_3 &= 38\,, \\ 2x_1 \qquad\quad + \ \ 3x_3 &= 6\,. \end{aligned}$$

Solution By subtracting 2 times the first equation from the second, we eliminate x_1 from the latter. Similarly, x_1 is eliminated from the third equation by subtracting 1 times the first equation from it:

$$\begin{aligned} 2x_1 + 6x_2 + 8x_3 &= 16\,, \\ 3x_2 + 3x_3 &= 6\,, \\ -6x_2 - 5x_3 &= -10\,. \end{aligned}$$

Next we subtract multiples of the second equation from the first and third to eliminate x_2 in them; the appropriate multiples are 2 and -2, respectively:

$$\begin{aligned} 2x_1 \qquad + 2x_3 &= 4\,, \\ 3x_2 + 3x_3 &= 6\,, \\ x_3 &= 2\,. \end{aligned}$$

Finally, we eliminate x_3 from the first two equations by subtracting multiples (2 and 3, respectively) of the third equation:

$$\begin{aligned} 2x_1 \qquad\qquad &= 0\,, \\ 3x_2 \qquad &= 0\,, \\ x_3 &= 2\,. \end{aligned}$$

The system is now uncoupled; i.e., we can solve each equation separately:

$$x_1 = 0\,, \quad x_2 = 0\,, \quad x_3 = 2\,. \ \blacklozenge$$

Two complications can disrupt the straightforward execution of the Gauss–Jordan algorithm. The first occurs when the impending variable to be eliminated (say, x_j) does not occur in the jth equation. The solution is usually obvious; we employ one of the subsequent equations to eliminate x_j. Example 2 illustrates this maneuver.

Example 2 Solve the system

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + \ \ x_4 &= 0\,, \\ -x_1 - 2x_2 - 2x_3 \qquad\quad &= 1\,, \\ -2x_1 - 4x_2 - 8x_3 + 2x_4 &= 4\,, \\ x_1 + 4x_2 + 2x_3 \qquad\quad &= -3\,. \end{aligned}$$

Solution The first unknown x_1 is eliminated from the last three equations by subtracting multiples of the first equation:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 &= 0\,, \\ 2x_3 + x_4 &= 1\,, \\ 4x_4 &= 4\,, \\ 2x_2 - 2x_3 - x_4 &= -3\,. \end{aligned}$$

Now, we cannot use the second equation to eliminate the second unknown because x_2 is not present. The next equation that *does* contain x_2 is the fourth, so we switch the second and fourth equation:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 &= 0\,, \\ 2x_2 - 2x_3 - x_4 &= -3\,, \\ 4x_4 &= 4\,, \\ 2x_3 + x_4 &= 1\,, \end{aligned}$$

and proceed to eliminate x_2:

$$\begin{aligned} x_1 \qquad + 6x_3 + 2x_4 &= 3\,, \\ 2x_2 - 2x_3 - x_4 &= -3\,, \\ 4x_4 &= 4\,, \\ 2x_3 + x_4 &= 1\,. \end{aligned}$$

To eliminate x_3, we have to switch again,

$$\begin{aligned} x_1 \qquad + 6x_3 + 2x_4 &= 3\,, \\ 2x_2 - 2x_3 - x_4 &= -3\,, \\ 2x_3 + x_4 &= 1\,, \\ 4x_4 &= 4\,, \end{aligned}$$

and eliminate, in turn, x_3 and x_4. This gives

$$\begin{aligned} x_1 \qquad\qquad - x_4 &= 0\,, \\ 2x_2 \qquad\qquad &= -2\,, \\ 2x_3 + x_4 &= 1\,, \\ 4x_4 &= 4\,, \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 \qquad\qquad\quad &= 1\,, \\ 2x_2 \qquad\quad &= -2\,, \\ 2x_3 \quad &= 0\,, \\ 4x_4 &= 4\,. \end{aligned}$$

The solution to the uncoupled equations is

$$x_1 = 1\,, \quad x_2 = -1\,, \quad x_3 = 0\,, \quad x_4 = 1\,. \quad ◆$$

The other complication that can disrupt the Gauss–Jordan algorithm is much more profound. What if, when we are "scheduled" to eliminate the unknown x_j, it is absent from *all* of the subsequent equations? The first thing to do is to move on to the elimination of the *next* unknown x_{j+1}, as demonstrated in Example 3.

Example 3 Apply the Gauss–Jordan algorithm to the system

$$\begin{aligned} 2x_1 + 4x_2 + x_3 &= 8\,, \\ 2x_1 + 4x_2 \qquad &= 6\,, \\ -4x_1 - 8x_2 + x_3 &= -10\,. \end{aligned} \tag{1}$$

Solution Elimination of x_1 proceeds as usual:

$$\begin{aligned} 2x_1 + 4x_2 + x_3 &= 8\,, \\ -x_3 &= -2\,, \\ 3x_3 &= 6\,. \end{aligned}$$

Now since x_2 is absent from the second *and* third equations, we use the second equation to eliminate x_3 :

$$\begin{aligned} 2x_1 + 4x_2 \qquad &= 6\,, \\ -x_3 &= -2\,, \\ 0 &= 0\,. \end{aligned} \tag{2}$$

How do we interpret the system (2)? The final equation contains no information, of course, and we ignore it.[†] The second equation implies that $x_3 = 2$.

The first equation implies that $x_1 = 3 - 2x_2$, but there is no equation for x_2. Evidently, x_2 is a "free" variable, and we can assign *any* value to it—as long as we take x_1 to be $3 - 2x_2$. Thus (1) has an infinite number of solutions, and a convenient way of characterizing them is

$$x_1 = 3 - 2s\,, \quad x_2 = s\,, \quad x_3 = 2\,; \quad -\infty < s < \infty\,.$$

We remark that an equivalent solution can be obtained by treating x_1 as the free variable, say $x_1 = s$, and taking $x_2 = (3 - s)/2$, $x_3 = 2$. ◆

The final example is contrived to demonstrate all the features that we have encountered.

Example 4 Find all solutions to the system

$$\begin{aligned} x_1 - x_2 + 2x_3 + 2x_4 &= 0\,, \\ 2x_1 - 2x_2 + 4x_3 + 3x_4 &= 1\,, \\ 3x_1 - 3x_2 + 6x_3 + 9x_4 &= -3\,, \\ 4x_1 - 4x_2 + 8x_3 + 8x_4 &= 0\,. \end{aligned}$$

Solution We use the first equation to eliminate x_1:

$$\begin{aligned} x_1 - x_2 + 2x_3 + 2x_4 &= 0\,, \\ -x_4 &= 1\,, \\ 3x_4 &= -3\,, \\ 0 &= 0\,. \end{aligned}$$

Now, both x_2 *and* x_3 are absent from all subsequent equations, so we use the second equation to eliminate x_4.

$$\begin{aligned} x_1 - x_2 + 2x_3 \qquad &= 2\,, \\ -x_4 &= 1\,, \\ 0 &= 0\,, \\ 0 &= 0\,. \end{aligned}$$

There are no constraints on either x_2 or x_3; thus we take them to be free variables and characterize the solutions by

$$x_1 = 2 + s - 2t\,, \quad x_2 = s\,, \quad x_3 = t\,, \quad x_4 = -1, \quad -\infty < s, t < \infty\,. \quad ◆$$

In closing, we note that if the execution of the Gauss–Jordan algorithm results in a display of the form $0 = 1$ (or $0 = k$, where $k \neq 0$), the original system has no solutions; it is *inconsistent.* This is explored in Problem 12.

[†]The occurrence of the identity $0 = 0$ in the Gauss–Jordan algorithm implies that one of the original equations was *redundant.* In this case you may observe that the final equation in (1) can be derived by subtracting 3 times the second equation from the first.

9.2 EXERCISES

In Problems 1–11, find all solutions to the system using the Gauss–Jordan elimination algorithm.

1. $$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 6\,, \\ 2x_1 + x_2 + x_3 &= 6\,, \\ x_1 + x_2 + 3x_3 &= 6 \end{aligned}$$

2. $$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1\,, \\ x_1 \qquad\qquad + x_4 &= 0\,, \\ 2x_1 + 2x_2 - x_3 + x_4 &= 0\,, \\ x_1 + 2x_2 - x_3 + x_4 &= 0 \end{aligned}$$

3. $$\begin{aligned} x_1 + x_2 - x_3 &= 0\,, \\ -x_1 - x_2 + x_3 &= 0\,, \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$$

4. $$\begin{aligned} x_3 + x_4 &= 0\,, \\ x_1 + x_2 + x_3 + x_4 &= 1\,, \\ 2x_1 - x_2 + x_3 + 2x_4 &= 0\,, \\ 2x_1 - x_2 + x_3 + x_4 &= 0 \end{aligned}$$

5. $$\begin{aligned} -x_1 + 2x_2 &= 0\,, \\ 2x_1 + 3x_2 &= 0 \end{aligned}$$

6. $$\begin{aligned} -2x_1 + 2x_2 - x_3 &= 0\,, \\ x_1 - 3x_2 + x_3 &= 0\,, \\ 4x_1 - 4x_2 + 2x_3 &= 0 \end{aligned}$$

7. $$\begin{aligned} -x_1 + 3x_2 &= 0\,, \\ -3x_1 + 9x_2 &= 0 \end{aligned}$$

8. $$\begin{aligned} x_1 + 2x_2 + x_3 &= -3\,, \\ 2x_1 + 4x_2 - x_3 &= 0\,, \\ x_1 + 3x_2 - 2x_3 &= 3 \end{aligned}$$

9. $$\begin{aligned} (1-i)x_1 + 2x_2 &= 0\,, \\ -x_1 - (1+i)x_2 &= 0 \end{aligned}$$

10. $$\begin{aligned} x_1 + x_2 + x_3 &= i\,, \\ 2x_1 + 3x_2 - ix_3 &= 0\,, \\ x_1 + 2x_2 + x_3 &= i \end{aligned}$$

11. $$\begin{aligned} 2x_1 \qquad + x_3 &= -1\,, \\ -3x_1 + x_2 + 4x_3 &= 1\,, \\ -x_1 + x_2 + 5x_3 &= 0 \end{aligned}$$

12. Use the Gauss–Jordan elimination algorithm to show that the following systems of equations are inconsistent. That is, demonstrate that the existence of a solution would imply a mathematical contradiction.

(a) $$\begin{aligned} 2x_1 - x_2 &= 2\,, \\ -6x_1 + 3x_2 &= 4 \end{aligned}$$

(b) $$\begin{aligned} 2x_1 \qquad + x_3 &= -1\,, \\ -3x_1 + x_2 + 4x_3 &= 1\,, \\ -x_1 + x_2 + 5x_3 &= 1 \end{aligned}$$

13. Use the Gauss–Jordan elimination algorithm to show that the following system of equations has a unique solution for $r = 2$, but an infinite number of solutions for $r = 1$.

$$\begin{aligned} 2x_1 - 3x_2 &= rx_1\,, \\ x_1 - 2x_2 &= rx_2 \end{aligned}$$

14. Use the Gauss–Jordan elimination algorithm to show that the following system of equations has a unique solution for $r = -1$, but an infinite number of solutions for $r = 2$.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= rx_1\,, \\ x_1 \qquad + x_3 &= rx_2\,, \\ 4x_1 - 4x_2 + 5x_3 &= rx_3 \end{aligned}$$

9.3 Review 2: Matrices and Vectors

A **matrix** is a rectangular array of numbers arranged in rows and columns. An $m \times n$ matrix—that is, a matrix with m rows and n columns—is usually denoted by

$$\mathbf{A} := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix},$$

where the element in the ith row and jth column is a_{ij}. The notation $[a_{ij}]$ is also used to designate $\mathbf{A}$. The matrices we will work with usually consist of real numbers, but in certain instances we allow complex-number entries.

Some matrices of special interest are **square matrices**, which have the same number of rows and columns; **diagonal matrices**, which are square matrices with only zero entries off the main diagonal (that is, $a_{ij} = 0$ if $i \neq j$); and (column) **vectors**, which are $n \times 1$ matrices. For example, if

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & -1 \\ 2 & 6 & 5 \\ 0 & 1 & 4 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix},$$

then $\mathbf{A}$ is a square matrix, $\mathbf{B}$ is a diagonal matrix, and $\mathbf{x}$ is a vector. An $m \times n$ matrix whose entries are all zero is called a **zero matrix** and is denoted by $\mathbf{0}$. For consistency, we denote matrices by boldfaced capitals, such as $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{I}$, $\mathbf{X}$, and $\mathbf{Y}$, and reserve boldfaced lowercase letters, such as $\mathbf{c}$, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, for vectors.

Algebra of Matrices

Matrix Addition and Scalar Multiplication. The operations of matrix addition and scalar multiplication are very straightforward. Addition is performed by adding corresponding elements:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}.$$

Formally, the *sum* of two $m \times n$ matrices is given by

$$\mathbf{A} + \mathbf{B} = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}].$$

(The sole novelty here is that addition is not defined for two matrices whose dimensions m, n differ.)

To multiply a matrix by a scalar (number), we simply multiply each element in the matrix by the number:

$$3\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}.$$

In other words, $r\mathbf{A} = r[a_{ij}] = [ra_{ij}]$. The notation $-\mathbf{A}$ stands for $(-1)\mathbf{A}$.

Properties of Matrix Addition and Scalar Multiplication. Matrix addition and scalar multiplication are nothing more than mere bookkeeping, and the usual algebraic properties hold. If **A, B**, and **C** are $m \times n$ matrices and r, s are scalars, then

$$\begin{aligned} &\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}, && \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \\ &\mathbf{A} + \mathbf{0} = \mathbf{A}, && \mathbf{A} + (-\mathbf{A}) = \mathbf{0}, \\ &r(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B}, && (r + s)\mathbf{A} = r\mathbf{A} + s\mathbf{A}, \\ &r(s\mathbf{A}) = (rs)\mathbf{A} = s(r\mathbf{A}). \end{aligned}$$

Matrix Multiplication. The matrix product is what makes matrix algebra interesting and useful. We indicated in Section 9.1 that the product of a matrix $\mathbf{A}$ and a column vector $\mathbf{x}$ is the column vector composed of dot products of the rows of $\mathbf{A}$ with $\mathbf{x}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 0 + 3 \cdot 2 \\ 4 \cdot 1 + 5 \cdot 0 + 6 \cdot 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}.$$

More generally, the product of two matrices **A** and **B** is formed by taking the array of dot products of the *rows* of the first "factor" **A** with the *columns* of the second factor **B**; the dot product of the ith row of **A** with the jth column of **B** is written as the ijth entry of the product **AB**:

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & x \\ -1 & -1 & y \\ 4 & 1 & z \end{bmatrix} = \begin{bmatrix} 1+0+4 & 2+0+1 & x+0+z \\ 3+1+8 & 6+1+2 & 3x-y+2z \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 & x+z \\ 12 & 9 & 3x-y+2z \end{bmatrix}.$$

Note that **AB** is only defined when the number of columns of **A** matches the number of rows of **B**. A useful formula for the product of an $m \times n$ matrix **A** and an $n \times p$ matrix **B** is

$$\mathbf{AB} := [c_{ij}], \qquad \text{where} \quad c_{ij} := \sum_{k=1}^{n} a_{ik}b_{kj}.$$

The dot product of the ith row of **A** and the jth column of **B** is seen in the "sum of products" expression for c_{ij}.

Since **AB** is computed in terms of the *rows* of the first factor and the *columns* of the second factor, it should not be surprising that, in general, **AB** does not equal **BA** (matrix multiplication does not *commute*):

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \qquad \text{but} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}.$$

In fact, the dimensions of **A** and **B** may render one or the other of these products undefined:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}; \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{not defined}.$$

By the same token, one might not expect (**AB**)**C** to equal **A**(**BC**), since in (**AB**)**C** we take dot products with the *columns* of **B**, whereas in **A**(**BC**) we employ the *rows* of **B**. So it is a pleasant surprise that *this complication does not arise*, and the "parenthesis grouping" rules are the customary ones:

Properties of Matrix Multiplication

$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$	(Associativity)
$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$	(Distributivity)
$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$	(Distributivity)
$(r\mathbf{A})\mathbf{B} = r(\mathbf{AB}) = \mathbf{A}(r\mathbf{B})$	(Associativity)

To summarize, the algebra of matrices proceeds much like the standard algebra of numbers, *except that we should never presume that we can switch the order of matrix factors.* (If you think $(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + \mathbf{B}^2 + 2\mathbf{AB}$, what error have you made?)

Matrices as Linear Operators. Let **A** be an $m \times n$ matrix and let **x** and **y** be $n \times 1$ vectors. Then **Ax** is an $m \times 1$ vector, and so we can think of multiplication by **A** as defining an operator that maps $n \times 1$ vectors into $m \times 1$ vectors. A consequence of the distributivity and associativity properties is that multiplication by **A** defines a **linear operator**, since $\mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{Ax}+\mathbf{Ay}$ and $\mathbf{A}(r\mathbf{x}) = r\mathbf{Ax}$. Moreover, if **A** is an $m \times n$ matrix and **B** is an $n \times p$ matrix, then the $m \times p$ matrix **AB** defines a linear operator that is the composition of the linear operator defined by **B**

with the linear operator defined by **A**. That is, $(\mathbf{AB})\mathbf{x} = \mathbf{A}(\mathbf{Bx})$, where **x** is a $p \times 1$ vector.

Examples of linear operations are

(i) stretching or contracting the components of a vector by constant factors;
(ii) rotating a vector through some angle about a fixed axis;
(iii) reflecting a vector in a plane mirror.

The Matrix Formulation of Linear Algebraic Systems. Matrix algebra was developed to provide a convenient tool for expressing and analyzing linear algebraic systems. Note that the set of equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1\,, \\ x_1 + 3x_2 + 2x_3 &= -1\,, \\ x_1 \phantom{{}+3x_2} + x_3 &= 0 \end{aligned}$$

can be written using the matrix product

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}. \tag{1}$$

In general, we express the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\,, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\,, \\ &\;\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

in matrix notation as $\mathbf{Ax} = \mathbf{b}$, where **A** is the *coefficient matrix,* **x** is the vector of unknowns, and **b** is the vector of constants occurring on the right-hand side:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

If $\mathbf{b} = \mathbf{0}$, the system $\mathbf{Ax} = \mathbf{b}$ is said to be *homogeneous* (analogous to the nomenclature of Section 4.2).

Matrix Transpose. The matrix obtained from **A** by interchanging its rows and columns is called the **transpose** of **A** and is denoted by $\mathbf{A}^T$. For example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 6 \\ -1 & 2 & -1 \end{bmatrix}, \quad \text{then}$$

$$\mathbf{A}^T = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 6 & -1 \end{bmatrix}.$$

In general, we have $[a_{ij}]^T = [b_{ij}]$, where $b_{ij} = a_{ji}$. Properties of the transpose are explored in Problem 7.

Matrix Identity. There is a "multiplicative identity" in matrix algebra, namely, a square diagonal matrix $\mathbf{I}$ with ones down the main diagonal. Multiplying $\mathbf{I}$ on the right or left by any other matrix (with compatible dimensions) reproduces the latter matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(The notation $\mathbf{I}_n$ is used if it is convenient to specify the dimensions, $n \times n$, of the identity matrix.)

Matrix Inverse. Some *square* matrices $\mathbf{A}$ can be paired with other (square) matrices $\mathbf{B}$ having the property that $\mathbf{BA} = \mathbf{I}$:

$$\begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

When this happens, it can be shown that

(i) $\mathbf{B}$ is the *unique* matrix satisfying $\mathbf{BA} = \mathbf{I}$, and
(ii) $\mathbf{B}$ also satisfies $\mathbf{AB} = \mathbf{I}$.

In such a case, we say that $\mathbf{B}$ is the **inverse** of $\mathbf{A}$ and write $\mathbf{B} = \mathbf{A}^{-1}$.

Not every matrix possesses an inverse; the zero matrix $\mathbf{0}$, for example, can never satisfy the equation $\mathbf{0B} = \mathbf{I}$. A matrix that has no inverse is said to be **singular.**

If we know an inverse for the coefficient matrix $\mathbf{A}$ in a system of linear equations $\mathbf{Ax} = \mathbf{b}$, the solution can be calculated directly by computing $\mathbf{A}^{-1}\mathbf{b}$, as the following derivation shows:

$$\mathbf{Ax} = \mathbf{b} \text{ implies } \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b} \text{ implies } \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}\,.$$

Using (2), for example, we can solve equation (1) quite efficiently:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \\ -\frac{5}{2} \end{bmatrix}.$$

On the other hand, the coefficient matrix for any *inconsistent* system has no inverse. For example, the coefficient matrices

$$\begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 & 1 \\ -3 & 1 & 4 \\ -1 & 1 & 5 \end{bmatrix}$$

for the inconsistent systems of Problem 12, Exercises 9.2 (page 504), are necessarily singular.

When $\mathbf{A}^{-1}$ is known, solving $\mathbf{Ax} = \mathbf{b}$ by multiplying $\mathbf{b}$ by $\mathbf{A}^{-1}$ is certainly easier than applying the Gauss–Jordan algorithm of the previous section†. So it appears advantageous to be able to find matrix inverses. Some inverses can be obtained directly from the interpretation of the matrix as a linear operator. For example, the inverse of a matrix that rotates a vector is

†(When the effort to compute the inverse is accounted for, Gauss–Jordan emerges as the winner.)

the matrix that rotates it in the opposite direction. A matrix that performs a mirror reflection is its own inverse (what do you get if you reflect twice?). But in general one must employ an algorithm to compute a matrix inverse. The underlying strategy for this algorithm is based on the observation that if $\mathbf{X}$ denotes the inverse of $\mathbf{A}$, then $\mathbf{X}$ must satisfy the equation $\mathbf{AX} = \mathbf{I}$; finding $\mathbf{X}$ amounts to solving n linear systems of equations for the columns $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of $\mathbf{X}$:

$$\mathbf{Ax}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{Ax}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \qquad \cdots, \quad \mathbf{Ax}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The implementation of this operation is neatly executed by the following variation of the Gauss–Jordan algorithm.

Finding the Inverse of a Matrix. By a **row operation**, we mean any one of the following:

(a) Interchanging two rows of the matrix
(b) Multiplying a row of the matrix by a nonzero scalar
(c) Adding a scalar multiple of one row of the matrix to another row.

If the $n \times n$ matrix $\mathbf{A}$ has an inverse, then $\mathbf{A}^{-1}$ can be determined by performing row operations on the $n \times 2n$ matrix $[\mathbf{A} \mid \mathbf{I}]$ obtained by writing $\mathbf{A}$ and $\mathbf{I}$ side by side. In particular, we perform row operations on the matrix $[\mathbf{A} \mid \mathbf{I}]$ until the first n rows and columns form the identity matrix; that is, the new matrix is $[\mathbf{I} \mid \mathbf{B}]$. Then $\mathbf{A}^{-1} = \mathbf{B}$. We remark that if this procedure fails to produce a matrix of the form $[\mathbf{I} \mid \mathbf{B}]$, then $\mathbf{A}$ has no inverse.

Example 1 Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$.

Solution We first form the matrix $[\mathbf{A} \mid \mathbf{I}]$ and *row-reduce* the matrix to $[\mathbf{I} \mid \mathbf{A}^{-1}]$. Computing, we find the following:

The matrix $[\mathbf{A} \mid \mathbf{I}]$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Subtract the first row from the second and third to obtain
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right].$$

Add 2 times the second row to the third row to obtain
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -3 & 2 & 1 \end{array}\right].$$

Subtract 2 times the second row from the first to obtain
$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -3 & 2 & 1 \end{array}\right].$$

Multiply the third row by $1/2$ to obtain
$$\left[\begin{array}{ccc:ccc} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 1 & \frac{1}{2} \end{array}\right].$$

Add the third row to the first and then subtract the third row from the second to obtain
$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & 1 & \frac{1}{2} \end{array}\right].$$

The matrix shown in color is $\mathbf{A}^{-1}$. [Compare equation (2).] ◆

It is convenient to have an expression for the inverse of a generic 2×2 matrix. The following formula is easily verified by mental arithmetic:

(3)
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{if } ad - bc \neq 0 .$$

The denominator in (3), whose nonvanishing is the crucial condition for the existence of the inverse, is known as the determinant.

Determinants. The **determinant** of a 2×2 matrix $\mathbf{A}$, denoted det $\mathbf{A}$ or $|\mathbf{A}|$, is defined by
$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

The determinants of higher-order square matrices can be defined recursively in terms of lower-order determinants, using the concept of the *minor*; the **minor** of a particular entry is the determinant of the submatrix formed when that entry's row and column are deleted. Then the determinant of an $n \times n$ matrix equals the alternating-sign sum of the products of the entries of the first row with their minors. For a 3×3 matrix $\mathbf{A}$ this looks like
$$\det \mathbf{A} := \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} .$$

For example,
$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 2 & 1 & -1 \end{vmatrix} = 1 \begin{vmatrix} 3 & 5 \\ 1 & -1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 5 \\ 2 & -1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix}$$
$$= 1(-3-5) - 2(0-10) + 1(0-6) = 6 .$$

For a 4×4 matrix, we have
$$\begin{vmatrix} 4 & 3 & 2 & -6 \\ -2 & 1 & 2 & 1 \\ 3 & 0 & 3 & 5 \\ 5 & 2 & 1 & -1 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 2 & 1 & -1 \end{vmatrix} - 3 \begin{vmatrix} -2 & 2 & 1 \\ 3 & 3 & 5 \\ 5 & 1 & -1 \end{vmatrix}$$
$$+ 2 \begin{vmatrix} -2 & 1 & 1 \\ 3 & 0 & 5 \\ 5 & 2 & -1 \end{vmatrix} - (-6) \begin{vmatrix} -2 & 1 & 2 \\ 3 & 0 & 3 \\ 5 & 2 & 1 \end{vmatrix} .$$

As we have shown, the first minor is 6; and the others are computed the same way, resulting in

$$\begin{vmatrix} 4 & 3 & 2 & -6 \\ -2 & 1 & 2 & 1 \\ 3 & 0 & 3 & 5 \\ 5 & 2 & 1 & -1 \end{vmatrix} = 4(6) - 3(60) + 2(54) - (-6)(36) = 168 \,.$$

Although higher-order determinants can be calculated similarly, a more practical way to evaluate them involves the row-reduction of the matrix to upper triangular form. Here we will deal mainly with low-order determinants, and direct the reader to a linear algebra text for further discussion.†

Determinants have a geometric interpretation: $\det \mathbf{A}$ is the volume (in n-dimensional space) of the parallelepiped whose edges are given by the column vectors of $\mathbf{A}$. But their chief value lies in the role they play in the following theorem, which summarizes many of the results from linear algebra that we shall need, and in Cramer's rule, described in Appendix D.

Matrices and Systems of Equations

Theorem 1. Let $\mathbf{A}$ be an $n \times n$ matrix. The following statements are equivalent:

(a) $\mathbf{A}$ is singular (does not have an inverse).

(b) The determinant of $\mathbf{A}$ is zero.

(c) $\mathbf{Ax} = \mathbf{0}$ has nontrivial solutions ($\mathbf{x} \neq \mathbf{0}$).

(d) The columns (rows) of $\mathbf{A}$ form a linearly dependent set.

In part (d), the statement that the n columns of $\mathbf{A}$ are linearly dependent means that there exist scalars $c_1, \ldots, c_n$, *not all zero*, such that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0} \,,$$

where $\mathbf{a}_j$ is the vector forming the jth column of $\mathbf{A}$.

If $\mathbf{A}$ is a singular square matrix (so $\det \mathbf{A} = 0$), then $\mathbf{Ax} = \mathbf{0}$ has infinitely many solutions. Indeed, Theorem 1 asserts that there is a vector $\mathbf{x}_0 \neq \mathbf{0}$ such that $\mathbf{Ax}_0 = \mathbf{0}$, and we can get infinitely many other solutions by multiplying $\mathbf{x}_0$ by any scalar, i.e., taking $\mathbf{x} = c\mathbf{x}_0$. Furthermore, $\mathbf{Ax} = \mathbf{b}$ either has no solutions or it has infinitely many of them of the form

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h \,,$$

where $\mathbf{x}_p$ is a *particular* solution to $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x}_h$ is any of the infinity of solutions to $\mathbf{Ax} = \mathbf{0}$ (see Problem 15). The resemblance of this situation to that of solving nonhomogeneous linear differential equations should be quite apparent.

To illustrate, in Example 3 of Section 9.2 (page 502) we saw that the system

$$\begin{bmatrix} 2 & 4 & 1 \\ 2 & 4 & 0 \\ -4 & -8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ -10 \end{bmatrix}$$

has solutions

$$x_1 = 3 - 2s \,, \qquad x_2 = s \,, \qquad x_3 = 2 \,; \quad -\infty < s < \infty \,.$$

†Your authors' personal favorite is *Fundamentals of Matrix Analysis with Applications*, by Edward Barry Saff and Arthur David Snider (John Wiley & Sons, Hoboken, New Jersey, 2016).

Writing these in matrix notation, we can identify the vectors $\mathbf{x}_p$ and $\mathbf{x}_h$ mentioned above:

$$\mathbf{x} = \begin{bmatrix} 3-2s \\ s \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{x}_p + \mathbf{x}_h .$$

Note further that the determinant of $\mathbf{A}$ is indeed zero,

$$\det \mathbf{A} = 2\begin{vmatrix} 4 & 0 \\ -8 & 1 \end{vmatrix} - 4\begin{vmatrix} 2 & 0 \\ -4 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ -4 & -8 \end{vmatrix} = 2 \cdot 4 - 4 \cdot 2 + 1 \cdot 0 = 0 ,$$

and that the linear dependence of the columns of $\mathbf{A}$ is exhibited by the identity

$$-2\begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 4 \\ -8 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} .$$

If $\mathbf{A}$ is a nonsingular square matrix (i.e., $\mathbf{A}$ has an inverse and $\det \mathbf{A} \neq 0$), then the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$ has $\mathbf{x} = \mathbf{0}$ as its only solution. More generally, when $\det \mathbf{A} \neq 0$, the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution (namely, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$).

Calculus of Matrices

If we allow the entries $a_{ij}(t)$ in a matrix $\mathbf{A}(t)$ to be functions of the variable t, then $\mathbf{A}(t)$ is a **matrix function of** t. Similarly, if the entries $x_i(t)$ of a vector $\mathbf{x}(t)$ are functions of t, then $\mathbf{x}(t)$ is a **vector function of** t.

These matrix and vector functions have a calculus much like that of real-valued functions. A matrix $\mathbf{A}(t)$ is said to be **continuous at** t_0 if each entry $a_{ij}(t)$ is continuous at t_0. Moreover, $\mathbf{A}(t)$ is **differentiable at** t_0 if each entry $a_{ij}(t)$ is differentiable at t_0, and we write

$$\text{(4)} \qquad \frac{d\mathbf{A}}{dt}(t_0) = \mathbf{A}'(t_0) := [a'_{ij}(t_0)] .$$

Similarly, we define

$$\text{(5)} \qquad \int_a^b \mathbf{A}(t)\,dt := \left[\int_a^b a_{ij}(t)\,dt\right] .$$

Example 2 Let $\mathbf{A}(t) = \begin{bmatrix} t^2+1 & \cos t \\ e^t & 1 \end{bmatrix}$.

Find: **(a)** $\mathbf{A}'(t)$. **(b)** $\int_0^1 \mathbf{A}(t)dt$.

Solution Using formulas (4) and (5), we compute

$$\textbf{(a)}\ \mathbf{A}'(t) = \begin{bmatrix} 2t & -\sin t \\ e^t & 0 \end{bmatrix} . \qquad \textbf{(b)}\ \int_0^1 \mathbf{A}(t)dt = \begin{bmatrix} \frac{4}{3} & \sin 1 \\ e-1 & 1 \end{bmatrix} . \ \blacklozenge$$

Example 3 Show that $\mathbf{x}(t) = \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix}$ is a solution of the matrix differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} .$$

Solution We simply verify that $\mathbf{x}'(t)$ and $\mathbf{A}\mathbf{x}(t)$ are the same vector function:

$$\mathbf{x}'(t) = \begin{bmatrix} -\omega \sin \omega t \\ \omega \cos \omega t \end{bmatrix}; \qquad \mathbf{A}\mathbf{x} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}\begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix} = \begin{bmatrix} -\omega \sin \omega t \\ \omega \cos \omega t \end{bmatrix}. \quad ◆$$

The basic properties of differentiation are valid for matrix functions.

Differentiation Formulas for Matrix Functions

$$\frac{d}{dt}(\mathbf{C}\mathbf{A}) = \mathbf{C}\frac{d\mathbf{A}}{dt} \qquad (\mathbf{C} \text{ a constant matrix}).$$

$$\frac{d}{dt}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}.$$

$$\frac{d}{dt}(\mathbf{A}\mathbf{B}) = \mathbf{A}\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\mathbf{B}.$$

In the last formula, the order in which the matrices are written is very important because, as we have emphasized, matrix multiplication does not always commute.

9.3 EXERCISES

1. Let $\mathbf{A} := \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ and $\mathbf{B} := \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}$.
   Find: **(a)** $\mathbf{A}+\mathbf{B}$. **(b)** $3\mathbf{A}-\mathbf{B}$.

2. Let $\mathbf{A} := \begin{bmatrix} 2 & 0 & 5 \\ 2 & 1 & 1 \end{bmatrix}$ and $\mathbf{B} := \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -2 \end{bmatrix}$.
   Find: **(a)** $\mathbf{A}+\mathbf{B}$. **(b)** $7\mathbf{A}-4\mathbf{B}$.

3. Let $\mathbf{A} := \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{B} := \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}$.
   Find: **(a)** $\mathbf{A}\mathbf{B}$. **(b)** $\mathbf{A}^2 = \mathbf{A}\mathbf{A}$. **(c)** $\mathbf{B}^2 = \mathbf{B}\mathbf{B}$.

4. Let $\mathbf{A} := \begin{bmatrix} 2 & 1 \\ 0 & 4 \\ -1 & 3 \end{bmatrix}$ and $\mathbf{B} := \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 1 \end{bmatrix}$.
   Find: **(a)** $\mathbf{A}\mathbf{B}$. **(b)** $\mathbf{B}\mathbf{A}$.

5. Let $\mathbf{A} := \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$, $\mathbf{B} := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and
   $\mathbf{C} := \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$.
   Find: **(a)** $\mathbf{A}\mathbf{B}$. **(b)** $\mathbf{A}\mathbf{C}$. **(c)** $\mathbf{A}(\mathbf{B}+\mathbf{C})$.

6. Let $\mathbf{A} := \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $\mathbf{B} := \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$, and
   $\mathbf{C} := \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$.
   Find: **(a)** $\mathbf{A}\mathbf{B}$. **(b)** $(\mathbf{A}\mathbf{B})\mathbf{C}$. **(c)** $(\mathbf{A}+\mathbf{B})\mathbf{C}$.

7. **(a)** Show that if $\mathbf{u}$ and $\mathbf{v}$ are each $n \times 1$ column vectors, then the matrix product $\mathbf{u}^T\mathbf{v}$ is the same as the dot product $\mathbf{u} \cdot \mathbf{v}$.
   (b) Let $\mathbf{v}$ be a 3×1 column vector with $\mathbf{v}^T = [2 \quad 3 \quad 5]$. Show that, for $\mathbf{A}$ as given in Example 1 (page 509), $(\mathbf{A}\mathbf{v})^T = \mathbf{v}^T\mathbf{A}^T$.
   (c) Does $(\mathbf{A}\mathbf{v})^T = \mathbf{v}^T\mathbf{A}^T$ hold for every $m \times n$ matrix $\mathbf{A}$ and $n \times 1$ vector $\mathbf{v}$?
   (d) Does $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$ hold for every pair of matrices $\mathbf{A}, \mathbf{B}$ such that both matrix products are defined? Justify your answer.

8. Let $\mathbf{A} := \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$ and $\mathbf{B} := \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.
   Verify that $\mathbf{A}\mathbf{B} \neq \mathbf{B}\mathbf{A}$.

In Problems 9–14, use the method of Example 1 to compute the inverse of the given matrix, if it exists. For Problems 9 and 10, confirm your answer by comparison with formula (3).

9. $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 1 \\ 5 & 9 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

13. $\begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{bmatrix}$

15. Prove that if $\mathbf{x}_p$ satisfies $\mathbf{A}\mathbf{x}_p = \mathbf{b}$, then every solution to the nonhomogeneous system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is of the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution to the corresponding homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$.

16. Let $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(a) Show that $\mathbf{A}$ is singular.

(b) Show that $\mathbf{A}\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$ has no solutions.

(c) Show that $\mathbf{A}\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$ has infinitely many solutions.

In Problems 17–20, find the matrix function $\mathbf{X}^{-1}(t)$ whose value at t is the inverse of the given matrix $\mathbf{X}(t)$.

17. $\mathbf{X}(t) = \begin{bmatrix} e^t & e^{4t} \\ e^t & 4e^{4t} \end{bmatrix}$

18. $\mathbf{X}(t) = \begin{bmatrix} \sin 2t & \cos 2t \\ 2\cos 2t & -2\sin 2t \end{bmatrix}$

19. $\mathbf{X}(t) = \begin{bmatrix} e^t & e^{-t} & e^{2t} \\ e^t & -e^{-t} & 2e^{2t} \\ e^t & e^{-t} & 4e^{2t} \end{bmatrix}$

20. $\mathbf{X}(t) = \begin{bmatrix} e^{3t} & 1 & t \\ 3e^{3t} & 0 & 1 \\ 9e^{3t} & 0 & 0 \end{bmatrix}$

In Problems 21–26, evaluate the given determinant.

21. $\begin{vmatrix} 4 & 3 \\ -1 & 2 \end{vmatrix}$

22. $\begin{vmatrix} 12 & 8 \\ 3 & 2 \end{vmatrix}$

23. $\begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & -2 \end{vmatrix}$

24. $\begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & -1 \\ -1 & 2 & 1 \end{vmatrix}$

25. $\begin{vmatrix} 1 & 4 & 3 \\ -1 & -1 & 2 \\ 4 & 5 & 2 \end{vmatrix}$

26. $\begin{vmatrix} 1 & 4 & 4 \\ 3 & 0 & -3 \\ 1 & 6 & 2 \end{vmatrix}$

In Problems 27–29, determine the values of r for which $\det(\mathbf{A} - r\mathbf{I}) = 0$.

27. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

28. $\mathbf{A} = \begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix}$

29. $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

30. Illustrate the equivalence of the assertions (a)–(d) in Theorem 1 (page 511) for the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

as follows.

(a) Show that the row-reduction procedure applied to $[\mathbf{A} \mid \mathbf{I}]$ fails to produce the inverse of $\mathbf{A}$.

(b) Calculate det $\mathbf{A}$.

(c) Determine a nontrivial solution $\mathbf{x}$ to $\mathbf{A}\mathbf{x} = \mathbf{0}$.

(d) Find scalars c_1, c_2, and c_3, *not all zero*, so that $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 = \mathbf{0}$, where $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ are the columns of $\mathbf{A}$.

In Problems 31 and 32, find $d\mathbf{x}/dt$ for the given vector functions.

31. $\mathbf{x}(t) = \begin{bmatrix} e^{3t} \\ 2e^{3t} \\ -e^{3t} \end{bmatrix}$

32. $\mathbf{x}(t) = \begin{bmatrix} e^{-t}\sin 3t \\ 0 \\ -e^{-t}\sin 3t \end{bmatrix}$

In Problems 33 and 34, find $d\mathbf{X}/dt$ for the given matrix functions.

33. $\mathbf{X}(t) = \begin{bmatrix} e^{5t} & 3e^{2t} \\ -2e^{5t} & -e^{2t} \end{bmatrix}$

34. $\mathbf{X}(t) = \begin{bmatrix} \sin 2t & \cos 2t & e^{-2t} \\ -\sin 2t & 2\cos 2t & 3e^{-2t} \\ 3\sin 2t & \cos 2t & e^{-2t} \end{bmatrix}$

In Problems 35 and 36, verify that the given vector function satisfies the given system.

35. $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(t) = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}$

36. $\mathbf{x}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(t) = \begin{bmatrix} 0 \\ e^t \\ -3e^t \end{bmatrix}$

In Problems 37 and 38, verify that the given matrix function satisfies the given matrix differential equation.

37. $\mathbf{X}' = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\mathbf{X}, \quad \mathbf{X}(t) = \begin{bmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{bmatrix}$

38. $\mathbf{X}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}\mathbf{X},$

$\mathbf{X}(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & e^{5t} \\ 0 & e^t & -e^{5t} \end{bmatrix}$

In Problems 39 and 40, the matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are given. Find

(a) $\int \mathbf{A}(t)dt$. **(b)** $\int_0^1 \mathbf{B}(t)dt$. **(c)** $\frac{d}{dt}[\mathbf{A}(t)\mathbf{B}(t)]$.

39. $\mathbf{A}(t) = \begin{bmatrix} t & e^t \\ 1 & e^t \end{bmatrix}, \quad \mathbf{B}(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$

40. $\mathbf{A}(t) = \begin{bmatrix} 1 & e^{-2t} \\ 3 & e^{-2t} \end{bmatrix}, \quad \mathbf{B}(t) = \begin{bmatrix} e^{-t} & e^{-t} \\ -e^{-t} & 3e^{-t} \end{bmatrix}$

41. An $n \times n$ matrix $\mathbf{A}$ is called **symmetric** if $\mathbf{A}^T = \mathbf{A}$; that is, if $a_{ij} = a_{ji}$, for all $i, j = 1, \ldots, n$. Show that if $\mathbf{A}$ is an $n \times n$ matrix, then $\mathbf{A} + \mathbf{A}^T$ is a symmetric matrix.

42. Let $\mathbf{A}$ be an $m \times n$ matrix. Show that $\mathbf{A}^T\mathbf{A}$ is a symmetric $n \times n$ matrix and $\mathbf{A}\mathbf{A}^T$ is a symmetric $m \times m$ matrix (see Problem 41).

43. The **inner product** of two vectors is a generalization of the dot product, for vectors with complex entries. It is defined by

$$(\mathbf{x}, \mathbf{y}) \coloneqq \sum_{i=1}^{n} x_i \bar{y}_i , \qquad \text{where}$$

$\mathbf{x} = \operatorname{col}(x_1, x_2, \ldots, x_n)$, $\mathbf{y} = \operatorname{col}(y_1, y_2, \ldots, y_n)$ are complex vectors and the overbar denotes complex conjugation.

(a) Show that $(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T\bar{\mathbf{y}}$, where $\bar{\mathbf{y}} = \operatorname{col}(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n)$.

(b) Prove that for any $n \times 1$ vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and any complex number λ, we have

$$(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})} ,$$
$$(\mathbf{x}, \mathbf{y} + \mathbf{z}) = (\mathbf{x}, \mathbf{y}) + (\mathbf{x}, \mathbf{z}) ,$$
$$(\lambda\mathbf{x}, \mathbf{y}) = \lambda(\mathbf{x}, \mathbf{y}) , \quad (\mathbf{x}, \lambda\mathbf{y}) = \bar{\lambda}(\mathbf{x}, \mathbf{y}) .$$

9.4 Linear Systems in Normal Form

In keeping with the introduction presented in Section 9.1, we say that a system of n linear differential equations is in **normal form** if it is expressed as

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) , \tag{1}$$

where $\mathbf{x}(t) = \operatorname{col}\big(x_1(t), \ldots, x_n(t)\big)$, $\mathbf{f}(t) = \operatorname{col}\big(f_1(t), \ldots, f_n(t)\big)$, and $\mathbf{A}(t) = [a_{ij}(t)]$ is an $n \times n$ matrix. As with a scalar linear differential equation, a system is called **homogeneous** when $\mathbf{f}(t) \equiv \mathbf{0}$; otherwise, it is called **nonhomogeneous**. When the elements of $\mathbf{A}$ are all constants, the system is said to have **constant coefficients**. Recall that an nth-order linear differential equation

$$y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_0(t)y(t) = g(t) \tag{2}$$

can be rewritten as a first-order system in normal form using the substitution $x_1(t) \coloneqq y(t)$, $x_2(t) \coloneqq y'(t), \ldots, x_n(t) \coloneqq y^{(n-1)}(t)$; indeed, equation (2) is equivalent to $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$, where $\mathbf{x}(t) = \operatorname{col}\big(x_1(t), \ldots, x_n(t)\big)$, $\mathbf{f}(t) \coloneqq \operatorname{col}\big(0, \ldots, 0, g(t)\big)$, and

$$\mathbf{A}(t) \coloneqq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -p_0(t) & -p_1(t) & -p_2(t) & \cdots & -p_{n-2}(t) & -p_{n-1}(t) \end{bmatrix} .$$

The theory for systems in normal form parallels very closely the theory of linear differential equations presented in Chapters 4 and 6. In many cases the proofs for scalar linear differential equations carry over to normal systems with appropriate modifications. Conversely, results for normal systems apply to scalar linear equations since, as we showed, any scalar linear equation can be expressed as a normal system. This is the case with the existence and uniqueness theorems for linear differential equations.

The **initial value problem** for the normal system (1) is the problem of finding a differentiable vector function $\mathbf{x}(t)$ that satisfies the system on an interval I and also satisfies the **initial condition** $\mathbf{x}(t_0) = \mathbf{x}_0$, where t_0 is a given point of I and $\mathbf{x}_0 = \operatorname{col}(x_{1,0}, \ldots, x_{n,0})$ is a given vector.

Existence and Uniqueness

Theorem 2. If $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are continuous on an open interval I that contains the point t_0, then for any choice of the initial vector $\mathbf{x}_0$, there exists a unique solution $\mathbf{x}(t)$ on the whole interval I to the initial value problem

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)\,, \qquad \mathbf{x}(t_0) = \mathbf{x}_0\,.$$

We give a proof of this result in Chapter 13[†] and obtain as corollaries the existence and uniqueness theorems for second-order equations (Theorem 4, Section 4.5, page 182) and higher-order linear equations (Theorem 1, Section 6.1, page 319).

If we rewrite system (1) as $\mathbf{x}' - \mathbf{A}\mathbf{x} = \mathbf{f}$ and define the operator $L[\mathbf{x}] \coloneqq \mathbf{x}' - \mathbf{A}\mathbf{x}$, then we can express system (1) in the operator form $L[\mathbf{x}] = \mathbf{f}$. Here the operator L maps vector functions into vector functions. Moreover, L is a *linear* operator in the sense that for any scalars a, b and differentiable vector functions $\mathbf{x}$, $\mathbf{y}$, we have

$$L[a\mathbf{x} + b\mathbf{y}] = aL[\mathbf{x}] + bL[\mathbf{y}]\,.$$

The proof of this linearity follows from the properties of matrix multiplication (see Problem 27).

As a consequence of the linearity of L, if $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are solutions to the *homogeneous* system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, or $L[\mathbf{x}] = \mathbf{0}$ in operator notation, then any linear combination of these vectors, $c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$, is also a solution. Moreover, we will see that if the solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent, then *every* solution to $L[\mathbf{x}] = \mathbf{0}$ can be expressed as $c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$ for an appropriate choice of the constants $c_1, \ldots, c_n$.

Linear Dependence of Vector Functions

Definition 1. The m vector functions $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are said to be **linearly dependent on an interval** I if there exist constants $c_1, \ldots, c_m$, not all zero, such that

$$c_1\mathbf{x}_1(t) + \cdots + c_m\mathbf{x}_m(t) = \mathbf{0} \tag{3}$$

for all t in I. If the vectors are not linearly dependent, they are said to be **linearly independent on** I.

Example 1 Show that the vector functions $\mathbf{x}_1(t) = \mathrm{col}(e^t, 0, e^t)$, $\mathbf{x}_2(t) = \mathrm{col}(3e^t, 0, 3e^t)$, and $\mathbf{x}_3(t) = \mathrm{col}(t, 1, 0)$ are linearly dependent on $(-\infty, \infty)$.

Solution Notice that $\mathbf{x}_2$ is just 3 times $\mathbf{x}_1$ and therefore $3\mathbf{x}_1(t) - \mathbf{x}_2(t) + 0\cdot\mathbf{x}_3(t) = \mathbf{0}$ for all t. Hence, $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are linearly dependent on $(-\infty, \infty)$. ◆

Example 2 Show that

$$\mathbf{x}_1(t) = \begin{bmatrix} t \\ |t| \end{bmatrix}\,, \qquad \mathbf{x}_2(t) = \begin{bmatrix} |t| \\ t \end{bmatrix}$$

are linearly independent on $(-\infty, \infty)$.

[†]All references to Chapters 11–13 refer to the expanded text, *Fundamentals of Differential Equations and Boundary Value Problems*, 7th ed.

Solution Note that at every instant t_0, the column vector $\mathbf{x}_1(t_0)$ is a multiple of $\mathbf{x}_2(t_0)$; indeed, $\mathbf{x}_1(t_0) = \mathbf{x}_2(t_0)$ for $t_0 \geq 0$, and $\mathbf{x}_1(t_0) = -\mathbf{x}_2(t_0)$ for $t_0 \leq 0$. Nonetheless, the vector functions are not dependent, because the c's in condition (3) are not allowed to change with t; for $t < 0$, the equation $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = \mathbf{0}$ implies $c_1 - c_2 = 0$, but for $t > 0$ it implies $c_1 + c_2 = 0$. Thus $c_1 = c_2 = 0$ and the functions are independent. ◆

Example 3 Show that the vector functions $\mathbf{x}_1(t) = \operatorname{col}(e^{2t}, 0, e^{2t})$, $\mathbf{x}_2(t) = \operatorname{col}(e^{2t}, e^{2t}, -e^{2t})$, and $\mathbf{x}_3(t) = \operatorname{col}(e^t, 2e^t, e^t)$ are linearly independent on $(-\infty, \infty)$.

Solution To prove independence, we *assume* c_1, c_2, and c_3 are constants for which

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t) = \mathbf{0}$$

holds at every t in $(-\infty, \infty)$ and show that this forces $c_1 = c_2 = c_3 = 0$. In particular, when $t = 0$ we obtain

$$c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\-1\end{bmatrix} + c_3\begin{bmatrix}1\\2\\1\end{bmatrix} = \mathbf{0}\,,$$

which is equivalent to the system of linear equations

$$\begin{aligned} c_1 + c_2 + c_3 &= 0\,, \\ c_2 + 2c_3 &= 0\,, \\ c_1 - c_2 + c_3 &= 0\,. \end{aligned} \tag{4}$$

Either by solving (4) or by checking that the determinant of its coefficients is nonzero (recall Theorem 1 on page 511), we can verify that (4) has only the trivial solution $c_1 = c_2 = c_3 = 0$. Therefore the vector functions $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are linearly independent on $(-\infty, \infty)$ (in fact, on any interval containing $t = 0$). ◆

As Example 3 illustrates, if $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are n vector functions, each having n components, we can establish their linear independence on an interval I if we can find *one point* t_0 in I where the determinant

$$\det[\mathbf{x}_1(t_0) \ldots \mathbf{x}_n(t_0)]$$

is not zero. Because of the analogy with scalar equations, we call this determinant the **Wronskian**.

Wronskian

Definition 2. The **Wronskian** of n vector functions $\mathbf{x}_1(t) = \operatorname{col}(x_{1,1}, \ldots, x_{n,1}), \ldots, \mathbf{x}_n(t) = \operatorname{col}(x_{1,n}, \ldots, x_{n,n})$ is defined to be the function

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) := \begin{vmatrix} x_{1,1}(t) & x_{1,2}(t) & \cdots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \cdots & x_{2,n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \cdots & x_{n,n}(t) \end{vmatrix}\,.$$

We see that n vector functions are linearly independent on an interval if their Wronskian is nonzero *at any point in the interval*. But now we show that if these functions happen to be independent solutions to a homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $n \times n$ matrix of continuous functions, then the Wronskian is *never* zero on I. For suppose to the contrary that $W(t_0) = 0$. Then by Theorem 1 the vanishing of the determinant implies that the column vectors $\mathbf{x}_1(t_0), \mathbf{x}_2(t_0), \ldots, \mathbf{x}_n(t_0)$ are linearly dependent. Thus there exist scalars $c_1, \ldots, c_n$ *not all zero*, such that at t_0

$$c_1\mathbf{x}_1(t_0) + \cdots + c_n\mathbf{x}_n(t_0) = \mathbf{0} .$$

However, $c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$ and the vector function $\mathbf{z}(t) \equiv \mathbf{0}$ are both solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ on I, and they agree at the point t_0. So these solutions must be identical on I according to the existence-uniqueness theorem (Theorem 2, page 516). That is,

$$c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) = \mathbf{0}$$

for *all* t in I. But this contradicts the given information that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent on I. We have shown that $W(t_0) \neq 0$, and since t_0 is an arbitrary point, it follows that $W(t) \neq 0$ for all $t \in I$.

The preceding argument has two important implications that parallel the scalar case. First, the Wronskian of *solutions* to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *is either identically zero or never zero on I* (see also Problem 33). Second, a set of *n solutions* $\mathbf{x}_1, \ldots, \mathbf{x}_n$ to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ on *I is linearly independent on I if and only if their Wronskian is never zero on I*. With these facts in hand, we can imitate the proof given for the scalar case in Section 6.1 (Theorem 2, page 322) to obtain the following representation theorem for the solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Representation of Solutions (Homogeneous Case)

Theorem 3. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be n linearly independent solutions to the homogeneous system

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) \tag{5}$$

on the interval I, where $\mathbf{A}(t)$ is an $n \times n$ matrix function continuous on I. Then every solution to (5) on I can be expressed in the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) , \tag{6}$$

where $c_1, \ldots, c_n$ are constants.

A set of solutions $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ that are linearly independent on I or, equivalently, whose Wronskian does not vanish on I, is called a **fundamental solution set** for (5) on I. The linear combination in (6), written with arbitrary constants, is referred to as a **general solution** to (5).

If we take the vectors in a fundamental solution set and let them form the columns of a matrix $\mathbf{X}(t)$, that is,

$$\mathbf{X}(t) = [\mathbf{x}_1(t)\,\mathbf{x}_2(t)\ldots\mathbf{x}_n(t)] = \begin{bmatrix} x_{1,1}(t) & x_{1,2}(t) & \cdots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \cdots & x_{2,n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \cdots & x_{n,n}(t) \end{bmatrix},$$

then the matrix $\mathbf{X}(t)$ is called a **fundamental matrix** for (5). We can use it to express the general solution (6) as

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} ,$$

where $\mathbf{c} = \text{col}(c_1, \ldots, c_n)$ is an arbitrary constant vector. Since det $\mathbf{X} = W[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ is never zero on I, it follows from Theorem 1 on page 511 that $\mathbf{X}(t)$ is invertible for every t in I.

Example 4 Verify that the set

$$S = \left\{ \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}, \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix} \right\}$$

is a fundamental solution set for the system

$$\textbf{(7)} \qquad \mathbf{x}'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

on the interval $(-\infty, \infty)$ and find a fundamental matrix for (7). Also determine a general solution for (7).

Solution Substituting the first vector in the set S into the right-hand side of (7) gives

$$\mathbf{Ax} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{bmatrix} = \mathbf{x}'(t) .$$

Hence this vector satisfies system (7) for all t. Similar computations verify that the remaining vectors in S are also solutions to (7) on $(-\infty, \infty)$. For us to show that S is a fundamental solution set, it is enough to observe that the Wronskian

$$W(t) = \begin{vmatrix} e^{2t} & -e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & e^{-t} & -e^{-t} \end{vmatrix} = e^{2t} \begin{vmatrix} 0 & e^{-t} \\ e^{-t} & -e^{-t} \end{vmatrix} + e^{-t} \begin{vmatrix} e^{2t} & e^{-t} \\ e^{2t} & -e^{-t} \end{vmatrix} = -3$$

is never zero.

A fundamental matrix $\mathbf{X}(t)$ for (7) is just the matrix we used to compute the Wronskian; that is,

$$\textbf{(8)} \qquad \mathbf{X}(t) := \begin{bmatrix} e^{2t} & -e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & e^{-t} & -e^{-t} \end{bmatrix} .$$

A general solution to (7) can now be expressed as

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix} . \quad \blacklozenge$$

It is easy to check that the fundamental matrix in (8) satisfies the equation

$$\mathbf{X}'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{X}(t) ;$$

indeed, this is equivalent to showing that $\mathbf{x}' = \mathbf{Ax}$ for each column $\mathbf{x}$ in S. In general, a fundamental matrix for a system $\mathbf{x}' = \mathbf{Ax}$ satisfies the corresponding **matrix differential equation** $\mathbf{X}' = \mathbf{AX}$.

Another consequence of the linearity of the operator L defined by $L[\mathbf{x}] := \mathbf{x}' - \mathbf{Ax}$ is the **superposition principle** for linear systems. It states that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, respectively, to the nonhomogeneous systems

$$L[\mathbf{x}] = \mathbf{g}_1 \text{ and } L[\mathbf{x}] = \mathbf{g}_2 ,$$

then $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is a solution to

$$L[\mathbf{x}] = c_1\mathbf{g}_1 + c_2\mathbf{g}_2 .$$

Using the superposition principle and the representation theorem for homogeneous systems, we can prove the following theorem.

Representation of Solutions (Nonhomogeneous Case)

Theorem 4. If $\mathbf{x}_p$ is a particular solution to the nonhomogeneous system

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) \tag{9}$$

on the interval I and $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a fundamental solution set on I for the corresponding homogeneous system $\mathbf{x}(t) = \mathbf{A}(t)\mathbf{x}(t)$, then every solution to (9) on I can be expressed in the form

$$\mathbf{x}(t) = \mathbf{x}_p(t) + c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) , \tag{10}$$

where $c_1, \ldots, c_n$ are constants.

The proof of this theorem is almost identical to the proofs of Theorem 4 in Section 4.5 (page 182) and Theorem 4 in Section 6.1 (page 325). We leave the proof as an exercise.

The linear combination of $\mathbf{x}_p, \mathbf{x}_1, \ldots, \mathbf{x}_n$ in (10) written with arbitrary constants $c_1, \ldots, c_n$ is called a **general solution** of (9). This general solution can also be expressed as $\mathbf{x} = \mathbf{x}_p + \mathbf{X}\mathbf{c}$, where $\mathbf{X}$ is a fundamental matrix for the homogeneous system and $\mathbf{c}$ is an arbitrary constant vector.

We now summarize the results of this section as they apply to the problem of finding a general solution to a system of n linear first-order differential equations in normal form.

Approach to Solving Normal Systems

1. To determine a general solution to the $n \times n$ homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$:

(a) Find a fundamental solution set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ that consists of n linearly independent solutions to the homogeneous system.

(b) Form the linear combination

$$\mathbf{x} = \mathbf{X}\mathbf{c} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n ,$$

where $\mathbf{c} = \operatorname{col}(c_1, \ldots, c_n)$ is any constant vector and $\mathbf{X} = [\mathbf{x}_1 \ldots \mathbf{x}_n]$ is the fundamental matrix, to obtain a general solution.

2. To determine a general solution to the nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$:

(a) Find a particular solution $\mathbf{x}_p$ to the nonhomogeneous system.

(b) Form the sum of the particular solution and the general solution $\mathbf{X}\mathbf{c} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$ to the corresponding homogeneous system in part 1,

$$\mathbf{x} = \mathbf{x}_p + \mathbf{X}\mathbf{c} = \mathbf{x}_p + c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n ,$$

to obtain a general solution to the given system.

We devote the rest of this chapter to methods for finding fundamental solution sets for homogeneous systems and particular solutions for nonhomogeneous systems.

9.4 EXERCISES

In Problems 1–4, write the given system in the matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$.

1. $x'(t) = 3x(t) - y(t) + t^2,$
$y'(t) = -x(t) + 2y(t) + e^t$

2. $r'(t) = 2r(t) + \sin t,$
$\theta'(t) = r(t) - \theta(t) + 1$

3. $\dfrac{dx}{dt} = t^2x - y - z + t,$
$\dfrac{dy}{dt} = e^t z + 5,$
$\dfrac{dz}{dt} = tx - y + 3z - e^t$

4. $\dfrac{dx}{dt} = x + y + z,$
$\dfrac{dy}{dt} = 2x - y + 3z,$
$\dfrac{dz}{dt} = x + 5z$

In Problems 5–8, rewrite the given scalar equation as a first-order system in normal form. Express the system in the matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$.

5. $y''(t) - 3y'(t) - 10y(t) = \sin t$

6. $x''(t) + x(t) = t^2$

7. $\dfrac{d^4w}{dt^4} + w = t^2$

8. $\dfrac{d^3y}{dt^3} - \dfrac{dy}{dt} + y = \cos t$

In Problems 9–12, write the given system as a set of scalar equations.

9. $\mathbf{x}' = \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}\mathbf{x} + e^{-2t}\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

10. $\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\mathbf{x} + e^{t}\begin{bmatrix} t \\ 1 \end{bmatrix}$

11. $\mathbf{x}' = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 5 \\ 0 & 5 & 1 \end{bmatrix}\mathbf{x} + e^{t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

12. $\mathbf{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}\mathbf{x} + t\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

In Problems 13–19, determine whether the given vector functions are linearly dependent (LD) or linearly independent (LI) on the interval $(-\infty, \infty)$.

13. $\begin{bmatrix} t \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

14. $\begin{bmatrix} te^{-t} \\ e^{-t} \end{bmatrix}, \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$

15. $e^t\begin{bmatrix} 1 \\ 5 \end{bmatrix}, e^t\begin{bmatrix} -3 \\ -15 \end{bmatrix}$

16. $\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}, \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}$

17. $e^{2t}\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, e^{2t}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, e^{3t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

18. $\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}, \begin{bmatrix} \sin t \\ \sin t \end{bmatrix}, \begin{bmatrix} \cos t \\ \cos t \end{bmatrix}$

19. $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}, \begin{bmatrix} t^2 \\ 0 \\ t^2 \end{bmatrix}$

20. Let

$$\mathbf{x}_1 = \begin{bmatrix} \cos t \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \sin t \\ \cos t \\ \cos t \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} \cos t \\ \sin t \\ \cos t \end{bmatrix}.$$

(a) Compute the Wronskian.
(b) Are these vector functions linearly independent on $(-\infty, \infty)$?
(c) Is there a first-order homogeneous linear system for which these functions are solutions?

In Problems 21–24, the given vector functions are solutions to a system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$. *Determine whether they form a fundamental solution set. If they do, find a fundamental matrix for the system and give a general solution.*

21. $\mathbf{x}_1 = e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \mathbf{x}_2 = e^{2t}\begin{bmatrix} -2 \\ 4 \end{bmatrix}$

22. $\mathbf{x}_1 = e^{-t}\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = e^{4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

23. $\mathbf{x}_1 = \begin{bmatrix} e^{-t} \\ 2e^{-t} \\ e^{-t} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} e^{t} \\ 0 \\ e^{t} \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} e^{3t} \\ -e^{3t} \\ 2e^{3t} \end{bmatrix}$

24. $\mathbf{x}_1 = \begin{bmatrix} e^{t} \\ e^{t} \\ e^{t} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \sin t \\ \cos t \\ -\sin t \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -\cos t \\ \sin t \\ \cos t \end{bmatrix}$

25. Verify that the vector functions

$$\mathbf{x}_1 = \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}$$

are solutions to the homogeneous system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\mathbf{x},$$

on $(-\infty, \infty)$, and that

$$\mathbf{x}_p = \frac{3}{2}\begin{bmatrix} te^{t} \\ te^{t} \end{bmatrix} - \frac{1}{4}\begin{bmatrix} e^{t} \\ 3e^{t} \end{bmatrix} + \begin{bmatrix} t \\ 2t \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is a particular solution to the nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$, where $\mathbf{f}(t) = \operatorname{col}(e^t, t)$. Find a general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$.

26. Verify that the vector functions

$$\mathbf{x}_1 = \begin{bmatrix} e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -e^{3t} \\ e^{3t} \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -e^{-3t} \\ -e^{-3t} \\ e^{-3t} \end{bmatrix}$$

are solutions to the homogeneous system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\mathbf{x}$$

on $(-\infty, \infty)$, and that

$$\mathbf{x}_p = \begin{bmatrix} 5t+1 \\ 2t \\ 4t+2 \end{bmatrix}$$

is a particular solution to $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$, where $\mathbf{f}(t) = \operatorname{col}(-9t, 0, -18t)$. Find a general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$.

27. Prove that the operator defined by $L[\mathbf{x}] := \mathbf{x}' - \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $n \times n$ matrix function and $\mathbf{x}$ is an $n \times 1$ differentiable vector function, is a linear operator.

28. Let $\mathbf{X}(t)$ be a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Show that $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0$ is the solution to the initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(t_0) = \mathbf{x}_0$.

In Problems 29–30, verify that $\mathbf{X}(t)$ is a fundamental matrix for the given system and compute $\mathbf{X}^{-1}(t)$. Use the result of Problem 28 to find the solution to the given initial value problem.

29. $\mathbf{x}' = \begin{bmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix};$

$$\mathbf{X}(t) = \begin{bmatrix} 6e^{-t} & -3e^{-2t} & 2e^{3t} \\ -e^{-t} & e^{-2t} & e^{3t} \\ -5e^{-t} & e^{-2t} & e^{3t} \end{bmatrix}$$

30. $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix};$

$$\mathbf{X}(t) = \begin{bmatrix} e^{-t} & e^{5t} \\ -e^{-t} & e^{5t} \end{bmatrix}$$

31. Show that

$$\begin{vmatrix} t^2 & t|t| \\ 2t & 2|t| \end{vmatrix} \equiv 0$$

on $(-\infty, \infty)$, but that the two vector functions

$$\begin{bmatrix} t^2 \\ 2t \end{bmatrix}, \quad \begin{bmatrix} t|t| \\ 2|t| \end{bmatrix}$$

are linearly independent on $(-\infty, \infty)$.

32. **Abel's Formula.** If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are any n solutions to the $n \times n$ system $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$, then Abel's formula gives a representation for the Wronskian $W(t) := W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t)$. Namely,

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} \{a_{11}(s) + \cdots + a_{nn}(s)\}\, ds\right),$$

where $a_{11}(s), \ldots, a_{nn}(s)$ are the main diagonal elements of $\mathbf{A}(s)$. Prove this formula in the special case when $n = 3$. [*Hint:* Follow the outline in Problem 30 of Exercises 6.1, page 327.]

33. Using Abel's formula (Problem 32), confirm that the Wronskian of n solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ on the interval I is either identically zero on I or never zero on I.

34. Prove that a fundamental solution set for the homogeneous system $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$ always exists on an interval I, provided $\mathbf{A}(t)$ is continuous on I. [*Hint:* Use the existence and uniqueness theorem (Theorem 2) and make judicious choices for $\mathbf{x}_0$.]

35. Prove Theorem 3 on the representation of solutions of the homogeneous system.

36. Prove Theorem 4 on the representation of solutions of the nonhomogeneous system.

37. To illustrate the connection between a higher-order equation and the equivalent first-order system, consider the equation

$$y'''(t) - 6y''(t) + 11y'(t) - 6y(t) = 0\,. \tag{11}$$

(a) Show that $\{e^t, e^{2t}, e^{3t}\}$ is a fundamental solution set for (11).

(b) Using the definition in Section 6.1, compute the Wronskian of $\{e^t, e^{2t}, e^{3t}\}$.

(c) Setting $x_1 = y, x_2 = y', x_3 = y''$, show that equation (11) is equivalent to the first-order system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}\,, \tag{12}$$

where

$$\mathbf{A} := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}.$$

(d) The substitution used in part (c) suggests that

$$S := \left\{ \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix}, \begin{bmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{bmatrix}, \begin{bmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{bmatrix} \right\}$$

is a fundamental solution set for system (12). Verify that this is the case.

(e) Compute the Wronskian of S. How does it compare with the Wronskian computed in part (b)?

38. Define $\mathbf{x}_1(t), \mathbf{x}_2(t)$, and $\mathbf{x}_3(t)$, for $-\infty < t < \infty$, by

$$\mathbf{x}_1(t) = \begin{bmatrix} \sin t \\ \sin t \\ 0 \end{bmatrix}, \quad \mathbf{x}_2(t) = \begin{bmatrix} \sin t \\ 0 \\ \sin t \end{bmatrix}, \quad \mathbf{x}_3(t) = \begin{bmatrix} 0 \\ \sin t \\ \sin t \end{bmatrix}.$$

(a) Show that for the three scalar functions *in each individual row* there are nontrivial linear combinations that sum to zero for all t.

(b) Show that, nonetheless, the three vector functions are linearly independent. (No single nontrivial combination works *for each row, for all t*.)

(c) Calculate the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](t)$.

(d) Is there a linear third-order homogeneous differential equation system having $\mathbf{x}_1(t), \mathbf{x}_2(t)$, and $\mathbf{x}_3(t)$ as solutions?

9.5 Homogeneous Linear Systems with Constant Coefficients

In this section we discuss a procedure for obtaining a general solution for the homogeneous system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)\,, \tag{1}$$

where $\mathbf{A}$ is a (real) *constant* $n \times n$ matrix. The general solution we seek will be defined for all t because the elements of $\mathbf{A}$ are just constant functions, which are continuous on $(-\infty, \infty)$ (recall Theorem 2, page 516). In Section 9.4 we showed that a general solution to (1) can be constructed from a fundamental solution set consisting of n linearly independent solutions to (1). Thus our goal is to find n such vector solutions.

In Chapter 4 we were successful in solving homogeneous linear equations with constant coefficients by guessing that the equation had a solution of the form e^{rt}. Because any scalar linear equation can be expressed as a system, it is reasonable to expect system (1) to have solutions of the form

$$\mathbf{x}(t) = e^{rt}\mathbf{u}\,,$$

where r is a constant and $\mathbf{u}$ is a constant vector, both of which must be determined. Substituting $e^{rt}\mathbf{u}$ for $\mathbf{x}(t)$ in (1) gives

$$re^{rt}\mathbf{u} = \mathbf{A}e^{rt}\mathbf{u} = e^{rt}\mathbf{A}\mathbf{u}\,.$$

Canceling the factor e^{rt} and rearranging terms, we find that

$$(\mathbf{A} - r\mathbf{I})\mathbf{u} = \mathbf{0}\,, \tag{2}$$

where $r\mathbf{I}$ denotes the diagonal matrix with r's along its main diagonal.

The preceding calculation shows that $\mathbf{x}(t) = e^{rt}\mathbf{u}$ is a solution to (1) if and only if r and $\mathbf{u}$ satisfy equation (2). Since the trivial case, $\mathbf{u} = \mathbf{0}$, is of no help in finding linearly independent solutions to (1), we require that $\mathbf{u} \neq \mathbf{0}$. Such vectors are given a special name, as follows.

Eigenvalues and Eigenvectors

Definition 3. Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ constant matrix. The **eigenvalues** of $\mathbf{A}$ are those (real or complex) numbers r for which $(\mathbf{A} - r\mathbf{I})\mathbf{u} = \mathbf{0}$ has at least one nontrivial (real or complex) solution $\mathbf{u}$. The corresponding nontrivial solutions $\mathbf{u}$ are called the **eigenvectors** of $\mathbf{A}$ associated with r.

As stated in Theorem 1 of Section 9.3, a linear homogeneous system of n *algebraic* equations in n unknowns has a nontrivial solution if and only if the determinant of its coefficients is zero. Hence, a necessary and sufficient condition for (2) to have a nontrivial solution is that

$$|\mathbf{A} - r\mathbf{I}| = 0\,. \tag{3}$$

Expanding the determinant of $\mathbf{A} - r\mathbf{I}$ in terms of its cofactors, we find that it is an nth-degree polynomial in r; that is,

$$|\mathbf{A} - r\mathbf{I}| = p(r)\,. \tag{4}$$

Therefore, *finding the eigenvalues of a matrix* $\mathbf{A}$ *is equivalent to finding the zeros of the polynomial* $p(r)$. Equation (3) is called the **characteristic equation** of $\mathbf{A}$, and $p(r)$ in (4) is the **characteristic polynomial** of $\mathbf{A}$. The characteristic equation plays a role for systems similar to the role played by the auxiliary equation for scalar equations.

Many commercially available software packages can be used to compute the eigenvalues and eigenvectors for a given matrix. Three such packages are MATLAB®, available from The MathWorks, Inc.; MATHEMATICA®, available from Wolfram Research; and MAPLESOFT®, available from Waterloo Maple Inc. Although you are encouraged to make use of such packages, the examples and most exercises in this text can be easily carried out without them. Those exercises for which a computer package is desirable are flagged with the icon .

Example 1 Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} := \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}.$$

Solution The characteristic equation for $\mathbf{A}$ is

$$|\mathbf{A} - r\mathbf{I}| = \begin{vmatrix} 2-r & -3 \\ 1 & -2-r \end{vmatrix} = (2-r)(-2-r) + 3 = r^2 - 1 = 0\,.$$

Hence the eigenvalues of $\mathbf{A}$ are $r_1 = 1$, $r_2 = -1$. To find the eigenvectors corresponding to $r_1 = 1$, we must solve $(\mathbf{A} - r_1\mathbf{I})\mathbf{u} = \mathbf{0}$. Substituting for $\mathbf{A}$ and r_1 gives

$$\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5}$$

Notice that this matrix equation is equivalent to the single scalar equation $u_1 - 3u_2 = 0$. Therefore, the solutions to (5) are obtained by assigning an arbitrary value for u_2 (say, $u_2 = s$) and setting $u_1 = 3u_2 = 3s$. Consequently, the eigenvectors associated with $r_1 = 1$ can be expressed as

$$\mathbf{u}_1 = s\begin{bmatrix} 3 \\ 1 \end{bmatrix}. \tag{6}$$

For $r_2 = -1$, the equation $(\mathbf{A} - r_2\mathbf{I})\mathbf{u} = \mathbf{0}$ becomes

$$\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving, we obtain $u_1 = s$ and $u_2 = s$, with s arbitrary. Therefore, the eigenvectors associated with the eigenvalue $r_2 = -1$ are

$$\mathbf{u}_2 = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{7}$$ ◆

We remark that in the above example the collection (6) of all eigenvectors associated with $r_1 = 1$ forms a one-dimensional subspace when the zero vector is adjoined. The same is true for $r_2 = -1$. These subspaces are called **eigenspaces.**

Example 2 Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} := \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

Solution The characteristic equation for $\mathbf{A}$ is

$$|\mathbf{A} - r\mathbf{I}| = \begin{vmatrix} 1-r & 2 & -1 \\ 1 & -r & 1 \\ 4 & -4 & 5-r \end{vmatrix} = 0\,,$$

which simplifies to $(r-1)(r-2)(r-3) = 0$. Hence, the eigenvalues of $\mathbf{A}$ are $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$. To find the eigenvectors corresponding to $r_1 = 1$, we set $r = 1$ in $(\mathbf{A} - r\mathbf{I})\mathbf{u} = \mathbf{0}$. This gives

$$\begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{8}$$

Using elementary row operations (Gaussian elimination), we see that (8) is equivalent to the two equations

$$\begin{aligned} u_1 - u_2 + u_3 &= 0\,, \\ 2u_2 - u_3 &= 0\,. \end{aligned}$$

Thus, we can obtain the solutions to (8) by assigning an arbitrary value to u_2 (say, $u_2 = s$), solving $2u_2 - u_3 = 0$ for u_3 to get $u_3 = 2s$, and then solving $u_1 - u_2 + u_3 = 0$ for u_1 to get $u_1 = -s$. Hence, the eigenvectors associated with $r_1 = 1$ are

$$\mathbf{u}_1 = s\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}. \tag{9}$$

For $r_2 = 2$, we solve

$$\begin{bmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 4 & -4 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

in a similar fashion to obtain the eigenvectors

$$\mathbf{u}_2 = s\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}. \tag{10}$$

Finally, for $r_3 = 3$, we solve

$$\begin{bmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and get the eigenvectors

$$\mathbf{u}_3 = s\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}. \tag{11}$$ ◆

Let's return to the problem of finding a general solution to a homogeneous system of differential equations. We have already shown that $e^{rt}\mathbf{u}$ is a solution to (1) if r is an eigenvalue and $\mathbf{u}$ a corresponding eigenvector. The question is: Can we obtain n linearly independent solutions to the homogeneous system by finding all the eigenvalues and eigenvectors of $\mathbf{A}$? The answer is yes, *if* $\mathbf{A}$ *has n linearly independent eigenvectors.*

n Linearly Independent Eigenvectors

Theorem 5. Suppose the $n \times n$ constant matrix $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Let r_i be the eigenvalue† corresponding to $\mathbf{u}_i$. Then

(12) $$\{e^{r_1 t}\mathbf{u}_1, e^{r_2 t}\mathbf{u}_2, \ldots, e^{r_n t}\mathbf{u}_n\}$$

is a fundamental solution set (and $\mathbf{X}(t) = [e^{r_1 t}\mathbf{u}_1\ e^{r_2 t}\mathbf{u}_2\ \cdots\ e^{r_n t}\mathbf{u}_n]$ is a fundamental matrix) on $(-\infty, \infty)$ for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Consequently, a general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

(13) $$\mathbf{x}(t) = c_1 e^{r_1 t}\mathbf{u}_1 + c_2 e^{r_2 t}\mathbf{u}_2 + \cdots + c_n e^{r_n t}\mathbf{u}_n ,$$

where $c_1, \ldots, c_n$ are arbitrary constants.

Proof. As we have seen, the vector functions listed in (12) are solutions to the homogeneous system. Moreover, their Wronskian is

$$W(t) = \det[e^{r_1 t}\mathbf{u}_1, \ldots, e^{r_n t}\mathbf{u}_n] = e^{(r_1 + \cdots + r_n)t}\det[\mathbf{u}_1, \ldots, \mathbf{u}_n] .$$

Since the eigenvectors are assumed to be linearly independent, it follows from Theorem 1 in Section 9.3 that $\det[\mathbf{u}_1, \ldots, \mathbf{u}_n]$ is not zero. Hence the Wronskian $W(t)$ is never zero. This shows that (12) is a fundamental solution set, and consequently a general solution is given by (13). ◆

An application of Theorem 5 is given in the next example.

Example 3 Find a general solution of

(14) $$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) , \text{ where } \mathbf{A} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} .$$

Solution In Example 1 we showed that the matrix $\mathbf{A}$ has eigenvalues $r_1 = 1$ and $r_2 = -1$. Taking, say, $s = 1$ in equations (6) and (7), we get the corresponding eigenvectors

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

Because $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent, it follows from Theorem 5 that a general solution to (14) is

(15) $$\mathbf{x}(t) = c_1 e^{t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$ ◆

†The eigenvalues $r_1, \ldots, r_n$ may be real or complex and need not be distinct. In this section the cases we discuss have real eigenvalues. We consider complex eigenvalues in Section 9.6.

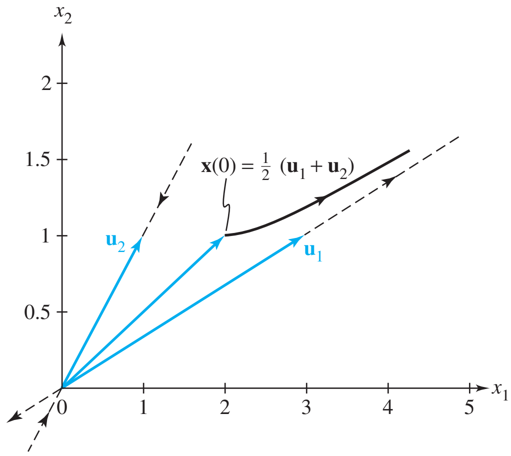

Figure 9.1 Trajectories of solutions for Example 3

If we sum the vectors on the right-hand side of equation (15) and then write out the expressions for the components of $\mathbf{x}(t) = \operatorname{col}\left(x_1(t), x_2(t)\right)$, we get

$$x_1(t) = 3c_1e^t + c_2e^{-t},$$
$$x_2(t) = c_1e^t + c_2e^{-t}.$$

This is the familiar form of a general solution for a system, as discussed in Section 5.2.

Example 3 nicely illustrates the geometric role played by the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. If the initial vector $\mathbf{x}(0)$ is a scalar multiple of $\mathbf{u}_1$ (i.e., $\mathbf{x}(0) = c_1\mathbf{u}_1$), then the vector solution to the system, $\mathbf{x}(t) = c_1e^t\mathbf{u}_1$, will always have the same or opposite direction as $\mathbf{u}_1$. That is, it will lie along the straight line determined by $\mathbf{u}_1$ (see Figure 9.1). Furthermore, the trajectory of this solution, as t increases, will tend to infinity, since the corresponding eigenvalue $r_1 = 1$ is positive (observe the e^t term). A similar assertion holds if the initial vector is a scalar multiple of $\mathbf{u}_2$, except that since $r_2 = -1$ is negative, the trajectory $\mathbf{x}(t) = c_2e^{-t}\mathbf{u}_2$ will approach the origin as t increases (because of e^{-t}). For an initial vector that involves both $\mathbf{u}_1$ and $\mathbf{u}_2$, such as $\mathbf{x}(0) = \frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2)$, the resulting trajectory is a blend of the above motions, with the contribution due to the larger eigenvalue $r_1 = 1$ dominating as t increases; see Figure 9.1.

The straight-line trajectories in the x_1x_2-plane (the phase plane), then, point along the directions of the eigenvectors of the matrix $\mathbf{A}$. (See Section 5.4, Figure 5.11, page 266, for example.)

A useful property of eigenvectors that concerns their linear independence is stated in the next theorem.

Linear Independence of Eigenvectors

Theorem 6. If $r_1, \ldots, r_m$ are *distinct* eigenvalues for the matrix $\mathbf{A}$ and $\mathbf{u}_i$ is an eigenvector associated with r_i, then $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linearly independent.

Proof. Let's first treat the case $m = 2$. Suppose, to the contrary, that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly dependent so that

(16) $\quad \mathbf{u}_1 = c\mathbf{u}_2$

for some constant c. Multiplying both sides of (16) by $\mathbf{A}$ and using the fact that $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigenvectors with corresponding eigenvalues r_1 and r_2, we obtain

$$r_1\mathbf{u}_1 = cr_2\mathbf{u}_2 . \tag{17}$$

Next we multiply (16) by r_2 and then subtract from (17) to get

$$(r_1 - r_2)\mathbf{u}_1 = \mathbf{0} .$$

Since $\mathbf{u}_1$ is not the zero vector, we must have $r_1 = r_2$. But this violates the assumption that the eigenvalues are distinct! Hence $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent.

The cases $m > 2$ follow by induction. The details of the proof are left as Problem 48. ◆

Combining Theorems 5 and 6, we get the following corollary.

n Distinct Eigenvalues

Corollary 1. If the $n \times n$ constant matrix $\mathbf{A}$ has n distinct eigenvalues $r_1, \ldots, r_n$ and $\mathbf{u}_i$ is an eigenvector associated with r_i, then

$$\{e^{r_1 t}\mathbf{u}_1, \ldots, e^{r_n t}\mathbf{u}_n\}$$

is a fundamental solution set for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Example 4 Solve the initial value problem

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}\mathbf{x}(t) , \qquad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} . \tag{18}$$

Solution In Example 2 we showed that the 3×3 coefficient matrix $\mathbf{A}$ has the three distinct eigenvalues $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$. If we set $s = 1$ in equations (9), (10), and (11), we obtain the corresponding eigenvectors

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \qquad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix},$$

whose linear independence is guaranteed by Theorem 6. Hence, a general solution to (18) is

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{t}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -e^{t} & -2e^{2t} & -e^{3t} \\ e^{t} & e^{2t} & e^{3t} \\ 2e^{t} & 4e^{2t} & 4e^{3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} . \end{aligned} \tag{19}$$

To satisfy the initial condition in (18), we solve

$$\mathbf{x}(0) = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

and find that $c_1 = 0$, $c_2 = 1$, and $c_3 = -1$. Inserting these values into (19) gives the desired solution. ◆

There is a special class of $n \times n$ matrices that *always* have n linearly independent eigenvectors. These are the real symmetric matrices.

Real Symmetric Matrices

Definition 4. A **real symmetric matrix A** is a matrix with real entries that satisfies $\mathbf{A}^T = \mathbf{A}$.

Taking the transpose of a matrix interchanges its rows and columns. Doing this is equivalent to "flipping" the matrix about its main diagonal. Consequently, $\mathbf{A}^T = \mathbf{A}$ if and only if $\mathbf{A}$ is symmetric about its main diagonal.

If $\mathbf{A}$ is an $n \times n$ real symmetric matrix, it is known† that all its eigenvalues are real and that there always exist n linearly independent eigenvectors. In such a case, Theorem 5 applies and a general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is given by (13).

Example 5 Find a general solution of

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)\ , \text{ where } \mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} . \tag{20}$$

Solution $\mathbf{A}$ is symmetric, so we are assured that $\mathbf{A}$ has three linearly independent eigenvectors. To find them, we first compute the characteristic equation for $\mathbf{A}$:

$$|\mathbf{A} - r\mathbf{I}| = \begin{vmatrix} 1-r & -2 & 2 \\ -2 & 1-r & 2 \\ 2 & 2 & 1-r \end{vmatrix} = -(r-3)^2(r+3) = 0\ .$$

Thus the eigenvalues of $\mathbf{A}$ are $r_1 = r_2 = 3$ and $r_3 = -3$.

Notice that the eigenvalue $r = 3$ has multiplicity 2 when considered as a root of the characteristic equation. Therefore, we must find *two* linearly independent eigenvectors associated with $r = 3$. Substituting $r = 3$ in $(\mathbf{A} - r\mathbf{I})\mathbf{u} = \mathbf{0}$ gives

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -2 & 2 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} .$$

This system is equivalent to the single equation $-u_1 - u_2 + u_3 = 0$, so we can obtain its solutions by assigning an arbitrary value to u_2, say $u_2 = v$, and an arbitrary value to u_3, say $u_3 = s$. Solving for u_1, we find $u_1 = u_3 - u_2 = s - v$. Therefore, the eigenvectors associated with $r_1 = r_2 = 3$ can be expressed as

$$\mathbf{u} = \begin{bmatrix} s - v \\ v \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + v \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} .$$

†See *Fundamentals of Matrix Analysis with Applications*, by Edward Barry Saff and Arthur David Snider (John Wiley & Sons, Hoboken, New Jersey, 2016).

By first taking $s = 1, v = 0$ and then taking $s = 0, v = 1$, we get the two linearly independent eigenvectors

$$\textbf{(21)} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

For $r_3 = -3$, we solve

$$(\mathbf{A} + 3\mathbf{I})\mathbf{u} = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

to obtain the eigenvectors $\operatorname{col}(-s, -s, s)$. Taking $s = 1$ gives

$$\mathbf{u}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Since the eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent, a general solution to (20) is

$$\mathbf{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}. \quad \blacklozenge$$

If a matrix **A** *is not symmetric, it is possible for* **A** *to have a repeated eigenvalue but not to have two linearly independent corresponding eigenvectors.* In particular, the matrix

$$\textbf{(22)} \qquad \mathbf{A} = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$$

has the repeated eigenvalue $r_1 = r_2 = -1$, but Problem 35 shows that all the eigenvectors associated with $r = -1$ are of the form $\mathbf{u} = s \operatorname{col}(1, 2)$. Consequently, no two eigenvectors are linearly independent.

A procedure for finding a general solution in such a case is illustrated in Problems 35–40, but the underlying theory is deferred to Section 9.8, where we discuss the matrix exponential.

A final note. If an $n \times n$ matrix **A** has n linearly independent eigenvectors $\mathbf{u}_i$ with eigenvalues r_i, a little inspection reveals that property (2) is expressed columnwise by the equation

$$\textbf{(23)} \qquad \begin{bmatrix} & & \\ & \mathbf{A} & \\ & & \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & r_n \end{bmatrix}$$

or $\mathbf{AU} = \mathbf{UD}$, where **U** is the matrix whose column vectors are eigenvectors and **D** is a diagonal matrix whose diagonal entries are the eigenvalues. Since **U**'s columns are independent, **U** is invertible and we can write

$$\textbf{(24)} \qquad \mathbf{A} = \mathbf{UDU}^{-1} \text{ or } \mathbf{D} = \mathbf{U}^{-1}\mathbf{AU},$$

and we say that **A** is *diagonalizable*. [In this context equation (24) expresses a *similarity transformation*.] Because the argument that leads from (2) to (23) to (24) can be reversed, we have a new characterization: *An $n \times n$ matrix has n linearly independent eigenvectors if, and only if, it is diagonalizable.*

9.5 EXERCISES

In Problems 1–8, find the eigenvalues and eigenvectors of the given matrix.

1. $\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ **2.** $\begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & 5 \\ 1 & -3 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$ **6.** $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix}$ **8.** $\begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 4 & -8 & 2 \end{bmatrix}$

In Problems 9 and 10, some of the eigenvalues of the given matrix are complex. Find all the eigenvalues and eigenvectors.

9. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

In Problems 11–16, find a general solution of the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ for the given matrix $\mathbf{A}$.

11. $\mathbf{A} = \begin{bmatrix} -1 & \frac{3}{4} \\ -5 & 3 \end{bmatrix}$ **12.** $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 12 & 1 \end{bmatrix}$

13. $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$ **14.** $\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix}$

15. $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix}$ **16.** $\mathbf{A} = \begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix}$

17. Consider the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, $t \geq 0$, with

$$\mathbf{A} = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}.$$

(a) Show that the matrix $\mathbf{A}$ has eigenvalues $r_1 = 2$ and $r_2 = -2$ with corresponding eigenvectors $\mathbf{u}_1 = \text{col}(\sqrt{3}, 1)$ and $\mathbf{u}_2 = \text{col}(1, -\sqrt{3})$.
(b) Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = -\mathbf{u}_1$.
(c) Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = \mathbf{u}_2$.
(d) Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = \mathbf{u}_2 - \mathbf{u}_1$.

18. Consider the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, $t \geq 0$, with

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

(a) Show that the matrix $\mathbf{A}$ has eigenvalues $r_1 = -1$ and $r_2 = -3$ with corresponding eigenvectors $\mathbf{u}_1 = \text{col}(1, 1)$ and $\mathbf{u}_2 = \text{col}(1, -1)$.
(b) Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = \mathbf{u}_1$.
(c) Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = -\mathbf{u}_2$.
(d) Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = \mathbf{u}_1 - \mathbf{u}_2$.

In Problems 19–24, find a fundamental matrix for the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ for the given matrix $\mathbf{A}$.

19. $\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 8 & 1 \end{bmatrix}$ **20.** $\mathbf{A} = \begin{bmatrix} 5 & 4 \\ -1 & 0 \end{bmatrix}$

21. $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -14 & 7 \end{bmatrix}$ **22.** $\mathbf{A} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix}$

23. $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

24. $\mathbf{A} = \begin{bmatrix} 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

25. Using matrix algebra techniques, find a general solution of the system

$$\begin{aligned} x' &= x + 2y - z\,, \\ y' &= x + z\,, \\ z' &= 4x - 4y + 5z\,. \end{aligned}$$

26. Using matrix algebra techniques, find a general solution of the system

$$\begin{aligned} x' &= 3x - 4y\,, \\ y' &= 4x - 7y\,. \end{aligned}$$

In Problems 27–30, use a linear algebra software package such as MATLAB®, MAPLESOFT®, *or* MATHEMATICA® *to compute the required eigenvalues and eigenvectors and then give a fundamental matrix for the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ for the given matrix $\mathbf{A}$.*

27. $\mathbf{A} = \begin{bmatrix} 0 & 1.1 & 0 \\ 0 & 0 & 1.3 \\ 0.9 & 1.1 & -6.9 \end{bmatrix}$ **28.** $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & 0 \\ 3 & 3 & 3 \end{bmatrix}$

29. $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & -6 & 3 & 3 \end{bmatrix}$

30. $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$

In Problems 31–34, solve the given initial value problem.

31. $\mathbf{x}'(t) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \mathbf{x}(t), \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

32. $\mathbf{x}'(t) = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} \mathbf{x}(t), \quad \mathbf{x}(0) = \begin{bmatrix} -10 \\ -6 \end{bmatrix}$

33. $\mathbf{x}'(t) = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \mathbf{x}(t), \quad \mathbf{x}(0) = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$

34. $\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}(t), \quad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$

35. **(a)** Show that the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$$

has the repeated eigenvalue $r = -1$ and that all the eigenvectors are of the form $\mathbf{u} = s\,\mathrm{col}(1, 2)$.

(b) Use the result of part (a) to obtain a nontrivial solution $\mathbf{x}_1(t)$ to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(c) To obtain a second linearly independent solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, try $\mathbf{x}_2(t) = te^{-t}\mathbf{u}_1 + e^{-t}\mathbf{u}_2$. [*Hint:* Substitute $\mathbf{x}_2$ into the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and derive the relations

$$(\mathbf{A} + \mathbf{I})\mathbf{u}_1 = \mathbf{0}, \qquad (\mathbf{A} + \mathbf{I})\mathbf{u}_2 = \mathbf{u}_1 .$$

Since $\mathbf{u}_1$ must be an eigenvector, set $\mathbf{u}_1 = \mathrm{col}(1, 2)$ and solve for $\mathbf{u}_2$.]

(d) What is $(\mathbf{A} + \mathbf{I})^2\mathbf{u}_2$? (In Section 9.8, $\mathbf{u}_2$ will be identified as a *generalized eigenvector*.)

36. Use the method discussed in Problem 35 to find a general solution to the system

$$\mathbf{x}'(t) = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix} \mathbf{x}(t) .$$

37. **(a)** Show that the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

has the repeated eigenvalue $r = 2$ with multiplicity 3 and that all the eigenvectors of $\mathbf{A}$ are of the form $\mathbf{u} = s\,\mathrm{col}(1, 0, 0)$.

(b) Use the result of part (a) to obtain a solution to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ of the form $\mathbf{x}_1(t) = e^{2t}\mathbf{u}_1$.

(c) To obtain a second linearly independent solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, try $\mathbf{x}_2(t) = te^{2t}\mathbf{u}_1 + e^{2t}\mathbf{u}_2$. [*Hint:* Show that $\mathbf{u}_1$ and $\mathbf{u}_2$ must satisfy

$$(\mathbf{A} - 2\mathbf{I})\mathbf{u}_1 = \mathbf{0}, \qquad (\mathbf{A} - 2\mathbf{I})\mathbf{u}_2 = \mathbf{u}_1.]$$

(d) To obtain a third linearly independent solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, try

$$\mathbf{x}_3(t) = \frac{t^2}{2}e^{2t}\mathbf{u}_1 + te^{2t}\mathbf{u}_2 + e^{2t}\mathbf{u}_3 .$$

[*Hint:* Show that $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ must satisfy

$$(\mathbf{A} - 2\mathbf{I})\mathbf{u}_1 = \mathbf{0}, \qquad (\mathbf{A} - 2\mathbf{I})\mathbf{u}_2 = \mathbf{u}_1,$$
$$(\mathbf{A} - 2\mathbf{I})\mathbf{u}_3 = \mathbf{u}_2.]$$

(e) Show that $(\mathbf{A} - 2\mathbf{I})^2\mathbf{u}_2 = (\mathbf{A} - 2\mathbf{I})^3\mathbf{u}_3 = \mathbf{0}$.

38. Use the method discussed in Problem 37 to find a general solution to the system

$$\mathbf{x}'(t) = \begin{bmatrix} 3 & -2 & 1 \\ 2 & -1 & 1 \\ -4 & 4 & 1 \end{bmatrix} \mathbf{x}(t).$$

39. **(a)** Show that the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$$

has the repeated eigenvalue $r = 1$ of multiplicity 3 and that all the eigenvectors of $\mathbf{A}$ are of the form $\mathbf{u} = s\,\mathrm{col}(-1, 1, 0) + v\,\mathrm{col}(-1, 0, 1)$.

(b) Use the result of part (a) to obtain two linearly independent solutions to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ of the form

$$\mathbf{x}_1(t) = e^t\mathbf{u}_1 \quad \text{and} \quad \mathbf{x}_2(t) = e^t\mathbf{u}_2.$$

(c) To obtain a third linearly independent solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, try $\mathbf{x}_3(t) = te^t\mathbf{u}_3 + e^t\mathbf{u}_4$. [*Hint:* Show that $\mathbf{u}_3$ and $\mathbf{u}_4$ must satisfy

$$(\mathbf{A} - \mathbf{I})\mathbf{u}_3 = \mathbf{0}, \qquad (\mathbf{A} - \mathbf{I})\mathbf{u}_4 = \mathbf{u}_3 .$$

Choose $\mathbf{u}_3$, an eigenvector of $\mathbf{A}$, so that you can solve for $\mathbf{u}_4$.]

(d) What is $(\mathbf{A} - \mathbf{I})^2\mathbf{u}_4$?

40. Use the method discussed in Problem 39 to find a general solution to the system

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 7 & -4 \\ 0 & 9 & -5 \end{bmatrix} \mathbf{x}(t) .$$

41. Use the substitution $x_1 = y$, $x_2 = y'$ to convert the linear equation $ay'' + by' + cy = 0$, where a, b, and c are constants, into a normal system. Show that the characteristic equation for this system is the same as the auxiliary equation for the original equation.

42. **(a)** Show that the Cauchy–Euler equation $at^2y'' + bty' + cy = 0$ can be written as a **Cauchy–Euler system**

$$tx' = Ax \tag{25}$$

with a constant coefficient matrix $\mathbf{A}$, by setting $x_1 = y/t$ and $x_2 = y'$.

(b) Show that for $t > 0$ any system of the form (25) with $\mathbf{A}$ an $n \times n$ constant matrix has nontrivial solutions of the form $\mathbf{x}(t) = t^r\mathbf{u}$ if and only if r is an eigenvalue of $\mathbf{A}$ and $\mathbf{u}$ is a corresponding eigenvector.

In Problems 43 and 44, use the result of Problem 42 to find a general solution of the given system.

43. $t\mathbf{x}'(t) = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}\mathbf{x}(t)\,, \quad t > 0$

44. $t\mathbf{x}'(t) = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\mathbf{x}(t)\,, \quad t > 0$

45. **Mixing Between Interconnected Tanks.** Two tanks, each holding 50 L of liquid, are interconnected by pipes with liquid flowing from tank A into tank B at a rate of 4 L/min and from tank B into tank A at 1 L/min (see Figure 9.2). The liquid inside each tank is kept well stirred. Pure water flows into tank A at a rate of 3 L/min, and the solution flows out of tank B at 3 L/min. If, initially, tank A contains 2.5 kg of salt and tank B contains no salt (only water), determine the mass of salt in each tank at time $t \geq 0$. Graph on the same axes the two quantities $x_1(t)$ and $x_2(t)$, where $x_1(t)$ is the mass of salt in tank A and $x_2(t)$ is the mass in tank B.

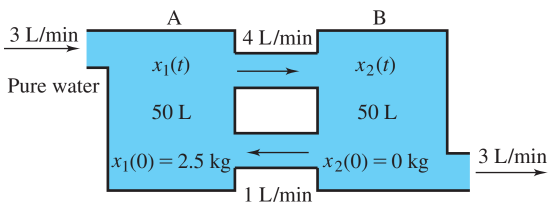

Figure 9.2 Mixing problem for interconnected tanks

46. **Mixing with a Common Drain.** Two tanks, each holding 1 L of liquid, are connected by a pipe through which liquid flows from tank A into tank B at a rate of $3 - \alpha$ L/min $(0 < \alpha < 3)$. The liquid inside each tank is kept well stirred. Pure water flows into tank A at a rate of 3 L/min. Solution flows out of tank A at α L/min and out of tank B at $3 - \alpha$ L/min. If, initially, tank B contains no salt (only water) and tank A contains 0.1 kg of salt, determine the mass of salt in each tank at time $t \geq 0$. How does the mass of salt in tank A depend on the choice of α? What is the maximum mass of salt in tank B? (See Figure 9.3.)

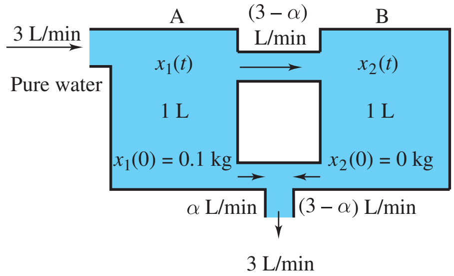

Figure 9.3 Mixing problem for a common drain, $0 < \alpha < 3$

47. To find a general solution to the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}\mathbf{x}\,,$$

proceed as follows:

(a) Use a numerical root-finding procedure to approximate the eigenvalues.

(b) If r is an eigenvalue, then let $\mathbf{u} = \text{col}\,(u_1, u_2, u_3)$ be an eigenvector associated with r. To solve for $\mathbf{u}$, assume $u_1 = 1$. (If not u_1, then either u_2 or u_3 may be chosen to be 1. Why?) Now solve the system

$$(\mathbf{A} - r\mathbf{I})\begin{bmatrix} 1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for u_2 and u_3. Use this procedure to find approximations for three linearly independent eigenvectors for $\mathbf{A}$.

(c) Use these approximations to give a general solution to the system.

48. To complete the proof of Theorem 6, page 527, assume the induction hypothesis that $\mathbf{u}_1, \ldots, \mathbf{u}_k$, $2 \leq k$, are linearly independent.

(a) Show that if

$$c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k + c_{k+1}\mathbf{u}_{k+1} = \mathbf{0}\,,$$

then

$$c_1(r_1 - r_{k+1})\mathbf{u}_1 + \cdots + c_k(r_k - r_{k+1})\mathbf{u}_k = \mathbf{0}\,.$$

(b) Use the result of part (a) and the induction hypothesis to conclude that $\mathbf{u}_1, \ldots, \mathbf{u}_{k+1}$ are linearly independent. The theorem follows by induction.

49. **Stability.** A homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with constant coefficients is **stable** if it has a fundamental matrix whose entries all remain bounded as $t \to +\infty$. (It will follow from Theorem 9 in Section 9.8 that if one fundamental matrix of the system has this property, then all fundamental matrices for the system do.) Otherwise, the system is **unstable**. A stable system is **asymptotically stable** if all solutions approach the zero solution as $t \to +\infty$. Stability is discussed in more detail in Chapter 12.[†]

[†]All references to Chapters 11–13 refer to the expanded text, *Fundamentals of Differential Equations and Boundary Value Problems*, 7th ed.

(a) Show that if $\mathbf{A}$ has all distinct real eigenvalues, then $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ is stable if and only if all eigenvalues are nonpositive.
(b) Show that if $\mathbf{A}$ has all distinct real eigenvalues, then $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ is asymptotically stable if and only if all eigenvalues are negative.
(c) Argue that in parts (a) and (b), we can replace "has distinct real eigenvalues" by "is symmetric" and the statements are still true.

50. In an ice tray, the water level in any particular ice cube cell will change at a rate proportional to the *difference* between that cell's water level and the level in the adjacent cells.

(a) Argue that a reasonable differentiable equation model for the water levels x, y, and z in the simplified three-cell tray depicted in Figure 9.4 is given by

$$x' = y - x\,, \quad y' = x + z - 2y\,, \quad z' = y - z\,.$$

(b) Use eigenvectors to solve this system for the initial conditions $x(0) = 3$, $y(0) = z(0) = 0$.

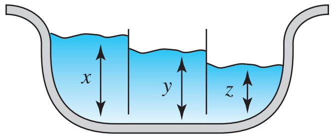

Figure 9.4 Ice tray

9.6 Complex Eigenvalues

In the previous section, we showed that the homogeneous system

$$\text{(1)} \qquad \mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)\,,$$

where $\mathbf{A}$ is a constant $n \times n$ matrix, has a solution of the form $\mathbf{x}(t) = e^{rt}\mathbf{u}$ if and only if r is an eigenvalue of $\mathbf{A}$ and $\mathbf{u}$ is a corresponding eigenvector. In this section we show how to obtain two real vector solutions to system (1) when $\mathbf{A}$ is real and has a pair† of complex conjugate eigenvalues $\alpha + i\beta$ and $\alpha - i\beta$.

Suppose $r_1 = \alpha + i\beta(\alpha$ and β real numbers) is an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{z} = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are real constant vectors. We first observe that the complex conjugate of $\mathbf{z}$, namely $\bar{\mathbf{z}} := \mathbf{a} - i\mathbf{b}$, is an eigenvector associated with the eigenvalue $r_2 = \alpha - i\beta$. To see this, note that taking the complex conjugate of $(\mathbf{A} - r_1\mathbf{I})\mathbf{z} = \mathbf{0}$ yields $(\mathbf{A} - \bar{r}_1\mathbf{I})\bar{\mathbf{z}} = \mathbf{0}$ because the conjugate of the product is the product of the conjugates and $\mathbf{A}$ and $\mathbf{I}$ have real entries $(\bar{\mathbf{A}} = \mathbf{A}, \bar{\mathbf{I}} = \mathbf{I})$. Since $r_2 = \bar{r}_1$, we see that $\bar{\mathbf{z}}$ is an eigenvector associated with r_2. Therefore, two linearly independent complex vector solutions to (1) are

$$\text{(2)} \qquad \mathbf{w}_1(t) = e^{r_1 t}\mathbf{z} = e^{(\alpha + i\beta)t}(\mathbf{a} + i\mathbf{b})\,,$$

$$\text{(3)} \qquad \mathbf{w}_2(t) = e^{r_2 t}\bar{\mathbf{z}} = e^{(\alpha - i\beta)t}(\mathbf{a} - i\mathbf{b})\,.$$

As in Section 4.3, where we handled complex roots to the auxiliary equation, let's use one of these complex solutions and Euler's formula to obtain two real vector solutions. With the aid of Euler's formula, we rewrite $\mathbf{w}_1(t)$ as

$$\begin{aligned}\mathbf{w}_1(t) &= e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{a} + i\mathbf{b})\\ &= e^{\alpha t}\{(\cos\beta t\,\mathbf{a} - \sin\beta t\,\mathbf{b}) + i(\sin\beta t\,\mathbf{a} + \cos\beta t\,\mathbf{b})\}\,.\end{aligned}$$

We have thereby expressed $\mathbf{w}_1(t)$ in the form $\mathbf{w}_1(t) = \mathbf{x}_1(t) + i\mathbf{x}_2(t)$, where $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are the two *real* vector functions

$$\text{(4)} \qquad \mathbf{x}_1(t) := e^{\alpha t}\cos\beta t\,\mathbf{a} - e^{\alpha t}\sin\beta t\,\mathbf{b}\,,$$

$$\text{(5)} \qquad \mathbf{x}_2(t) := e^{\alpha t}\sin\beta t\,\mathbf{a} + e^{\alpha t}\cos\beta t\,\mathbf{b}\,.$$

†Recall that the complex roots of a polynomial equation with real coefficients must occur in complex conjugate pairs.

Since $\mathbf{w}_1(t)$ is a solution to (1), then

$$\mathbf{w}_1'(t) = \mathbf{A}\mathbf{w}_1(t) ,$$
$$\mathbf{x}_1' + i\mathbf{x}_2' = \mathbf{A}\mathbf{x}_1 + i\mathbf{A}\mathbf{x}_2 .$$

Equating the real and imaginary parts yields

$$\mathbf{x}_1'(t) = \mathbf{A}\mathbf{x}_1(t) \quad \text{and} \quad \mathbf{x}_2'(t) = \mathbf{A}\mathbf{x}_2(t) .$$

Hence, $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are real vector solutions to (1) associated with the complex conjugate eigenvalues $\alpha \pm i\beta$. Because $\mathbf{a}$ and $\mathbf{b}$ are not both the zero vector, it can be shown that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent vector functions on $(-\infty, \infty)$ (see Problem 15).

Let's summarize our findings.

Complex Eigenvalues

If the real matrix $\mathbf{A}$ has complex conjugate eigenvalues $\alpha \pm i\beta$ with corresponding eigenvectors $\mathbf{a} \pm i\mathbf{b}$, then two linearly independent real vector solutions to $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ are

(6) $$e^{\alpha t}\cos\beta t\,\mathbf{a} - e^{\alpha t}\sin\beta t\,\mathbf{b} ,$$

(7) $$e^{\alpha t}\sin\beta t\,\mathbf{a} + e^{\alpha t}\cos\beta t\,\mathbf{b} .$$

Example 1 Find a general solution of

(8) $$\mathbf{x}'(t) = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} \mathbf{x}(t) .$$

Solution The characteristic equation for $\mathbf{A}$ is

$$|\mathbf{A} - r\mathbf{I}| = \begin{vmatrix} -1-r & 2 \\ -1 & -3-r \end{vmatrix} = r^2 + 4r + 5 = 0 .$$

Hence, $\mathbf{A}$ has eigenvalues $r = -2 \pm i$.

To find a general solution, we need only find an eigenvector associated with the eigenvalue $r = -2 + i$. Substituting $r = -2 + i$ into $(\mathbf{A} - r\mathbf{I})\mathbf{z} = \mathbf{0}$ gives

$$\begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

The solutions can be expressed as $z_1 = 2s$ and $z_2 = (-1+i)s$, with s arbitrary. Hence, the eigenvectors associated with $r = -2 + i$ are $\mathbf{z} = s\,\mathrm{col}(2, -1+i)$. Taking $s = 1$ gives the eigenvector

$$\mathbf{z} = \begin{bmatrix} 2 \\ -1+i \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

We have found that $\alpha = -2, \beta = 1$, and $\mathbf{z} = \mathbf{a} + i\mathbf{b}$ with $\mathbf{a} = \mathrm{col}(2, -1)$, and $\mathbf{b} = \mathrm{col}(0, 1)$, so a general solution to (8) is

$$\mathbf{x}(t) = c_1 \left\{ e^{-2t}\cos t \begin{bmatrix} 2 \\ -1 \end{bmatrix} - e^{-2t}\sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$
$$+ c_2 \left\{ e^{-2t}\sin t \begin{bmatrix} 2 \\ -1 \end{bmatrix} + e^{-2t}\cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

(9) $$\mathbf{x}(t) = c_1 \begin{bmatrix} 2e^{-2t}\cos t \\ -e^{-2t}(\cos t + \sin t) \end{bmatrix} + c_2 \begin{bmatrix} 2e^{-2t}\sin t \\ e^{-2t}(\cos t - \sin t) \end{bmatrix} . \quad ◆$$

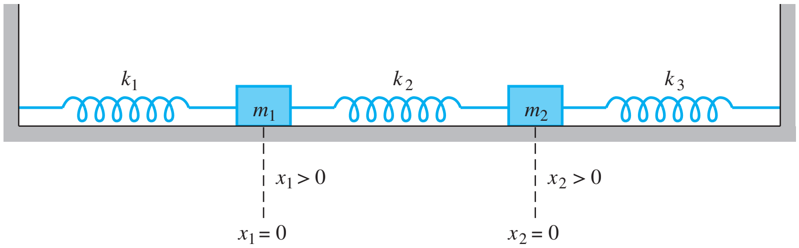

Figure 9.5 Coupled mass–spring system with fixed ends

Complex eigenvalues occur in modeling coupled mass–spring systems. For example, the motion of the mass–spring system illustrated in Figure 9.5 is governed by the second-order system

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1)\ , \\ m_2 x_2'' &= -k_2(x_2 - x_1) - k_3 x_2\ , \end{aligned} \tag{10}$$

where x_1 and x_2 represent the displacements of the masses m_1 and m_2 to the right of their equilibrium positions and k_1, k_2, k_3 are the spring constants of the three springs (see the discussion in Section 5.6). If we introduce the new variables $y_1 := x_1$, $y_2 := x_1'$, $y_3 := x_2$, $y_4 := x_2'$, then we can rewrite the system in the normal form

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(k_1+k_2)/m_1 & 0 & k_2/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k_2/m_2 & 0 & -(k_2+k_3)/m_2 & 0 \end{bmatrix} \mathbf{y}(t)\ . \tag{11}$$

For such a system, it turns out that **A** has only imaginary eigenvalues and they occur in complex conjugate pairs: $\pm i\beta_1$, $\pm i\beta_2$. Hence, any solution will consist of sums of sine and cosine functions. The frequencies of these functions

$$f_1 := \frac{\beta_1}{2\pi} \quad \text{and} \quad f_2 := \frac{\beta_2}{2\pi}$$

are called the **normal** or **natural frequencies** of the system (β_1 and β_2 are the **angular frequencies** of the system).

In some engineering applications, the only information that is required about a particular device is a knowledge of its normal frequencies; one must ensure that they are far from the frequencies that occur naturally in the device's operating environment (so that no resonances will be excited).

Example 2 Determine the normal frequencies for the coupled mass–spring system governed by system (11) when $m_1 = m_2 = 1$ kg, $k_1 = 1$ N/m, $k_2 = 2$ N/m, and $k_3 = 3$ N/m.

Solution To find the eigenvalues of **A**, we must solve the characteristic equation

$$|\mathbf{A} - r\mathbf{I}| = \begin{vmatrix} -r & 1 & 0 & 0 \\ -3 & -r & 2 & 0 \\ 0 & 0 & -r & 1 \\ 2 & 0 & -5 & -r \end{vmatrix} = r^4 + 8r^2 + 11 = 0\ .$$

From the quadratic formula we find $r^2 = -4 \pm \sqrt{5}$, so the four eigenvalues of $\mathbf{A}$ are $\pm i\sqrt{4-\sqrt{5}}$ and $\pm i\sqrt{4+\sqrt{5}}$. Hence, the two normal frequencies for this system are

$$\frac{\sqrt{4-\sqrt{5}}}{2\pi} \approx 0.211 \quad \text{and} \quad \frac{\sqrt{4+\sqrt{5}}}{2\pi} \approx 0.397 \text{ cycles per second.} \ \blacklozenge$$

9.6 EXERCISES

In Problems 1–4, find a general solution of the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ for the given matrix $\mathbf{A}$.

1. $\mathbf{A} = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix}$ **2.** $\mathbf{A} = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$

3. $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

4. $\mathbf{A} = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix}$

In Problems 5–9, find a fundamental matrix for the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ for the given matrix $\mathbf{A}$.

5. $\mathbf{A} = \begin{bmatrix} -1 & -2 \\ 8 & -1 \end{bmatrix}$ **6.** $\mathbf{A} = \begin{bmatrix} -2 & -2 \\ 4 & 2 \end{bmatrix}$

7. $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

8. $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -13 & 4 \end{bmatrix}$

9. $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$

In Problems 10–12, use a linear algebra software package to compute the required eigenvalues and eigenvectors for the given matrix $\mathbf{A}$ and then give a fundamental matrix for the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.

10. $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 13 & -4 & -12 & 4 \end{bmatrix}$

11. $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 2 \end{bmatrix}$

12. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -29 & -4 \end{bmatrix}$

In Problems 13 and 14, find the solution to the given system that satisfies the given initial condition.

13. $\mathbf{x}'(t) = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} \mathbf{x}(t)$,

(a) $\mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ **(b)** $\mathbf{x}(\pi) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(c) $\mathbf{x}(-2\pi) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **(d)** $\mathbf{x}(\pi/2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

14. $\mathbf{x}'(t) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{x}(t)$,

(a) $\mathbf{x}(0) = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$ **(b)** $\mathbf{x}(-\pi) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

15. Show that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ given by equations (4) and (5) are linearly independent on $(-\infty, \infty)$, provided $\beta \neq 0$ and $\mathbf{a}$ and $\mathbf{b}$ are not both the zero vector.

16. Show that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ given by equations (4) and (5) can be obtained as linear combinations of $\mathbf{w}_1(t)$ and $\mathbf{w}_2(t)$ given by equations (2) and (3). [*Hint:* Show that

$$\mathbf{x}_1(t) = \frac{\mathbf{w}_1(t) + \mathbf{w}_2(t)}{2}, \qquad \mathbf{x}_2(t) = \frac{\mathbf{w}_1(t) - \mathbf{w}_2(t)}{2i}.\Big]$$

In Problems 17 and 18, use the results of Problem 42 in Exercises 9.5 to find a general solution to the given Cauchy–Euler system for $t > 0$.

17. $t\mathbf{x}'(t) = \begin{bmatrix} -1 & -1 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{x}(t)$

18. $t\mathbf{x}'(t) = \begin{bmatrix} -1 & -1 \\ 9 & -1 \end{bmatrix} \mathbf{x}(t)$

19. For the coupled mass–spring system governed by system (10), assume $m_1 = m_2 = 1$ kg, $k_1 = k_2 = 2$ N/m, and $k_3 = 3$ N/m. Determine the normal frequencies for this coupled mass–spring system.

20. For the coupled mass–spring system governed by system (10), assume $m_1 = m_2 = 1$ kg, $k_1 = k_2 = k_3 = 1$ N/m, and assume initially that $x_1(0) = 0$ m, $x_1'(0) = 0$ m/sec, $x_2(0) = 2$ m, and $x_2'(0) = 0$ m/sec. Using matrix algebra techniques, solve this initial value problem.

21. ***RLC* Network.** The currents in the *RLC* network given by the schematic diagram in Figure 9.6 are governed by the following equations:

$$\begin{aligned} 4I_2'(t) + 52q_1(t) &= 10\,, \\ 13I_3(t) + 52q_1(t) &= 10\,, \\ I_1(t) &= I_2(t) + I_3(t)\,, \end{aligned}$$

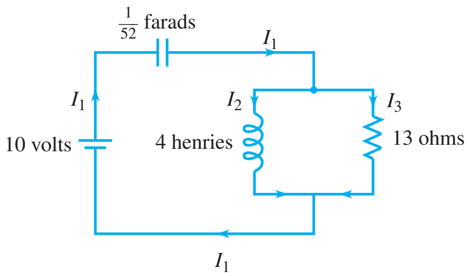

Figure 9.6 *RLC* network for Problem 21

where $q_1(t)$ is the charge on the capacitor, $I_1(t) = q_1'(t)$, and initially $q_1(0) = 0$ coulombs and $I_1(0) = 0$ amps. Solve for the currents I_1, I_2, and I_3. [*Hint:* Differentiate the first two equations, eliminate I_1, and form a normal system with $x_1 = I_2$, $x_2 = I_2'$, and $x_3 = I_3$.]

22. ***RLC* Network.** The currents in the *RLC* network given by the schematic diagram in Figure 9.7 are governed by the following equations:

$$\begin{aligned} 50I_1'(t) + 80I_2(t) &= 160\,, \\ 50I_1'(t) + 800q_3(t) &= 160\,, \\ I_1(t) &= I_2(t) + I_3(t)\,, \end{aligned}$$

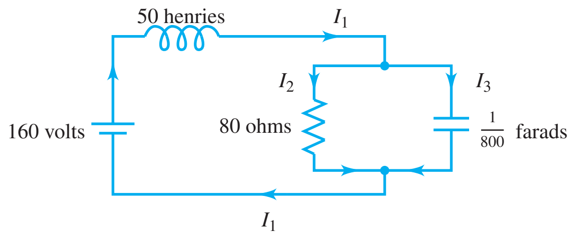

Figure 9.7 *RLC* network for Problem 22

where $q_3(t)$ is the charge on the capacitor, $I_3(t) = q_3'(t)$, and initially $q_3(0) = 0.5$ coulombs and $I_3(0) = 0$ amps. Solve for the currents I_1, I_2, and I_3. [*Hint:* Differentiate the first two equations, use the third equation to eliminate I_3, and form a normal system with $x_1 = I_1$, $x_2 = I_1'$, and $x_3 = I_2$.]

23. **Stability.** In Problem 49 of Exercises 9.5, (page 542), we discussed the notion of stability and asymptotic stability for a linear system of the form $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$. Assume that $\mathbf{A}$ has all distinct eigenvalues (real or complex).

(a) Show that the system is stable if and only if all the eigenvalues of $\mathbf{A}$ have nonpositive real part.

(b) Show that the system is asymptotically stable if and only if all the eigenvalues of $\mathbf{A}$ have negative real part.

24. **(a)** For Example 1, page 535, verify that

$$\mathbf{x}(t) = c_1\begin{bmatrix} -e^{-2t}\cos t + e^{-2t}\sin t \\ e^{-2t}\cos t \end{bmatrix} + c_2\begin{bmatrix} -e^{-2t}\sin t - e^{-2t}\cos t \\ e^{-2t}\sin t \end{bmatrix}$$

is another general solution to equation (8).

(b) How can the general solution of part (a) be directly obtained from the general solution derived in (9) on page 535?

9.7 Nonhomogeneous Linear Systems

The techniques discussed in Chapters 4 and 6 for finding a particular solution to the nonhomogeneous equation $y'' + p(x)y' + q(x)y = g(x)$ have natural extensions to nonhomogeneous linear systems.

Undetermined Coefficients

The method of undetermined coefficients can be used to find a particular solution to the nonhomogeneous linear system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$$

when $\mathbf{A}$ is an $n \times n$ *constant* matrix and the entries of $\mathbf{f}(t)$ are polynomials, exponential functions, sines and cosines, or finite sums and products of these functions. We can use the procedure box in Section 4.5 (page 184) and reproduced at the back of the book as a *guide* in choosing the form of a particular solution $\mathbf{x}_p(t)$. Some exceptions are discussed in the exercises (see Problems 25–28).

Example 1 Find a general solution of

$$\text{(1)} \qquad \mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + t\mathbf{g}\,, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{g} = \begin{bmatrix} -9 \\ 0 \\ -18 \end{bmatrix}.$$

Solution In Example 5 in Section 9.5, page 529, we found that a general solution to the corresponding homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\text{(2)} \qquad \mathbf{x}_h(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-3t}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Since the entries in $\mathbf{f}(t) := t\mathbf{g}$ are just linear functions of t, we are inclined to seek a particular solution of the form

$$\mathbf{x}_p(t) = t\mathbf{a} + \mathbf{b} = t\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

where the constant vectors $\mathbf{a}$ and $\mathbf{b}$ are to be determined. Substituting this expression for $\mathbf{x}_p(t)$ into system (1) yields

$$\mathbf{a} = \mathbf{A}(t\mathbf{a} + \mathbf{b}) + t\mathbf{g}\,,$$

which can be written as

$$t(\mathbf{A}\mathbf{a} + \mathbf{g}) + (\mathbf{A}\mathbf{b} - \mathbf{a}) = \mathbf{0}\,.$$

Setting the "coefficients" of this vector polynomial equal to zero yields the two systems

$$\text{(3)} \qquad \mathbf{A}\mathbf{a} = -\mathbf{g}\,,$$

$$\text{(4)} \qquad \mathbf{A}\mathbf{b} = \mathbf{a}\,.$$

By Gaussian elimination or by using a linear algebra software package, we can solve (3) for $\mathbf{a}$ and we find $\mathbf{a} = \operatorname{col}(5, 2, 4)$. Next we substitute for $\mathbf{a}$ in (4) and solve for $\mathbf{b}$ to obtain $\mathbf{b} = \operatorname{col}(1, 0, 2)$. Hence a particular solution for (1) is

$$\text{(5)} \qquad \mathbf{x}_p(t) = t\mathbf{a} + \mathbf{b} = t\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5t + 1 \\ 2t \\ 4t + 2 \end{bmatrix}.$$

A general solution for (1) is $\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$, where $\mathbf{x}_h(t)$ is given in (2) and $\mathbf{x}_p(t)$ in (5). ◆

In the preceding example, the nonhomogeneous term $\mathbf{f}(t)$ was a vector polynomial. If, instead, $\mathbf{f}(t)$ has the form

$$\mathbf{f}(t) = \operatorname{col}(1, t, \sin t)\,,$$

then, using the superposition principle, we would seek a particular solution of the form

$$\mathbf{x}_p(t) = t\mathbf{a} + \mathbf{b} + (\sin t)\mathbf{c} + (\cos t)\mathbf{d}\,.$$

Similarly, if

$$\mathbf{f}(t) = \operatorname{col}(t, e^t, t^2),$$

we would take

$$\mathbf{x}_p(t) = t^2\mathbf{a} + t\mathbf{b} + \mathbf{c} + e^t\mathbf{d}.$$

Of course, we must modify our guess, should one of the terms be a solution to the corresponding homogeneous system. If this is the case, the annihilator method [equations (15) and (16) of Section 6.3, page 337] would appear to suggest that for a nonhomogeneity $\mathbf{f}(t)$ of the form $e^{rt}t^m\mathbf{g}$, where r is an eigenvalue of $\mathbf{A}$, m is a nonnegative integer, and $\mathbf{g}$ is a constant vector, a particular solution of $\mathbf{x}' = \mathbf{Ax} + \mathbf{f}$ can be found in the form

$$\mathbf{x}_p(t) = e^{rt}\{t^{m+s}\mathbf{a}_{m+s} + t^{m+s-1}\mathbf{a}_{m+s-1} + \cdots + t\mathbf{a}_1 + \mathbf{a}_0\},$$

for a suitable choice of s. We omit the details.

Variation of Parameters

In Section 4.6 we discussed the method of variation of parameters for a general constant-coefficient second-order linear equation. Simply put, the idea is that if a general solution to the homogeneous equation has the form $x_h(t) = c_1x_1(t) + c_2x_2(t)$, where $x_1(t)$ and $x_2(t)$ are linearly independent solutions to the homogeneous equation, then a particular solution to the nonhomogeneous equation would have the form $x_p(t) = v_1(t)x_1(t) + v_2(t)x_2(t)$, where $v_1(t)$ and $v_2(t)$ are certain functions of t. A similar idea can be used for systems.

Let $\mathbf{X}(t)$ be a fundamental matrix for the homogeneous system

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t), \tag{6}$$

where now *the entries of* $\mathbf{A}$ *may be any continuous functions of* t. Because a general solution to (6) is given by $\mathbf{X}(t)\mathbf{c}$, where $\mathbf{c}$ is a constant $n \times 1$ vector, we seek a particular solution to the nonhomogeneous system

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) \tag{7}$$

of the form

$$\mathbf{x}_p(t) = \mathbf{X}(t)\boldsymbol{v}(t), \tag{8}$$

where $\boldsymbol{v}(t) = \operatorname{col}\big(v_1(t), \ldots, v_n(t)\big)$ is a vector function of t to be determined.

To derive a formula for $\boldsymbol{v}(t)$, we first differentiate (8) using the matrix version of the product rule to obtain

$$\mathbf{x}_p'(t) = \mathbf{X}(t)\boldsymbol{v}'(t) + \mathbf{X}'(t)\boldsymbol{v}(t).$$

Substituting the expressions for $\mathbf{x}_p(t)$ and $\mathbf{x}_p'(t)$ into (7) yields

$$\mathbf{X}(t)\boldsymbol{v}'(t) + \mathbf{X}'(t)\boldsymbol{v}(t) = \mathbf{A}(t)\mathbf{X}(t)\boldsymbol{v}(t) + \mathbf{f}(t). \tag{9}$$

Since $\mathbf{X}(t)$ satisfies the matrix equation $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$, equation (9) becomes

$$\mathbf{X}\boldsymbol{v}' + \mathbf{AX}\boldsymbol{v} = \mathbf{AX}\boldsymbol{v} + \mathbf{f},$$
$$\mathbf{X}\boldsymbol{v}' = \mathbf{f}.$$

Multiplying both sides of the last equation by $\mathbf{X}^{-1}(t)$ [which exists since the columns of $\mathbf{X}(t)$ are linearly independent] gives

$$\boldsymbol{v}'(t) = \mathbf{X}^{-1}(t)\mathbf{f}(t).$$

Integrating, we obtain

$$\boldsymbol{v}(t) = \int \mathbf{X}^{-1}(t)\mathbf{f}(t)dt\,.$$

Hence, a particular solution to (7) is

$$\mathbf{x}_p(t) = \mathbf{X}(t)\boldsymbol{v}(t) = \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\mathbf{f}(t)dt\,. \tag{10}$$

Combining (10) with the solution $\mathbf{X}(t)\mathbf{c}$ to the homogeneous system yields the following general solution to (7):

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\mathbf{f}(t)dt\,. \tag{11}$$

The elegance of the derivation of the variation of parameters formula (10) for systems becomes evident when one compares it with the more lengthy derivations for the scalar case in Sections 4.6 and 6.4.

Given an initial value problem of the form

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)\,, \qquad \mathbf{x}(t_0) = \mathbf{x}_0\,, \tag{12}$$

we can use the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ to solve for $\mathbf{c}$ in (11). Expressing $\mathbf{x}(t)$ using a definite integral, we have

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t)\int_{t_0}^{t} \mathbf{X}^{-1}(s)\mathbf{f}(s)ds\,.$$

From the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, we find

$$\mathbf{x}_0 = \mathbf{x}(t_0) = \mathbf{X}(t_0)\mathbf{c} + \mathbf{X}(t_0)\int_{t_0}^{t_0} \mathbf{X}^{-1}(s)\mathbf{f}(s)ds = \mathbf{X}(t_0)\mathbf{c}\,.$$

Solving for $\mathbf{c}$, we have $\mathbf{c} = \mathbf{X}^{-1}(t_0)\mathbf{x}_0$. Thus, the solution to (12) is given by the formula

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0 + \mathbf{X}(t)\int_{t_0}^{t} \mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds\,. \tag{13}$$

To apply the variation of parameters formulas, we first must determine a fundamental matrix $\mathbf{X}(t)$ for the homogeneous system. In the case when the coefficient matrix $\mathbf{A}$ is constant, we have discussed methods for finding $\mathbf{X}(t)$. However, if the entries of $\mathbf{A}$ depend on t, the determination of $\mathbf{X}(t)$ may be extremely difficult (entailing, perhaps, a matrix power series!).

Example 2 Find the solution to the initial value problem

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \tag{14}$$

Solution In Example 3 in Section 9.5, we found two linearly independent solutions to the corresponding homogeneous system; namely,

$$\mathbf{x}_1(t) = \begin{bmatrix} 3e^t \\ e^t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}.$$

Hence a fundamental matrix for the homogeneous system is

$$\mathbf{X}(t) = \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix}.$$

Although the solution to (14) can be found via the method of undetermined coefficients, we shall find it directly from formula (13). For this purpose, we need $\mathbf{X}^{-1}(t)$. By formula (3) of Section 9.3 (page 510):

$$\mathbf{X}^{-1}(t) = \begin{bmatrix} \frac{1}{2}e^{-t} & -\frac{1}{2}e^{-t} \\ -\frac{1}{2}e^t & \frac{3}{2}e^t \end{bmatrix}.$$

Substituting into formula (13), we obtain the solution

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix} \int_0^t \begin{bmatrix} \frac{1}{2}e^{-s} & -\frac{1}{2}e^{-s} \\ -\frac{1}{2}e^s & \frac{3}{2}e^s \end{bmatrix} \begin{bmatrix} e^{2s} \\ 1 \end{bmatrix} ds \\ &= \begin{bmatrix} -\frac{3}{2}e^t + \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^t + \frac{1}{2}e^{-t} \end{bmatrix} + \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix} \int_0^t \begin{bmatrix} \frac{1}{2}e^s - \frac{1}{2}e^{-s} \\ -\frac{1}{2}e^{3s} + \frac{3}{2}e^s \end{bmatrix} ds \\ &= \begin{bmatrix} -\frac{3}{2}e^t + \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^t + \frac{1}{2}e^{-t} \end{bmatrix} + \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2}e^t + \frac{1}{2}e^{-t} - 1 \\ \frac{3}{2}e^t - \frac{1}{6}e^{3t} - \frac{4}{3} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{9}{2}e^t - \frac{5}{6}e^{-t} + \frac{4}{3}e^{2t} + 3 \\ -\frac{3}{2}e^t - \frac{5}{6}e^{-t} + \frac{1}{3}e^{2t} + 2 \end{bmatrix}. \quad \blacklozenge \end{aligned}$$

9.7 EXERCISES

In Problems 1–6, use the method of undetermined coefficients to find a general solution to the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$, where $\mathbf{A}$ and $\mathbf{f}(t)$ are given.

1. $\mathbf{A} = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} -11 \\ -5 \end{bmatrix}$

2. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} -t-1 \\ -4t-2 \end{bmatrix}$

3. $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 2e^t \\ 4e^t \\ -2e^t \end{bmatrix}$

4. $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} -4\cos t \\ -\sin t \end{bmatrix}$

5. $\mathbf{A} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} e^{2t} \\ \sin t \\ t \end{bmatrix}$

6. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{f}(t) = e^{-2t}\begin{bmatrix} t \\ 3 \end{bmatrix}$

In Problems 7–10, use the method of undetermined coefficients to determine only the form of a particular solution for the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$, *where* $\mathbf{A}$ *and* $\mathbf{f}(t)$ *are given.*

7. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} \sin 3t \\ t \end{bmatrix}$

8. $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} t^2 \\ t+1 \end{bmatrix}$

9. $\mathbf{A} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} e^{2t} \\ \sin t \\ t \end{bmatrix}$

10. $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} te^{-t} \\ 3e^{-t} \end{bmatrix}$

In Problems 11–16, use the variation of parameters formula (11) to find a general solution of the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$, *where* $\mathbf{A}$ *and* $\mathbf{f}(t)$ *are given.*

11. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

12. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

13. $\mathbf{A} = \begin{bmatrix} 8 & -4 \\ 4 & -2 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} t^{-2}/2 \\ t^{-2} \end{bmatrix}$

14. $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} t^2 \\ 1 \end{bmatrix}$

15. $\mathbf{A} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} t^{-1} \\ 4 + 2t^{-1} \end{bmatrix}$

16. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 8 \sin t \\ 0 \end{bmatrix}$

In Problems 17–20, use the variation of parameters formulas (11) and possibly a linear algebra software package to find a general solution of the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t)$, *where* $\mathbf{A}$ *and* $\mathbf{f}(t)$ *are given.*

17. $\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 3e^t \\ -e^t \\ -e^t \end{bmatrix}$

18. $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ e^t \\ e^t \end{bmatrix}$

19. $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} t \\ 0 \\ e^{-t} \\ t \end{bmatrix}$

20. $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & -4 & -2 & -1 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} e^t \\ 0 \\ 1 \\ 0 \end{bmatrix}$

In Problems 21 and 22, find the solution to the given system that satisfies the given initial condition.

21. $\mathbf{x}'(t) = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} e^t \\ -e^t \end{bmatrix},$

(a) $\mathbf{x}(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ **(b)** $\mathbf{x}(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) $\mathbf{x}(5) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **(d)** $\mathbf{x}(-1) = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$

22. $\mathbf{x}'(t) = \begin{bmatrix} 0 & 2 \\ 4 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 4t \\ -4t-2 \end{bmatrix},$

(a) $\mathbf{x}(0) = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$ **(b)** $\mathbf{x}(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

23. Using matrix algebra techniques and the method of undetermined coefficients, find a general solution for

$$x''(t) + y'(t) - x(t) + y(t) = -1\,,$$
$$x'(t) + y'(t) - x(t) = t^2\,.$$

Compare your solution with the solution in Example 4 in Section 5.2, page 247.

24. Using matrix algebra techniques and the method of undetermined coefficients, solve the initial value problem

$$x'(t) - 2y(t) = 4t\,, \qquad x(0) = 4\,;$$
$$y'(t) + 2y(t) - 4x(t) = -4t - 2\,, \qquad y(0) = -5\,.$$

Compare your solution with the solution in Example 1 in Section 7.10, page 412.

25. To find a general solution to the system

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\mathbf{x}(t) + \mathbf{f}(t)\,, \quad \text{where} \quad \mathbf{f}(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix},$$

proceed as follows:

(a) Find a fundamental solution set for the corresponding homogeneous system.

(b) The obvious choice for a particular solution would be a vector function of the form $\mathbf{x}_p(t) = e^t\mathbf{a}$; however, the homogeneous system has a solution of this form. The next choice would be $\mathbf{x}_p(t) = te^t\mathbf{a}$. Show that this choice does *not* work.

(c) For systems, multiplying by t is not always sufficient. The proper guess is

$$\mathbf{x}_p(t) = te^t\mathbf{a} + e^t\mathbf{b}\,.$$

Use this guess to find a particular solution of the given system.

(d) Use the results of parts (a) and (c) to find a general solution of the given system.

26. For the system of Problem 25, we found that a proper guess for a particular solution is $\mathbf{x}_p(t) = te^t\mathbf{a} + e^t\mathbf{b}$. In some cases $\mathbf{a}$ or $\mathbf{b}$ may be zero.

(a) Find a particular solution for the system of Problem 25 if $\mathbf{f}(t) = \text{col}(3e^t, 6e^t)$.

(b) Find a particular solution for the system of Problem 25 if $\mathbf{f}(t) = \text{col}(e^t, e^t)$.

27. Find a general solution of the system

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -1 \\ -1 - e^{-t} \\ -2e^{-t} \end{bmatrix}.$$

[*Hint:* Try $\mathbf{x}_p(t) = e^{-t}\mathbf{a} + te^{-t}\mathbf{b} + \mathbf{c}$.]

28. Find a particular solution for the system

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

[*Hint:* Try $\mathbf{x}_p(t) = t\mathbf{a} + \mathbf{b}$.]

In Problems 29 and 30, find a general solution to the given Cauchy–Euler system for $t > 0$. (See Problem 42 in Exercises 9.5, page 533.) Remember to express the system in the form $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$ before using the variation of parameters formula.

29. $t\mathbf{x}'(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} t^{-1} \\ 1 \end{bmatrix}$

30. $t\mathbf{x}'(t) = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} t \\ 2t \end{bmatrix}$

31. Use the variation of parameters formula (10) to derive a formula for a particular solution y_p to the scalar equation $y'' + p(t)y' + q(t)y = g(t)$ in terms of two linearly independent solutions $y_1(t), y_2(t)$ of the corresponding homogeneous equation. Show that your answer agrees with the formulas derived in Section 4.6. [*Hint:* First write the scalar equation in system form.]

32. Conventional Combat Model. A simplistic model of a pair of conventional forces in combat yields the following system:

$$\mathbf{x}' = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \mathbf{x} + \begin{bmatrix} p \\ q \end{bmatrix},$$

where $\mathbf{x} = \text{col}(x_1, x_2)$. The variables $x_1(t)$ and $x_2(t)$ represent the strengths of opposing forces at time t. The terms $-ax_1$ and $-dx_2$ represent the *operational loss rates*, and the terms $-bx_2$ and $-cx_1$ represent the *combat loss rates* for the troops x_1 and x_2, respectively. The constants p and q represent the respective rates of reinforcement. Let $a = 1$, $b = 4$, $c = 3$, $d = 2$, and $p = q = 5$. By solving the appropriate initial value problem, determine which forces will win if

(a) $x_1(0) = 20$, $\quad x_2(0) = 20$.

(b) $x_1(0) = 21$, $\quad x_2(0) = 20$.

(c) $x_1(0) = 20$, $\quad x_2(0) = 21$.

33. *RL* Network. The currents in the *RL* network given by the schematic diagram in Figure 9.8 are governed by the following equations:

$$\begin{aligned} 2I_1'(t) + 90I_2(t) &= 9, \\ I_3'(t) + 30I_4(t) - 90I_2(t) &= 0, \\ 60I_5(t) - 30I_4(t) &= 0, \\ I_1(t) &= I_2(t) + I_3(t), \\ I_3(t) &= I_4(t) + I_5(t). \end{aligned}$$

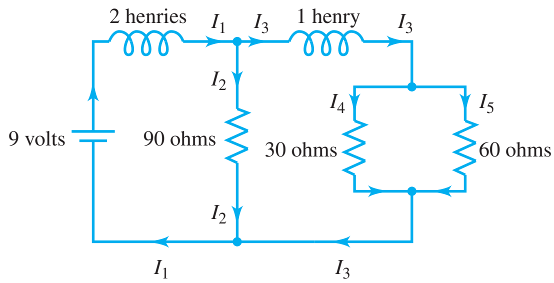

Figure 9.8 *RL* network for Problem 33

Assume the currents are initially zero. Solve for the five currents $I_1, \ldots, I_5$. [*Hint:* Eliminate all unknowns except I_2 and I_5, and form a normal system with $x_1 = I_2$ and $x_2 = I_5$.]

34. Mixing Problem. Two tanks A and B, each holding 50 L of liquid, are interconnected by pipes. The liquid flows from tank A into tank B at a rate of 4 L/min and from B into A at a rate of 1 L/min (see Figure 9.9). The liquid inside each tank is kept well stirred. A brine solution that has a concentration of 0.2 kg/L of salt flows into tank A at a rate of 4 L/min. A brine solution that has a concentration of 0.1 kg/L of salt flows into tank B at a rate of 1 L/min. The solutions flow out of the system from both tanks—from tank A at 1 L/min and from tank B at 4 L/min. If, initially, tank A contains pure water and tank B contains 0.5 kg of salt, determine the mass of salt in each tank at time $t \geq 0$. After several minutes have elapsed, which tank has the higher concentration of salt? What is its limiting concentration?

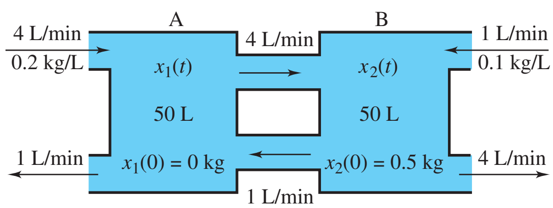

Figure 9.9 Mixing problem for interconnected tanks